Gravitational Waves to Thermal CFT Data

A work in progress by BC[1] and Sean Colin-Ellerin[1]

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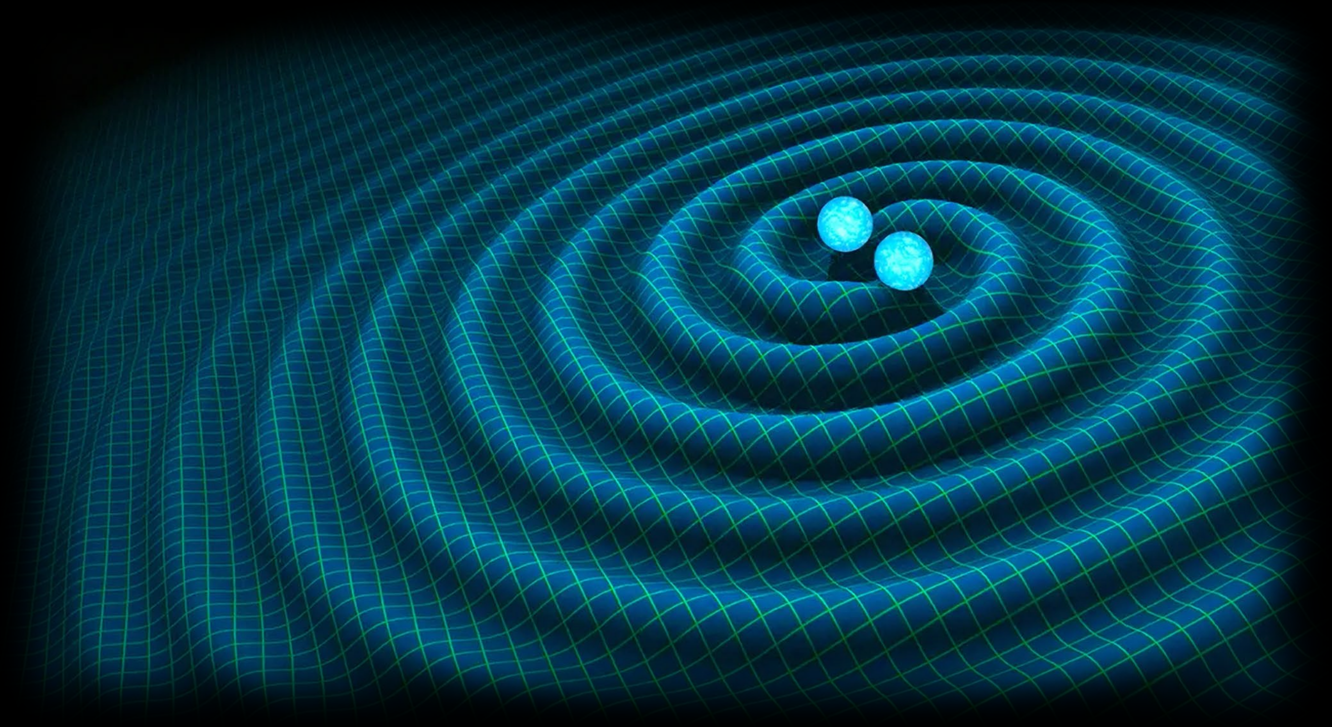
The Big Idea:

Let's use what we know about the **Quasi Normal Modes** of a Black Hole to discover something interesting about **Thermal Conformal Field Theories**!

VIRGO



Talk outline

- What is a Quasi Normal Mode?
- What is a Thermal CFT?
- Why might the two be related?
- What do we learn?

What is a Normal Mode?

- We can imagine a classical, **finite**, linear oscillating system like an electromagnetic cavity, string/membrane, etc.

$$\chi_n(t,x) = e^{i\omega_n t}\chi_n(x), \quad n = 1,2,3\ldots$$

$$\chi(t,x) = \sum_{n=1}^{\infty} a_n e^{i\omega_n t}\chi_n(x)$$

- The problem becomes a boundary value problem

What is a Normal Mode?

- If the system is instead "open" like an infinite string

$$\frac{\partial^2}{\partial t^2}\chi + \left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\chi = 0$$

- We now only need information about the function at an initial time (and its first and second derivatives)
- Two types of solutions: spatially compact or infinite energy
- "improper eigenfunctions" are not bounded, like plane waves

What is a Quasi Normal Mode (QNM)?

- Let's consider the wave equation for $V \geq 0$ that vanishes for $|x| < x_0$
- The Laplace transform is

$$\hat{\chi}(s,x) = \int_0^\infty e^{-st}\chi(t,x)dt$$

- And therefore

$$\underbrace{s^2\hat{\chi} - \hat{\chi}'' + V\hat{\chi}}_{=0 \text{ is the homogeneous equation}} = +s\chi(0,x) + \partial_t\chi(0,x)$$

What is a Quasi Normal Mode (QNM)?

- If we choose two independent solutions of the homogeneous equation f_+ and f_- we get the Green's function

$$G(s,x,x') = \frac{1}{W(s)} \begin{array}{ll} f_-(s,x')f_+(s,x) & (x' < x), \\ f_-(s,x)f_+(s,x') & (x' > x), \end{array}$$

- We can then define an "inhomogeneity" *j*

$$\hat{\chi}(s,x) = \int_{-\infty}^{\infty} G(s,x,x')j(s,x')dx'$$

What is a Quasi Normal Mode (QNM)?

- Solving the homogeneous equation gives us

$$s^2\hat{\chi} - \hat{\chi}'' + V\hat{\chi} = +s\chi(0,x) + \partial_t\chi(0,x)$$

=0 is the homogeneous equation

$$f_+ = e^{-sx} \quad \text{for } x > x_0, \qquad f_- = e^{+sx} \quad \text{for } x < -x_0,$$

but for small x, f_+ and f_- will be some linear combination of exponentials

What is a Quasi Normal Mode (QNM)?

- A quasi-normal mode is a value s_n such that

$$f_+(s_n, x) = c(s_n) f_-(s_n, x)$$

- In these places the Green's function is singular

$$G(s,x,x') = \frac{1}{W(s)} \begin{cases} f_-(s,x') f_+(s,x) & (x' < x), \\ f_-(s,x) f_+(s,x') & (x' > x), \end{cases}$$

QNMs for Black Holes

- Black Hole potential for “axial” perturbations

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2\sigma M}{r^3}\right]$$

- Black Hole potential for “polar” perturbations

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\frac{2n^2(n+1)r^3 + 6n^2Mr^2 + 18nM^2r + 18M^3}{r^3(nr+3M)^2},$$

these both go to zero at the horizon (r = 2M) and infinity

Kokkotas and Schmidt https://arxiv.org/pdf/gr-qc/9909058.pdf

QNMs for Black Holes

- Then one can already see that we must look for QNMs of the Black Hole

- Oscillations can correspond to scalar fields, photons, or gravitational perturbations!

Kokkotas and Schmidt https://arxiv.org/pdf/gr-qc/9909058.pdf

Black Holes

- Black Holes have…
  - Energy $E = M$ [Einstein, Schwarzschild]
  - Entropy $S = A/4$ [Bekenstein]
  - Inverse temperature $\beta = 8\pi M$ [Hawking]

Schwarzschild, K. (1916). "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie"
Bekenstein, J. (1974). Generalized second law of thermodynamics in black-hole physics
Hawking, SW. (1975). Particle Creation by Black Holes

Black Holes

- Black Holes in flat spacetimes evaporate [Hawking]

- We would like to think about Black Holes that are equilibrium solutions, so we need to look at Black Holes in curved spacetime

Anti-deSitter Space (AdS)

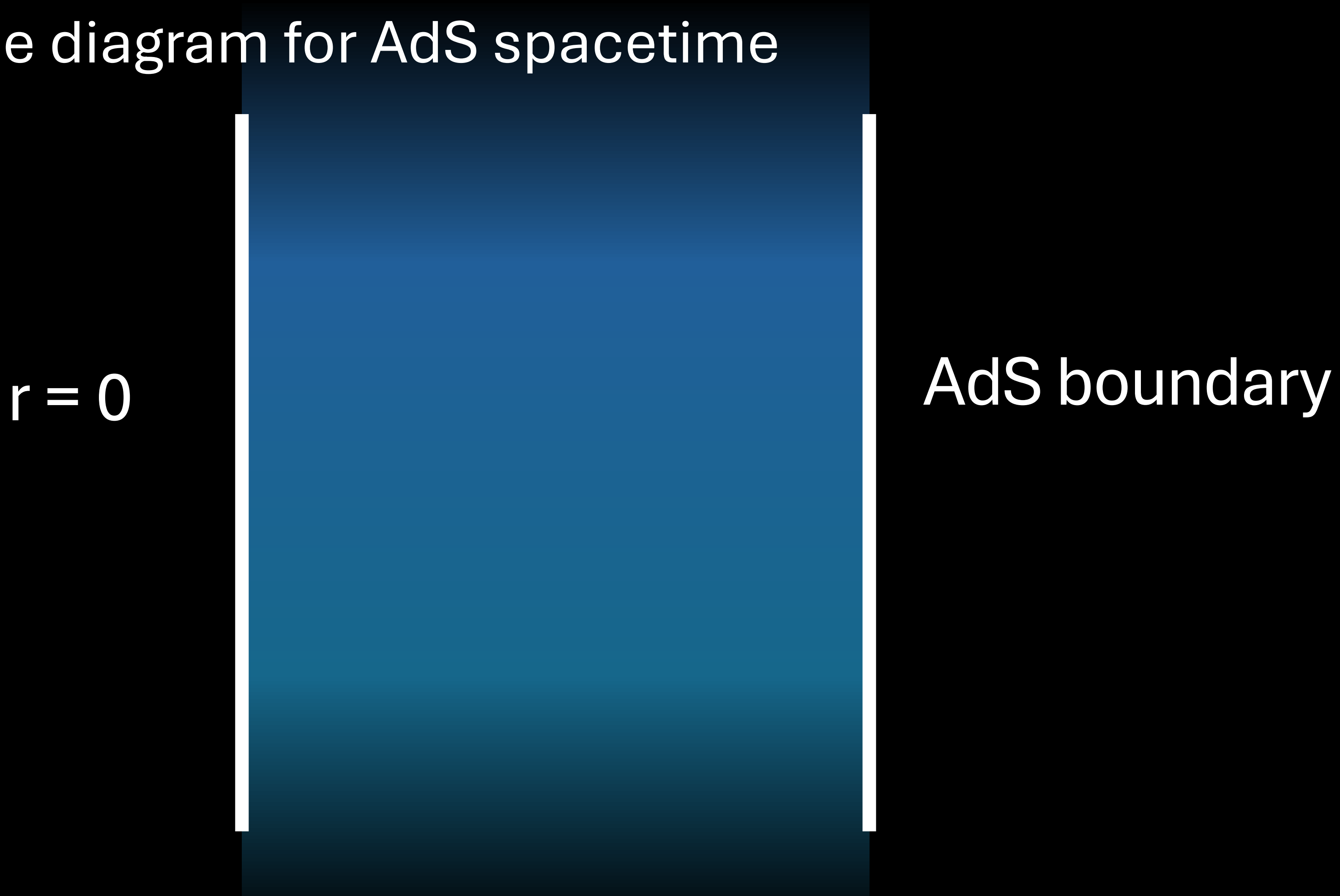
- Maximally symmetric spacetime with constant negative curvature

$$ds^2 = \frac{1}{y^2}\left(-dt^2 + dy^2 + \sum_i dx_i^2\right)$$

$$ds^2 = -\left(k^2 r^2 + 1\right) dt^2 + \frac{1}{k^2 r^2 + 1} dr^2 + r^2 d\Omega^2$$

Anti-deSitter Space (AdS)

- Penrose diagram for AdS spacetime



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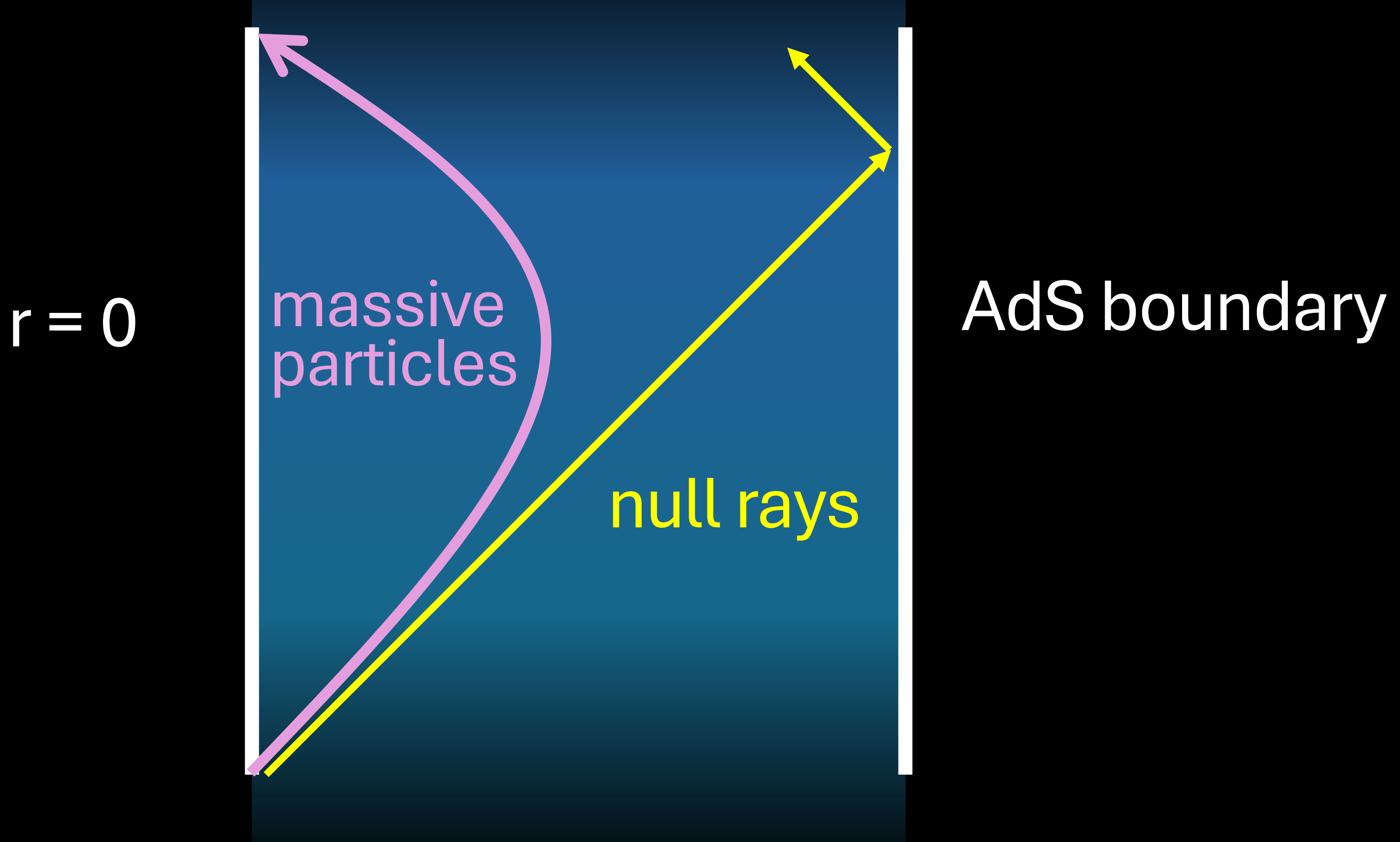
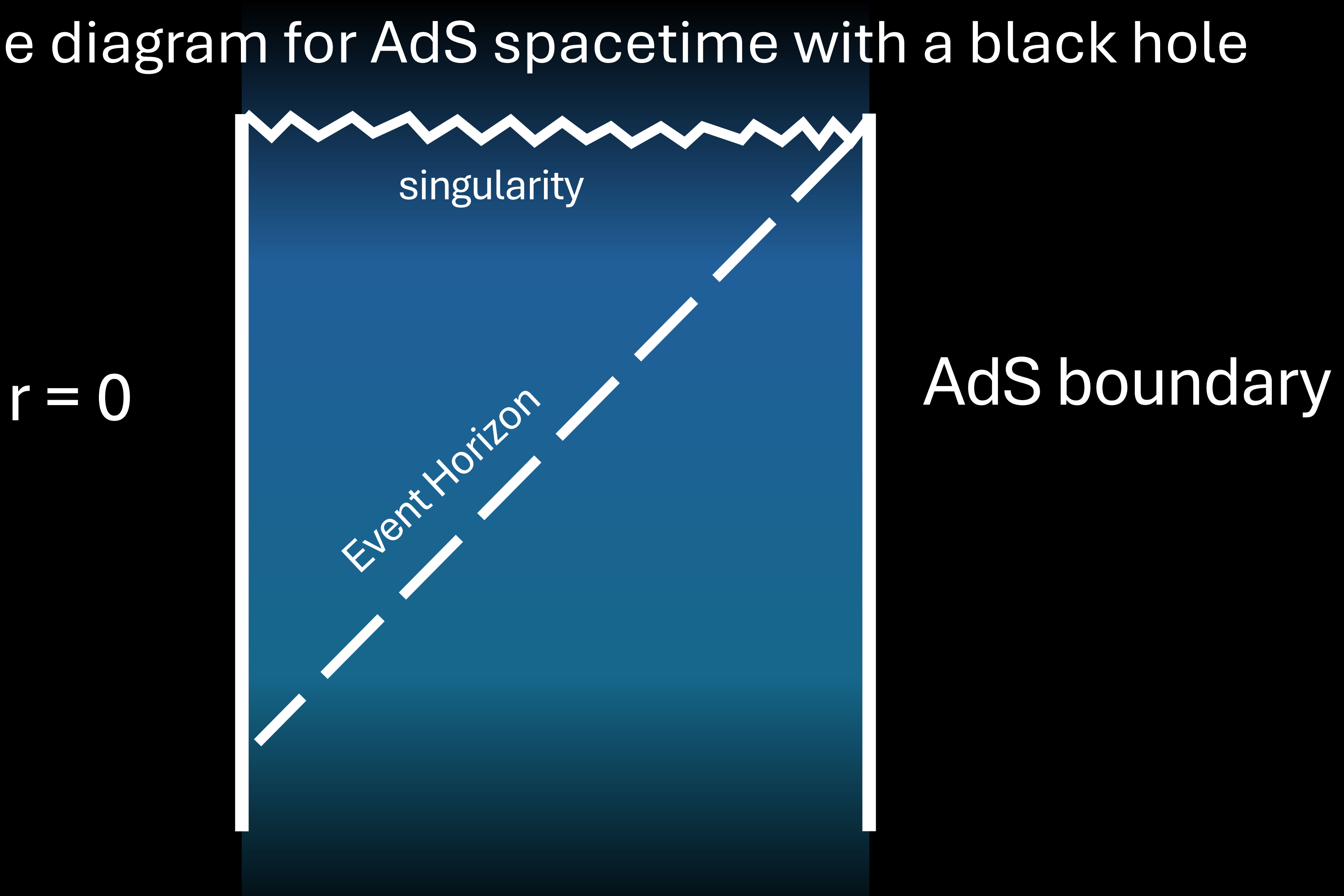


Anti-deSitter Space (AdS)

- Penrose diagram for AdS spacetime with a black hole



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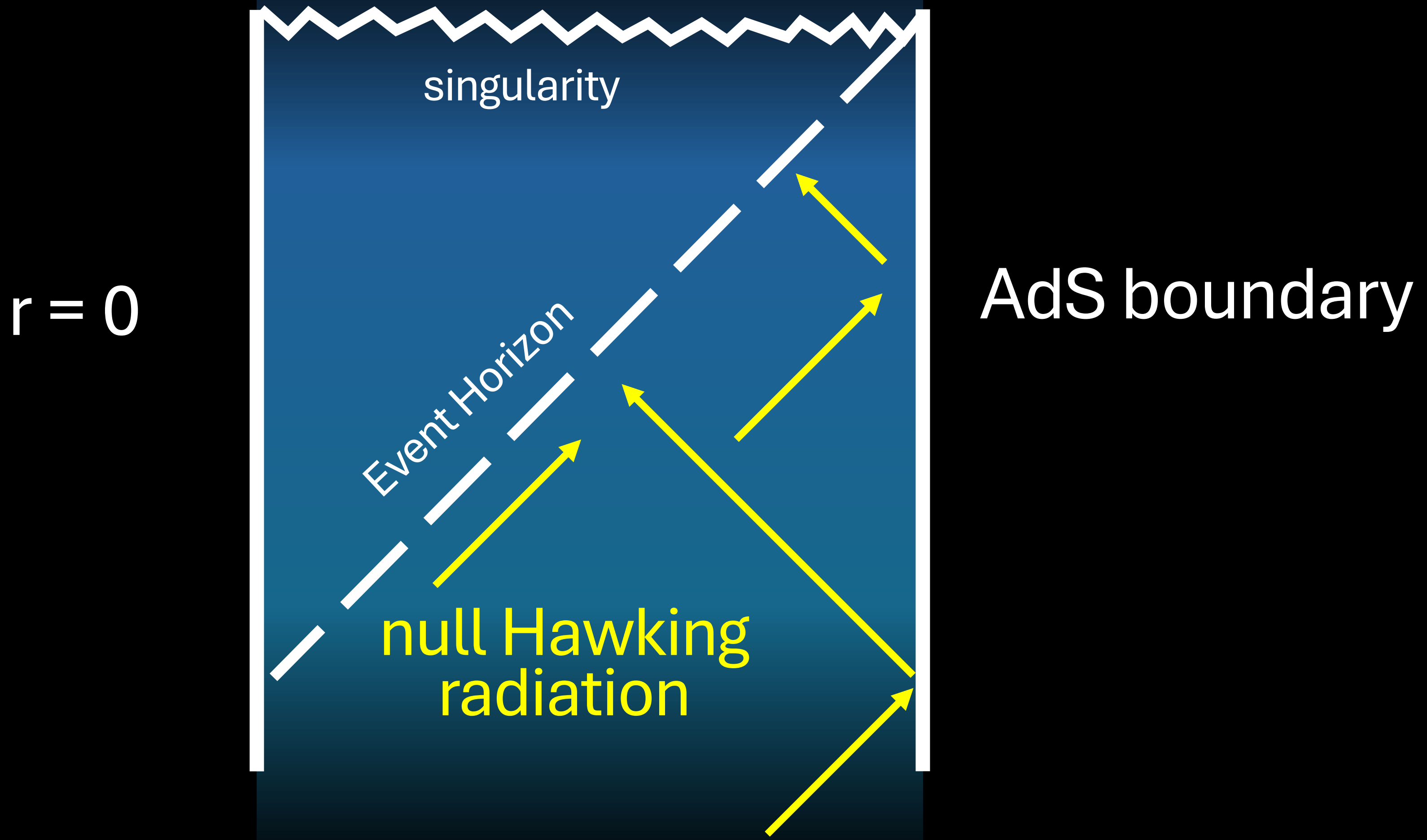

Black Holes

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- In AdS, null rays return from the boundary in finite time, so a black hole can come into thermal equilibrium with its own radiation

Holography: AdS/CFT

- AdS/CFT [Maldacena] is the statement that there is a "dictionary" that takes the physics of an Anti-deSitter (AdS) spacetime to the physics of a Conformal Field Theory (CFT) in one less spatial dimension

Maldacena, https://arxiv.org/abs/hep-th/9711200

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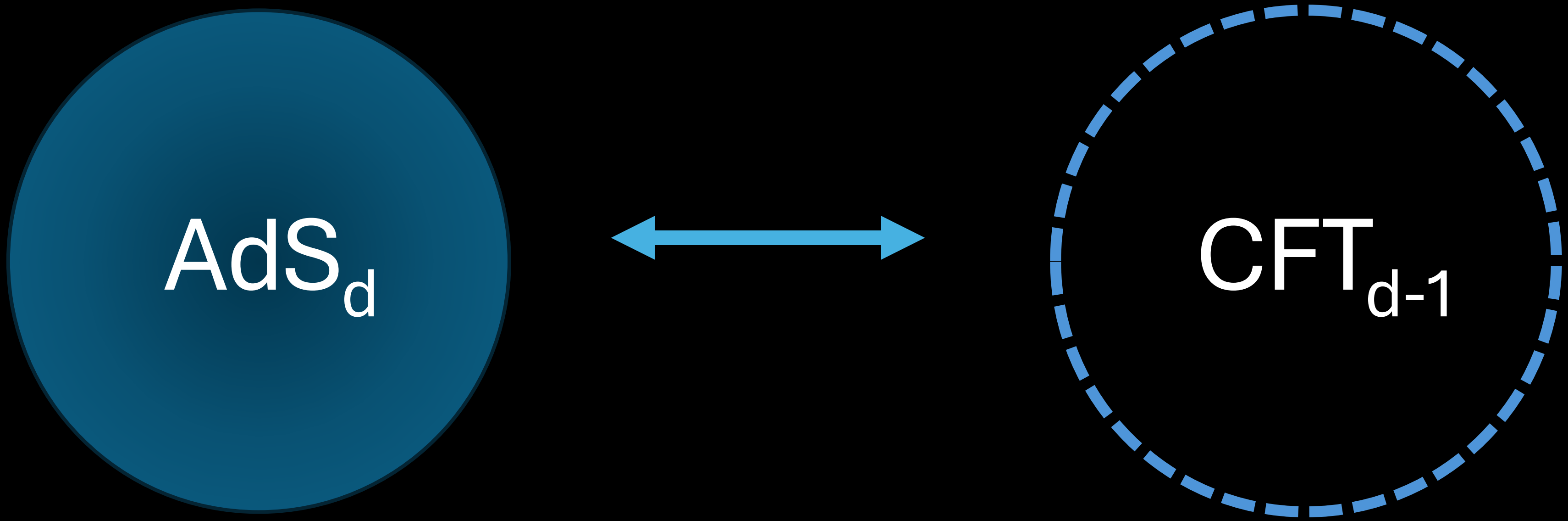
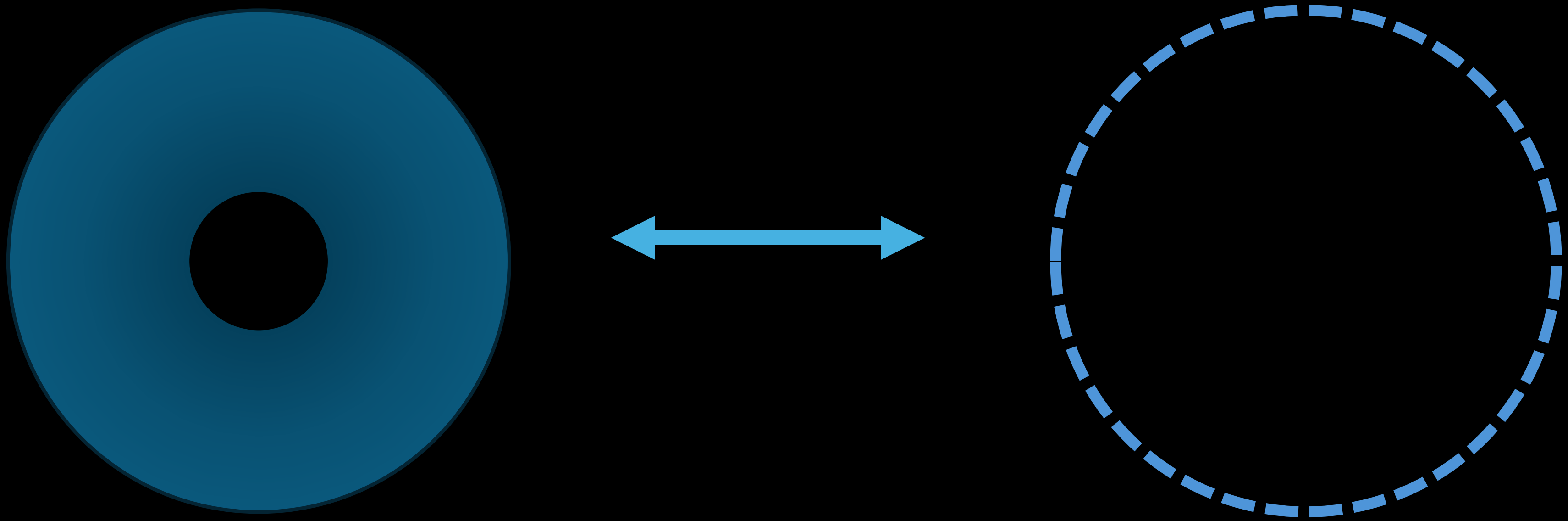


Holography with a Black Hole

A black hole of mass M in AdS_d

A thermal CFT_{d-1} with $\beta = 8\pi M$



Conformal Field Theory (CFT)

- CFTs are Quantum Field Theories with conformal symmetries

- An important thing to know is the state operator correspondence:

  Every state can be written in terms of an operator acting on the vacuum state near the origin

$$|\phi\rangle = \lim_{z,\bar{z}\mapsto 0} \phi(z,\bar{z})|0\rangle$$

Conformal Field Theory (CFT)

- Operator Product Expansion:

  A product of two operators can be written as a sum of contributions from all the operators in the theory

$$\phi(x)\phi(y) = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} P(x-y, \partial_y)\mathcal{O}(y)$$

One point functions of all non-identity operators vanish due to translation/scaling invariance

Conformal Field Theory (CFT)

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primary operator

fixed by conformal symmetry

“CFT data”

One point functions of all non-identity operators vanish due to translation/scaling invariance

Thermal CFT

- In thermal CFT, operator one point functions are not trivial

$$\langle \mathcal{O} \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}$$

We interpret temperature as Euclidean time, β is the length of the circle

L. Iliesiu, M. Kologlu, R. Mahajan, E. Perlmutter, D. Simmons-Duffin. “The Conformal Bootstrap at Finite Temperature”

Thermal CFT

- We can relate one point, two point, and three point data:

$$a_{\mathcal{O}} \equiv \frac{f_{\phi\phi\mathcal{O}} b_{\mathcal{O}}}{c_{\mathcal{O}}} \frac{J!}{2^J (\nu)_J}.$$

$$\langle \phi(x)\phi(0) \rangle_\beta = \sum_{\mathcal{O} \in \phi\times\phi} \frac{a_{\mathcal{O}}}{\beta^\Delta} C_J^{(\nu)}(\eta) |x|^{\Delta - 2\Delta_\phi}$$

In a holographic CFT, we can find one point data “easily”

If we can learn the $a_{\mathcal{O}}$’s, we learn all the data

L. Iliesiu, M. Kologlu, R. Mahajan, E. Perlmutter, D. Simmons-Duffin. “The Conformal Bootstrap at Finite Temperature”

Thermal CFT

From the two point function, we learn the Thermal Correlator Green's Function

We know some things about the Green's Function (in frequency space)

- It is a meromorphic function of frequency ω
- It is an entire function (it has no zeroes)
- It has poles at every QNM ω_n

M. Dodelson, C. Iossa, R. Karlsson, A. Zhiboedova. A Thermal Product Formula

Thermal CFT

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- It is a meromorphic function of frequency ω
- It is an entire function (it has no zeroes)
- It has poles at every QNM ω_n

This fixes the entire Green's Function up to a function of momentum ***k***

"Hadamard Factorization Theorem"

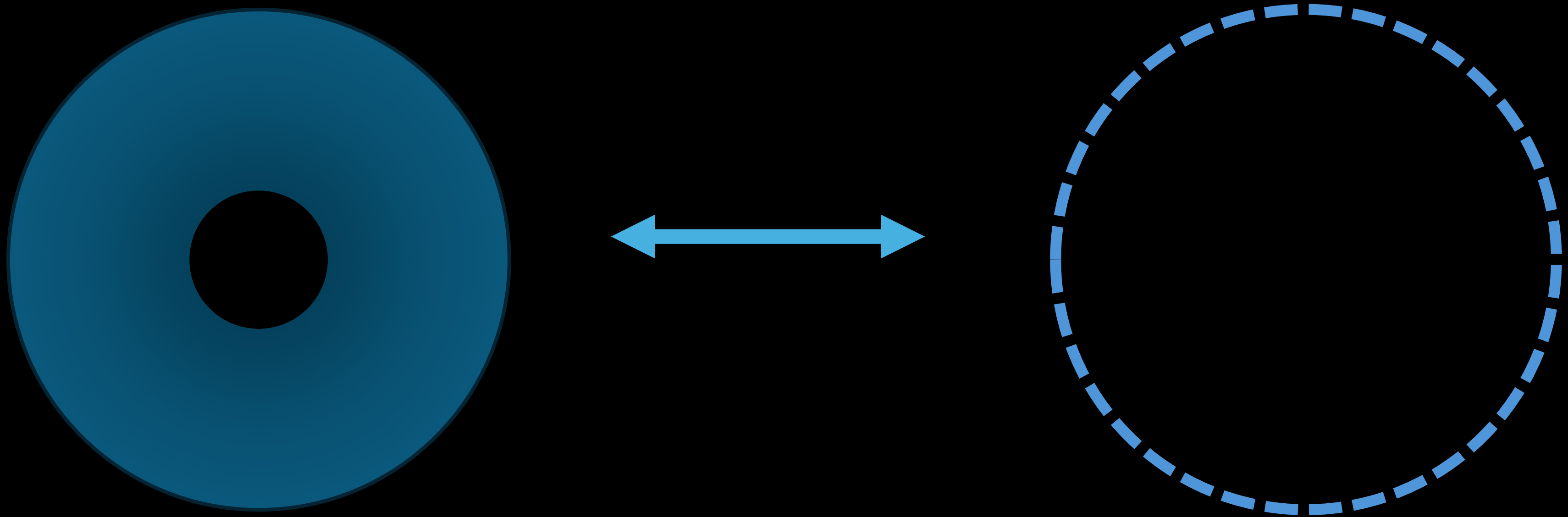
$$G_{12}(\omega) = \frac{G_{12}(0)}{\prod_{n=1}^{\infty}(1 - \frac{\omega^2}{\omega_n^2})(1 - \frac{\omega^2}{(\omega_n^*)^2})}, \qquad G_{12}(0) > 0.$$

M. Dodelson, C. Iossa, R. Karlsson, A. Zhiboedova. A Thermal Product Formula

QNMs to Thermal CFT data test: 2D

A "BTZ" black hole of mass M in AdS_3

A thermal CFT_2 with $\beta = 8\pi M$



Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli

QNMs to Thermal CFT data test: 2D

We want to look at a scalar operator that perturbs the AdS BTZ black hole, and the QNMs describe the decay of the 2 pt. function in time in the black hole background

QNMs to Thermal CFT data test: 2D

CFT data is fixed by conformal symmetry

In 2D the OPE expansion looks like: ($\omega > k >> 1$)

$$G_{12}^{(d=2)}(\omega, k) = \theta(\omega)\theta(\omega^2 - k^2)e^{-\frac{\beta\omega}{2}}\omega^{2\Delta_{\mathcal{O}}-2} \sum_{\mathcal{V}\in\mathcal{O}\times\mathcal{O}} \frac{a_{\mathcal{V}}^{(d=2)}}{(\beta\omega)^{\Delta_{\mathcal{V}}}} G_{\Delta,J}^{(d=2)}(\zeta)$$

$$G_{\Delta,J}^{(d=2)}(\zeta) = \frac{2^{2-2\tilde{\Delta}}\pi^2}{\Gamma\left(\frac{J}{2}+\tilde{\Delta}\right)\Gamma\left(\tilde{\Delta}-\frac{J}{2}\right)}(1-\zeta^2)^{\tilde{\Delta}-\frac{J}{2}-1}\left((1-\zeta)^J + (1+\zeta)^J\right)$$

$$\zeta = \frac{k}{\omega}$$

QNMs to Thermal CFT data test: 2D

We can solve for the QNMs in a BTZ black hole

$$\omega_n = k + i(2n + \Delta_O - 2)$$

$$\zeta = \frac{k}{\omega}$$

$$G_{12}(\zeta, k) = \frac{G_{12}(\omega = 0, k)}{\prod_n^\infty \left(1 - \frac{\zeta_n^2}{\zeta^2}\right)\left(1 - \frac{(\zeta_n^*)^2}{\zeta^2}\right)}$$

$$G_{12}(\zeta, k) = G_{12}(\omega = 0, k) \exp\left(\sum_n^\infty \log\left(1 - \frac{\zeta_n^2}{\zeta^2}\right) + \sum_n^\infty \log\left(1 - \frac{(\zeta_n^*)^2}{\zeta^2}\right)\right)$$

$$G_{12}(\zeta, k) = G_{12}(\omega = 0, k) \sum_{m=0}^\infty K_m(\zeta_n, \zeta_n^*, k)(\zeta^2 - \epsilon^2)^m \prod_n^\infty \left|\frac{\varepsilon^2}{\zeta_n^2 - \varepsilon^2}\right|^2$$

QNMs to Thermal CFT data test: 2D

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$$\omega_n = k + i(2n + \Delta_O - 2)$$

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$$G_{12}(\omega = 0, k) = \frac{\Gamma\left(\frac{1}{2}\left(\Delta_O \pm ik\right)\right)}{\pi\Gamma\left(\left(\Delta_O - 1\right)\right)^2}$$

$$\prod_n^{\infty} \left| \frac{1/2}{\zeta_n^2 - 1/2} \right|^2 = \frac{\Gamma\left(\frac{1}{2}\left(\Delta_O - i\left(\pm 1 \pm \sqrt{2}\right) k\right)\right)}{\Gamma\left(\frac{1}{2}(\Delta_O \pm ik)\right)^2}$$

Then, we can expand in powers of $1/k$

The first term in the expansion:

From the OPE expansion:

$$G_{12}(\zeta, k)_{\Delta=0,J=0} = \frac{4^{3-2\Delta_O} e^{-\sqrt{2}\pi k} k^{-2+2\Delta_O} \pi^2}{\Gamma\left(\Delta_O\right)^2}$$

From the QNMs:

$$G_{12}(\zeta, k) \approx \frac{4^{2-\Delta_O} e^{-\sqrt{2}\pi k} k^{-2+2\Delta_O} \pi}{\Gamma\left(\Delta_O - 1\right)^2}$$

This first term is from the Identity operator because $\Delta = 0$, $J = 0$

And from this we learn the difference in normalization

$$\frac{\pi}{2(\Delta_O - 1)^2}$$

The second term in the expansion:

From the OPE expansion:

(after normalization)

$$\frac{3 \times 4^{2-\Delta_O} e^{-\sqrt{2}\pi k} k^{-4+2\Delta_O} (\Delta_O - 2)(\Delta_O - 1) a_{\Delta=2,J=2}}{\pi \Gamma\left(\Delta_O - 1\right)^2}$$

From the QNMs:

$$\frac{4^{2-\Delta_O} e^{-\sqrt{2}\pi k} k^{-4+2\Delta_O} \pi (\Delta_O - 2)(\Delta_O - 1)(\Delta_O)}{\Gamma\left(\Delta_O - 1\right)^2}$$

This term is from the Stress Energy Tensor T: $\Delta = 2$, $J = 2$

And so we learn some CFT data:

$$a_{\Delta=2,J=2} = \frac{\pi^2 \Delta_O}{3}$$

QNMs to Thermal CFT data test: 2D

Did we learn anything new?

No. It was already fixed by Conformal Symmetry.

But this should apply just as well to higher dimensional black holes, which will correspond to thermal CFTs that are notoriously hard to compute and little is known.

QNMs to Thermal CFT data: d>2

Problems with higher dimensions:

- Thermal CFT
  - What operators contribute to the product?
  - What are the conformal blocks?

- AdS Black Hole
  - Can we find the QNMs analytically?
  - What is $G(\omega=0,k)$?

QNMs to Thermal CFT data: d>2

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Well-known model: Black Brane in AdS_5

QNMs to Thermal CFT data: d>2

Well-known model: Black Brane in AdS_5

- Thermal CFT
  - What operators contribute to the product?
  - What are the conformal blocks?

Just like in 2D, one can show that the only operators that contribute in the holographic limit is the stress tensor and its descendants

M. Grinberg, J. Maldacena. Proper time to the black hole singularity from thermal one-point functions

QNMs to Thermal CFT data: d>2

Well-known model: Black Brane in AdS_5

- Thermal CFT
  - What operators contribute to the product?
  - What are the conformal blocks?

$$G_{\Delta,J}(\zeta) = \frac{2^{d-2\tilde{\Delta}+1}\pi^{\frac{d}{2}+1}\Gamma(d+J-2)}{\Gamma(d-2)\Gamma\left(\frac{J}{2}+\tilde{\Delta}\right)\Gamma(J+1)\Gamma\left(\tilde{\Delta}-\frac{J}{2}-\frac{(d-2)}{2}\right)}(1-\zeta^2)^{\tilde{\Delta}-\frac{J}{2}-\frac{d}{2}}{}_2F_1\left(-\frac{J}{2},\frac{1-J}{2},\frac{d-1}{2};\zeta^2\right)$$

(in the k >> 1 limit)

QNMs to Thermal CFT data: d>2

Well-known model: Black Brane in AdS_5

- AdS Black Hole
  - Can we find the QNMs analytically?
  - What is $G(\omega=0,k)$?

$$\omega = p \pm 0.344\, e^{\mp \frac{i\pi}{3}} (\pi T)^{\frac{4}{3}}\, p^{-\frac{1}{3}} (\Delta - 1 + 2n)^{\frac{4}{3}}, \quad n = 0, 1, \cdots$$

(in the p >> 1 limit)

G. Festuccia, H. Liu. A Bohr-Sommerfeld quantization formula for quasinormal frequencies of AdS black holes

QNMs to Thermal CFT data: d>2

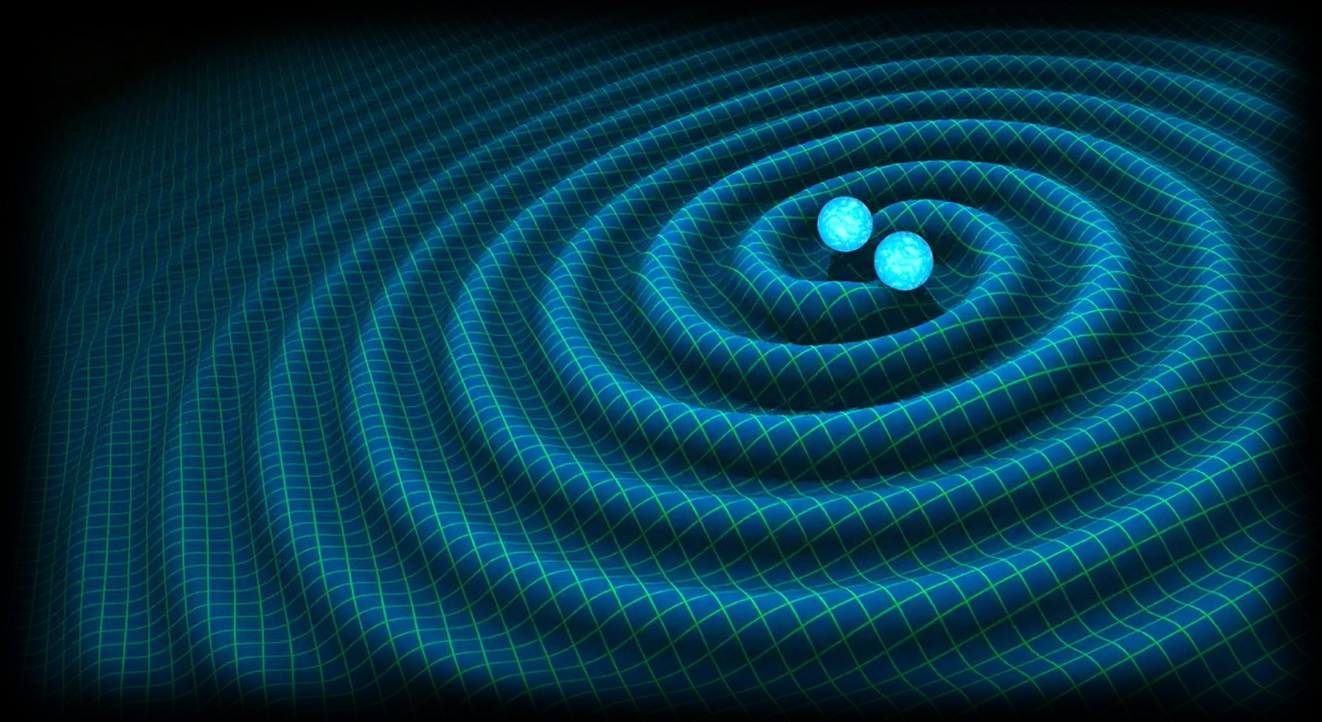
Well-known model: Black Brane in AdS_5

- AdS Black Hole
  - Can we find the QNMs analytically?
  - What is $G(\omega=0,k)$?

We're still working on it

Summary:

Let's use what we know about the **Quasi Normal Modes** of a Black Hole to discover something interesting about **Thermal Conformal Field Theories**!



Ongoing work:

We'd like to finish up this work on the AdS_5 black brane

Look at other spacetimes

Investigate restrictions on QNM asymptotics at large *n* based on the form of the OPE expansion

Thank you