Simulation of PBH formation using the COSMOS code

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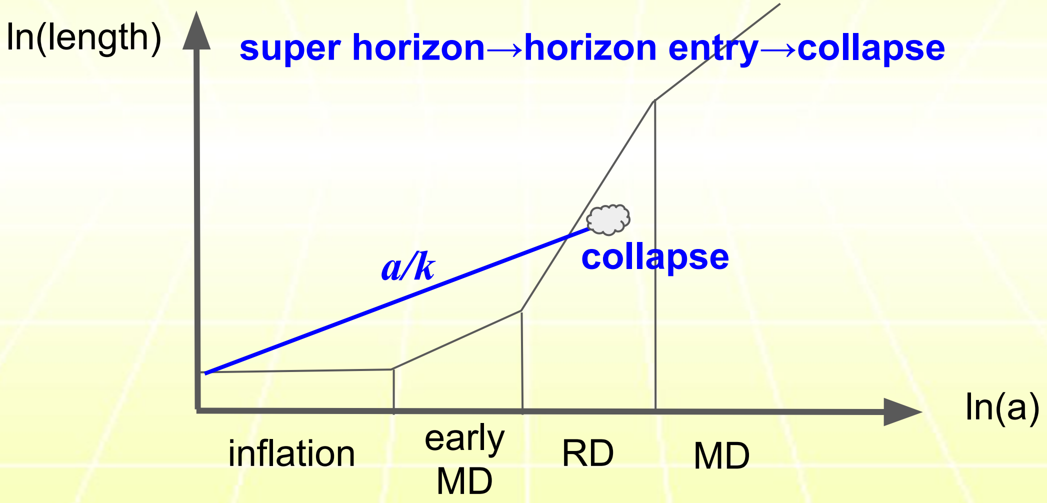
Numerical Simulation of PBH Formation

Introduction:PBH formation

◎Focus on PBH formation in radiation dominated era

◎Comoving scale of an inhomogeneity ~ $1/k$



◎GR simulation starting from a super-horizon non-linear initial data

Key features of PBH formation simulation

◎Initially super-horizon inhomogeneity

⇒ Cosmological expansion is essential

◎Nonlinear evolution in an expanding background

⇒ Not asymptotically flat

⇒ Periodic or asymptotically FLRW boundary cond.

◎Scale hierarchy between the collapsing region and Hubble scale

⇒An efficient refinement procedure in the collapsing region is needed

◎Critical behavior near the threshold (at least in spherical sym.)

$$M \sim (\delta - \delta_{\rm th})^{\gamma}$$

⇒Extremely high resolution is needed to check it

Spherically symmetric simulations

◎Many many previous works in spherical sym.

[Niemeyer, Jedamzik astro-ph/9901292], [Shibata, Sasaki gr-qc/9905064], [Hawke, Stewart CQG19(2002)3687]
[Musco, Miller, Rezzolla gr-qc/0412063], [Polnarev, Musco gr-qc/0605122], [Musco, Miller, Polnarev 0811.1452]
[Musco and Miller arXiv:1201.2379], [Polnarev, Nakama, Yokoyama arXiv:1204.6601], [Nakama, Harada, Polnarev, Yokoyama arXiv:1310.3007]
[Nakama arXiv:1408.0955], [Harada, CY, Nakama, Koga arXiv:1503.03934], [Musco arXiv:1809.02127], [Escriva arXiv:1907.13065]
[Escriva,Germani, Sheth arXiv:1907.13311], [Escriva,Germani, Sheth arXiv:2007.05564], [Escriva, Bagui, Clesse arXiv:2209.06196],
[CY, Harada, Hirano, Okawa, Sasaki arXiv:2112.12335], [Escriva, Tada, Yokoyama, CY arXiv:2202.01028],
[Franciolini, Musco, Pani, Urbano arXiv:2209.05959], [Escriva, CY arXiv:2310.16482], [Escriva, Tada, CY arXiv:2311.17760],
[Uehara, Escriva, Harada, Saito, CY arXiv:2401.06329], [Shimada, Escriva, Saito, Uehara, CY arXiv:2411.07648],
[Inui, Joana, Motohashi, Pi, Tada, Yokoyama arXiv:2411.07648] probably and more

*Blue: Including critical behavior but with decaying modes, Red: Growing mode but without critical behavior, Purple: Both are addressed

◎Two main schemes

- **Based on Misner-Sharp formulation with comoving gauge**
- **BSSN like formulation (used in Shibata Sasaki(1999) and COSMOS-S code)**

***If you are interested in only spherical cases,**
I recommend the Misner-Sharp based numerical simulation. (Probably it's faster)

Non-spherical simulations

◎Several works so far (focusing on PBH formation)

[CY, Ikeda, Okawa arXiv:1811.00762] Gravitational Collapse of a Massless Scalar Field in a Periodic Box
[CY, Harada, Okawa arXiv:2004.01042] Threshold of Primordial Black Hole Formation in Nonspherical Collapse
[de Jong, Aurrekoetxea, Lim arXiv:2109.04896] Primordial black hole formation with full numerical relativity
[de Jong, Aurrekoetxea, Lim, França arXiv: 2306.11810] Spinning primordial black holes formed during a matter-dominated era
[CY, Harada, Okawa arXiv:2004.01042] Threshold of Primordial Black Hole Formation in Nonspherical Collapse
[CY arXiv:2403.11147] Primordial black hole formation from a nonspherical density profile with a misaligned deformation tensor
[Escriva, CY arXiv:2410.03451] Non-spherical effects on the mass function of Primordial Black Holes
[Escriva, CY arXiv:2410.03452] Simulations of Ellipsoidal Primordial Black Hole Formation
[Kitajima JPS meeting@Sapporo]
[Joana private communication]

*Blue: scalar field system, Red: Fluid, long-wavelength growing mode

◎Two main groups

- GRChombo (public without fluid)
- COSMOS (not yet public, to be public?)

about COSMOS

About COSMOS

◎Originally provided by Hirotada Okawa (for E-eqs and real scalar field w/ periodic BC)

◎COSMOS(秋桜) code by C++ [CY, Hirotada Okawa, Ken-ichi Nakao(1306.1389), Hirotada Okawa, Helvi Witek, Vitor Cardoso(1401.1548)]

◎Basically follows the SACRA(桜) code by Fortran
[Tetsuro Yamamoto, Masaru Shibata, Keisuke Taniguchi(arXiv:0806.4007)]

◎Independently developed by CY and Okawa-san

◎In the CY side, it is mainly dedicated to PBH formation as follows

- Inhomogeneous coordinate system has been implemented
  [CY, Taishi Ikeda,Hirotada Okawa(arXiv:1811.00762)]
- Fluid evolution code has been implemented
  [CY, Tomohiro Harada,Hirotada Okawa(arXiv:2004.01042)]
- 1+1 code for spherical systems has been developed based on COSMOS (COSMOS-S)
  [CY, Harada, Hirano, Okawa, Sasaki(arXiv:2112.12335)]
- Recently, a mesh refinement procedure has been implemented

Baumgarte-Shapiro-Shibata-Nakamura formalism

Shibata, Nakamura(1995)
Baumgarte, Shapiro(1999)

◎Spatial metric

$$\left(\partial_t - \beta^i\partial_i\right)\psi = \frac{1}{6}\psi\left(\partial_i\beta^i - \alpha K\right)$$

$$\left(\partial_t - \beta^k\partial_k\right)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\partial_k\beta^k\tilde{\gamma}_{ij}$$

◎Extrinsic curvature

$$\left(\partial_t - \beta^k\partial_k\right)\mathrm{tr}K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}\mathrm{tr}K^2\right) - \triangle\alpha$$

$$\left(\partial_t - \beta^k\partial_k\right)\tilde{A}_{ij} = \text{functions of }[\psi, \tilde{\gamma}, \mathrm{tr}K, \tilde{A}, \tilde{\Gamma}, \alpha, \beta, \partial\psi, \triangle\psi, \cdots]$$

◎Auxiliary variable for numerical stability $\tilde{\Gamma}^i := -\mathcal{D}_j\tilde{\gamma}^{ij}$

$$\left(\partial_t - \beta^k\partial_k\right)\tilde{\Gamma}^i = \text{functions of }[\psi, \tilde{\gamma}, \mathrm{tr}K, \tilde{A}, \tilde{\Gamma}, \alpha, \beta, \partial\psi, \triangle\psi, \cdots]$$

17 evolution eqs.

◎Eqs. for gauge fixing

Dynamical Gauge Conditions

◎Time slicing condition(modified version of the "1+log slice" specialized for cosmological settings)

$$\left(\partial_t - \beta^i \partial_i\right) \alpha = -2\alpha(\mathrm{tr}K + 3H_b)$$

◎Spatial coordinates(～"Hyperbolic Gamma driver")

$$\left(\partial_t - \beta^k \partial_k\right) \beta^i = \frac{3}{4} B^i$$

$$\left(\partial_t - \beta^k \partial_k\right) B^i = \partial_t \tilde{\Gamma}^i - 3H_b B^i$$

† $\left(\partial_t - \beta^k \partial_k\right) B^i = \partial_t \tilde{\Gamma}^i - \beta^k \partial_k \tilde{\Gamma}^i - 3H_b B^i$ might be better…

◎17 + 1 + 6 = 24 variables for geometry

Relativistic Hydro-dynamics

◎Energy momentum tensor

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$$

◎Lorentz factor for n^μ

$$\Gamma = -u^\mu n_\mu$$

◎Velocity U^μ relative to n^μ

$$u^\mu = \Gamma(n^\mu + U^\mu)$$

◎Rest mass density(ρ_0), specific int. ene.(ε)

$$\rho = \rho_0(1+\varepsilon)$$

◎Rest mass density measured by n^μ

$$D = \rho_0 \Gamma$$

◎"Dynamical" variables

$$\rho_* := \sqrt{\gamma} D,\ S_0 := \sqrt{\gamma} E,\ S_i := \sqrt{\gamma} p_i$$

◎Fluid equations

$$\partial_t \rho_* + \partial_i f^i_{\rho_*} = 0$$

$$\partial_t S_0 + \partial_i f^i_{S_0} = -S^i \partial_i \alpha + \alpha \sqrt{\gamma} S_{ij} K^{ij}$$

$$\partial_t S_i + \partial_j f^j_{S_i} = -S_0 \partial_i \alpha + S_j \partial_i \beta^j - \frac{1}{2} \alpha \sqrt{\gamma} S_{jk} \partial_i \gamma^{jk}$$

with

$$f^i_{\rho_*} = \rho_* V^i = \rho_* (\alpha U^i - \beta^i)$$

$$f^i_{S_0} = S_0 V^i + \sqrt{\gamma} P (V^i + \beta^i)$$

$$f^i_{S_j} = S_j V^i + \alpha \sqrt{\gamma} \delta^i_j P$$

◎"Primitive" variables

$$\rho,\ V^i := u^i / u^0,\ \varepsilon$$

Barotropic EoS case

◎Relation between the variables

$$\rho_* = \sqrt{\gamma}\Gamma\frac{\rho}{1+\varepsilon}$$
$$S_0 = \sqrt{\gamma}[\Gamma^2(\rho+P)-P]$$
$$S_i = \sqrt{\gamma}(E+P)U_i = \frac{1}{\alpha}(S_0+\sqrt{\gamma}P)\gamma_{ij}(V^i+\beta^i)$$

$$p^\mu p_\mu - E^2 - (P-\rho)E + \rho P = 0$$

dynamical var

EoS P=(ρ,ε)

primitive var

◎Barotropic EoS P=P(ρ)

$$\rho = \rho(\text{dynamical variables})$$

$$V^i = \alpha U^i - \beta^i = \alpha\frac{\gamma^{ij}S_j}{S_0+\sqrt{\gamma}P} - \beta^i$$

Equations are closed without ρ_* (or equivalently ε)
⇒we don't need to solve the continuity eq.

◎P=$w\rho$

$$\rho = \frac{1}{2w}\left[-(1-w)E + \sqrt{E^2(1-w)^2+4w(E^2-p^\mu p_\mu)}\right]$$

Equations for fluid

◎4(+1) equations

$$\partial_t S_0 + \partial_i f^i_{S_0} = -S^i \partial_i \alpha + \alpha \sqrt{\gamma} S_{ij} K^{ij}$$

$$\partial_t S_i + \partial_j f^i_{S_j} = -S_0 \partial_i \alpha + S_j \partial_i \beta^j - \frac{1}{2} \alpha \sqrt{\gamma} S_{jk} \partial_i \gamma^{jk}$$

$$(\partial_t \rho_* + \partial_i f^i_{\rho_*} = 0)$$

◎Scheme for the flux calculation

A central scheme with MUSCL(Mono Upstream-centered Scheme for Conservation Laws)

Kurganov, Tadmor(2000)
Shibata, Font(2005)

◎Totally 24 equations for geometry and 4+1 equations for fluid = 28 +1 equations

Equations for scalar field

◎2 equations

$$(\partial_t - \beta^i \partial_i)\phi = -\alpha \Pi$$
$$(\partial_t - \beta^i \partial_i)\Pi = -\alpha \triangle \phi - \gamma^{\mu\nu}\partial_\mu \alpha \partial_\nu \phi + \alpha K \Pi + \alpha V'(\phi)$$

◎Stress-energy tensor

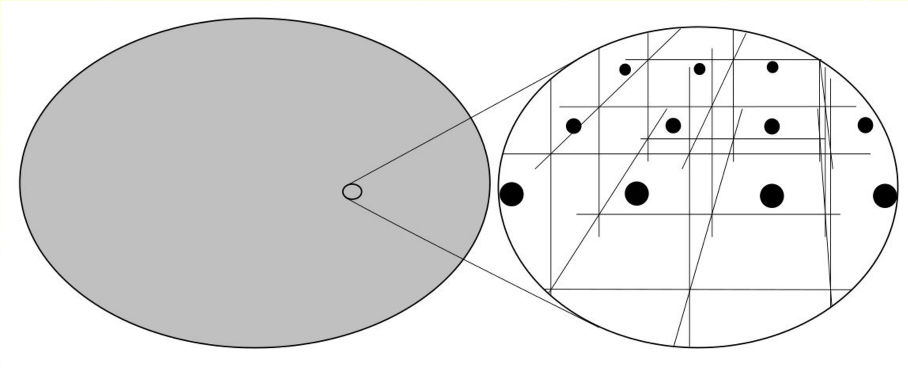
$$E^{\rm sc} = n^\mu n^\nu T^{\rm sc}_{\mu\nu} = \frac{1}{2}\Pi^2 + \frac{1}{2}\psi^{-4}\tilde{\gamma}^{ij}\partial_i\phi\partial_j\phi + V,$$
$$J^{\rm sc}_i = -\gamma_i^{\ \mu} n^\nu T^{\rm sc}_{\mu\nu} = \Pi \partial_i \phi,$$
$$S^{\rm sc}_{ij} = \gamma_i^{\ \mu}\gamma_j^{\ \nu} T^{\rm sc}_{\mu\nu} = \partial_i\phi\partial_j\phi - \frac{1}{2}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\partial_k\phi\partial_l\phi + \frac{1}{2}\psi^4\tilde{\gamma}_{ij}\Pi^2 - \psi^4\tilde{\gamma}_{ij}V$$

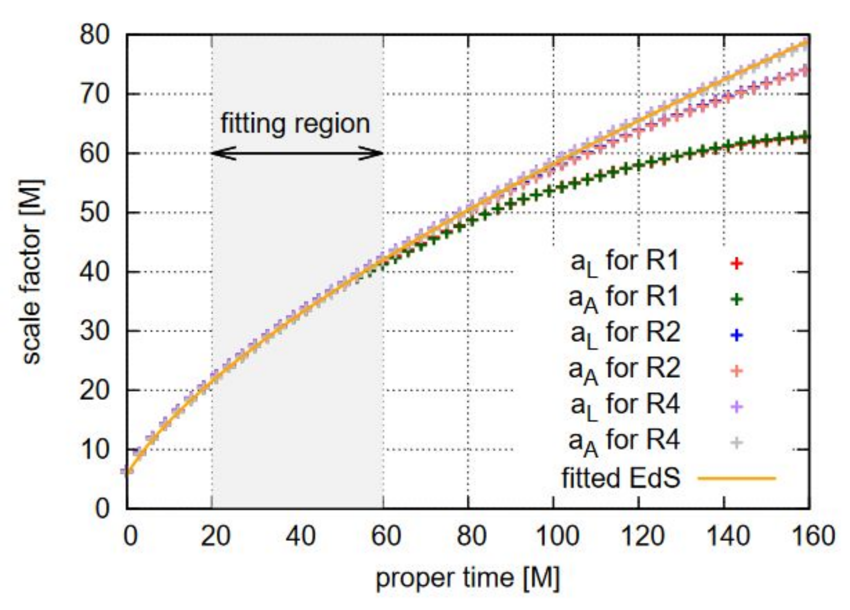
◎Totally 24 equations for geometry
4+1 equations for fluid
2 equations for scalar = 30 +1 equations

Black Hole Universe

[CY, Okawa, Nakao arXiv:1306.1389]

Black Hole Universe





◎**The evolution is well fitted with EdS**

Massless scalar field in a periodic box

[CY, Ikeda, Okawa arXiv:1811.00762]

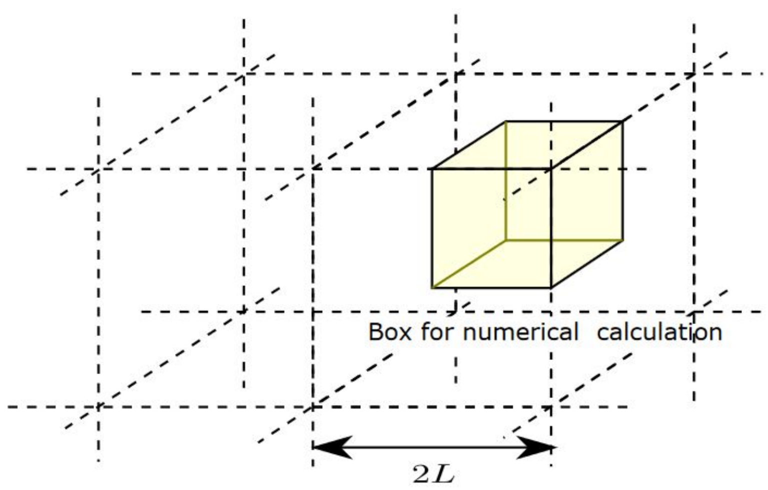
Summary of schemes

◎Geometry + massless scalar field

◎Reflection boundary condition

◎for geometry

BSSN ＋ 1+log slice + Gamma driver



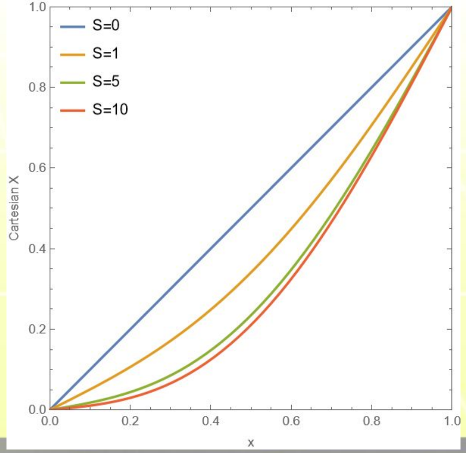
◎Resolution

・Scale-up reference coordinates x^i related to the Cartesian coord. X^i by

$$X^i = x^i - \frac{S}{1+S}\frac{L}{\pi}\sin\left(\frac{\pi}{L}x^i\right)$$

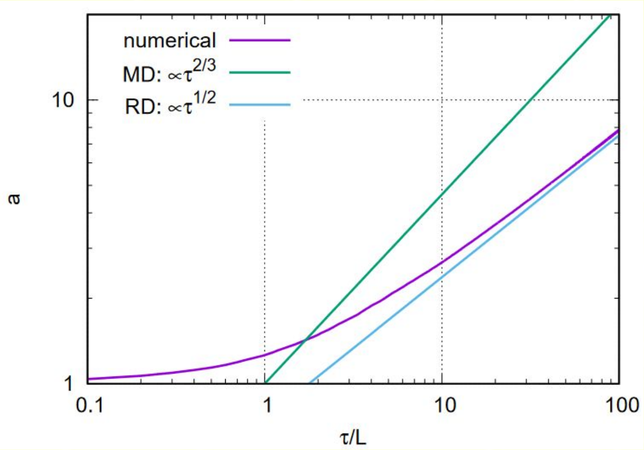
・Resolution at the center

$$\Delta X|_{\rm center} = \frac{1}{1+S}\Delta x$$



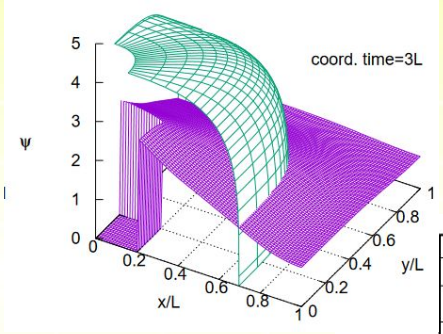
Horizon formation and background evolution

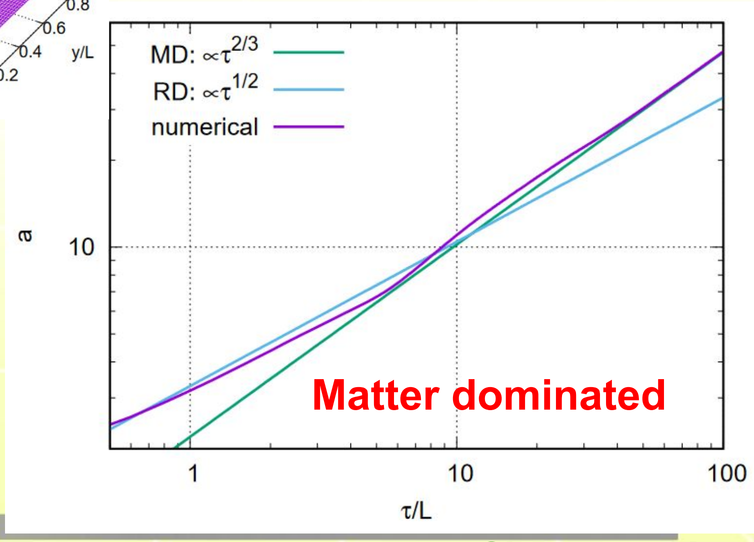
◎Without BH formation



Radiation dominated

◎With BH formation

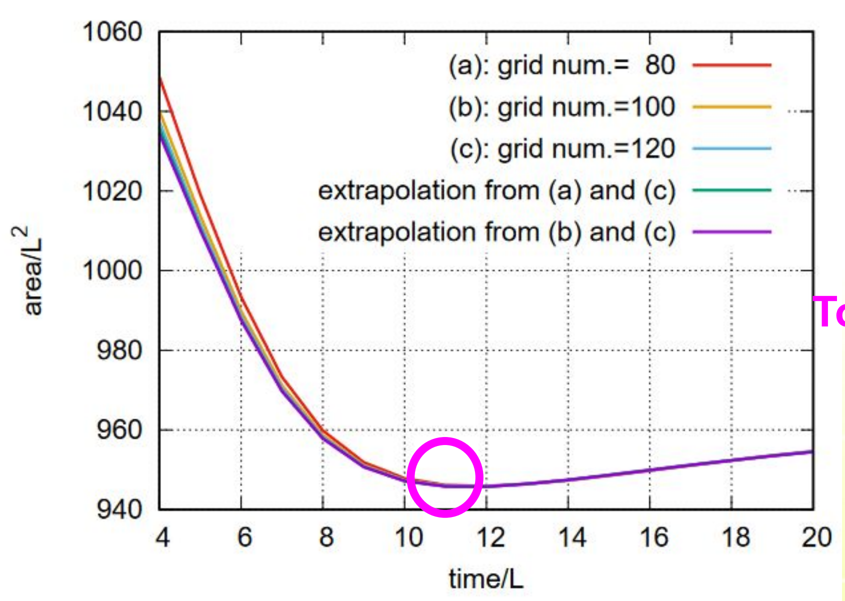


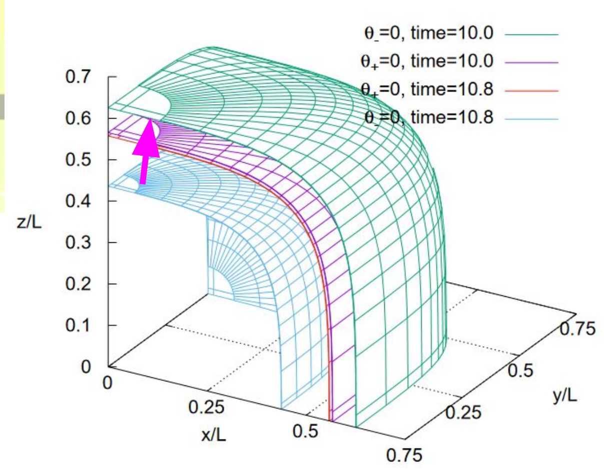


Matter dominated

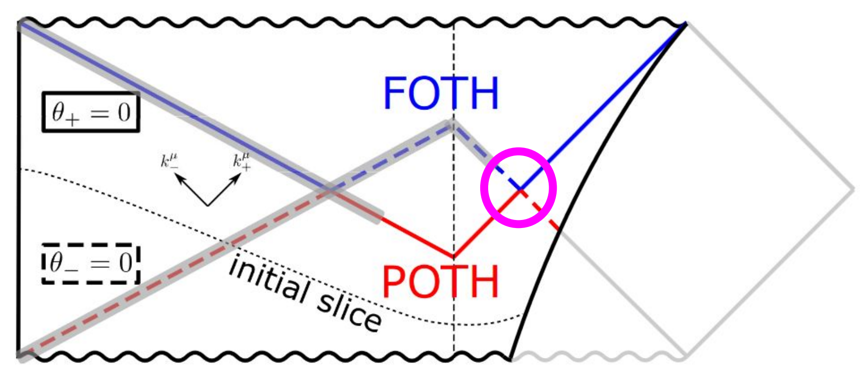
Horizon evolution

◎Area





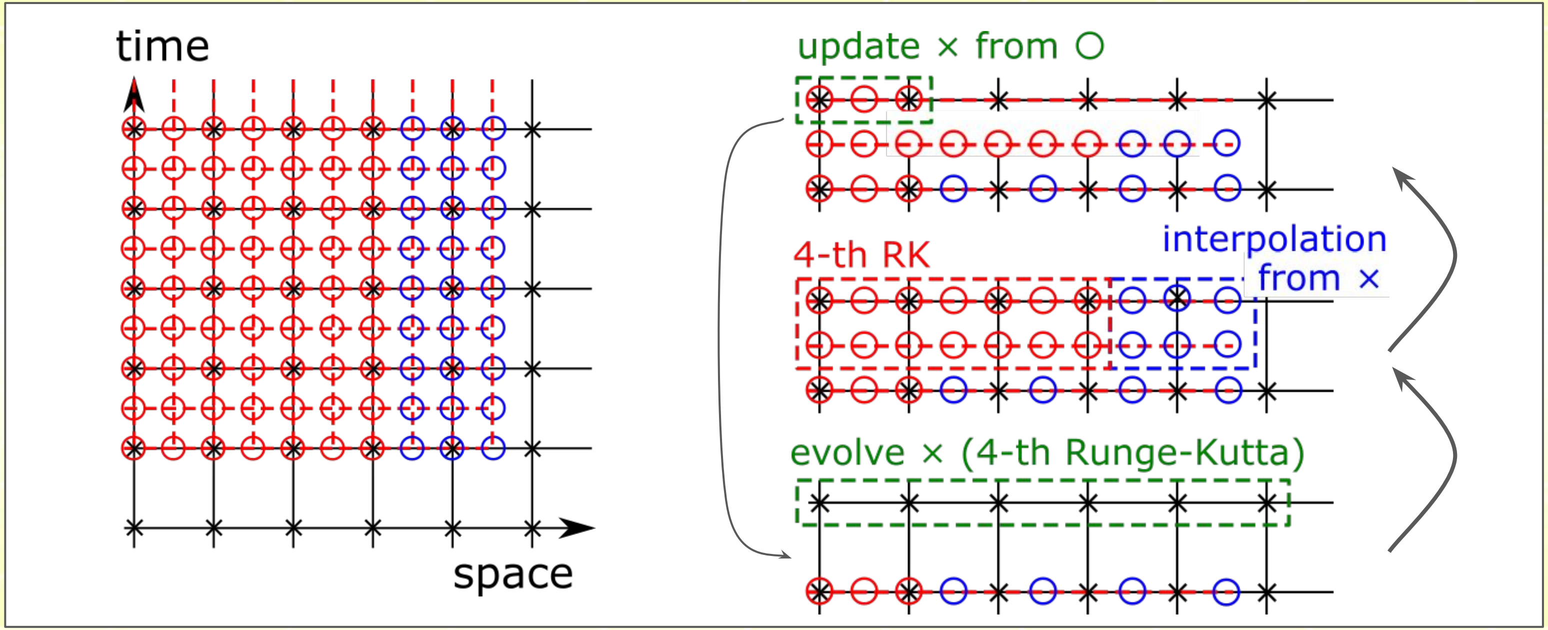
Tow horizons ($\Theta_{\pm}$=0) cross each other ~ “Type B”



PBH spin with $p=w\rho$ ($w=1/5$ and $1/3$)

[CY arXiv:2403.11147]

Rough sketch of mesh refinement



◎Twice finer resolution in a local spacetime patch

Summary of resolution difference

◎Resolution in previous simulation [CY, Harada, Okawa arXiv:2004.01042]

- Scale-up reference coordinates x^i related to the Cartesian coord. X^i by

$$X^i = x^i - \frac{S}{1+S}\frac{L}{\pi}\sin\left(\frac{\pi}{L}x^i\right) \text{ with } S = 15$$

- Resolution at the center ($\Delta x = L/100$)

$$\Delta X|_{\rm center} = \frac{1}{1+S}\Delta x = \frac{1}{16}\frac{L}{100} = \frac{L}{1600}$$

◎New simulation with mesh refinement [CY arXiv:2403.11147]

- $S = 10, \Delta x = L/60$
- Two additional layers for the mesh refinement

$$\Delta X|_{\rm center} = \frac{1}{1+S} \times \frac{1}{2^2} \times \Delta x = \frac{1}{44} \times \frac{L}{60} = \frac{L}{2640}$$

Initial condition for full numerical simulation

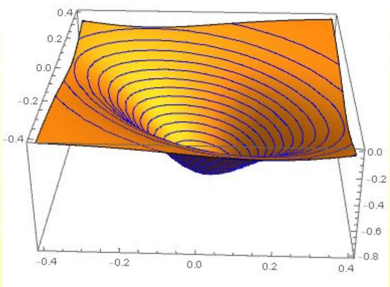
◎Initial curvature perturbation

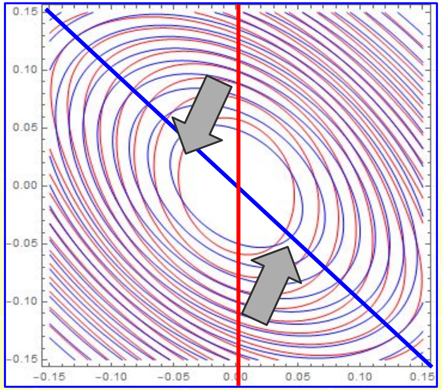
$$\frac{\zeta}{\mu} := -\frac{2}{\mu}\ln\Psi \simeq -1+\frac{1}{2}\left(k_1^2(x+y)^2/2+k_2^2(x-y)^2/2+k_3^2z^2\right)+\mathcal{O}(r^4)$$

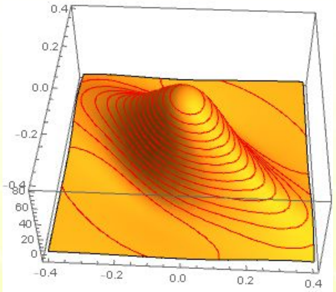
$$\frac{\triangle\zeta}{\mu k^2} \simeq 1-\frac{1}{2}\left(\kappa_1^2x^2+\kappa_2^2y^2+\kappa_3^2z^2\right)+\mathcal{O}(r^4)$$

ζ～gravitational potential on (x,y) plane

Δζ～energy density on (x,y) plane







tidal torque ⇒ angular momentum transfer ⇒ spinning PBH

Parameter setting

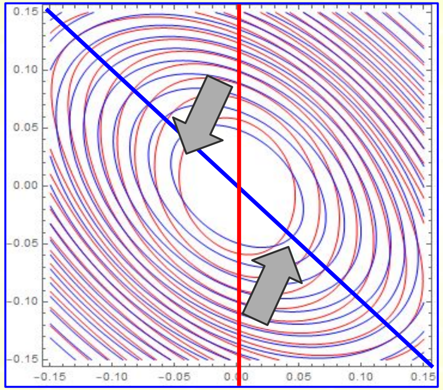
◎Initial curvature perturbation

$$\frac{\zeta}{\mu} \simeq -1 + \frac{1}{2}\left(k_1^2(x+y)^2/2 + k_2^2(x-y)^2/2 + k_3^2 z^2\right) + \mathcal{O}(r^4)$$

$$k_1^2 = \frac{1}{3}(\xi_1 + 3\xi_2 + \xi_3) \qquad k_2^2 = \frac{1}{3}(\xi_1 - 3\xi_2 + \xi_3) \qquad k_3^2 = \frac{1}{3}(\xi_1 - 2\xi_3)$$

$$\frac{\triangle\zeta}{\mu k^2} \simeq 1 - \frac{1}{2}\left(\kappa_1^2 x^2 + \kappa_2^2 y^2 + \kappa_3^2 z^2\right) + \mathcal{O}(r^4)$$

$$\kappa_1^2 = \frac{1}{3}\left(\tilde{\xi}_1 + 3\tilde{\xi}_2 + \tilde{\xi}_3\right) \qquad \kappa_2^2 = \frac{1}{3}\left(\tilde{\xi}_1 - 3\tilde{\xi}_2 + \tilde{\xi}_3\right) \qquad \kappa_3^2 = \frac{1}{3}\left(\tilde{\xi}_1 - 2\tilde{\xi}_3\right)$$



◎With reference to the peak theory

Strong correlation between $\vec{k}^2 = \xi_1$ and $\vec{\kappa}^2 = \tilde{\xi}_1$

⇒ We set $\vec{k}^2 = \vec{\kappa}^2 = \xi_1 = \tilde{\xi}_1 = 100/L^2$

Probability for $\hat{\xi}_2 := \xi_2/\sigma_2$ and $\hat{\xi}_3 = \xi_3/\sigma_2$ $\quad \xi_3 = \tilde{\xi}_3 = 0$ The most probable values

$$P(\hat{\xi}_2, \hat{\xi}_3) = \frac{5^{5/2}3^2}{\sqrt{2\pi}}\hat{\xi}_2(\hat{\xi}_2^2 - \hat{\xi}_3^2)\exp\left[-\frac{5}{2}(3\hat{\xi}_2^2 + \hat{\xi}_3^2)\right]$$

$$\Rightarrow \hat{\xi}_3 \approx 0,\ \xi_2 = \hat{\xi}_2\sigma_2 \sim \hat{\xi}_2\sigma k^2 \sim 5/L^2$$

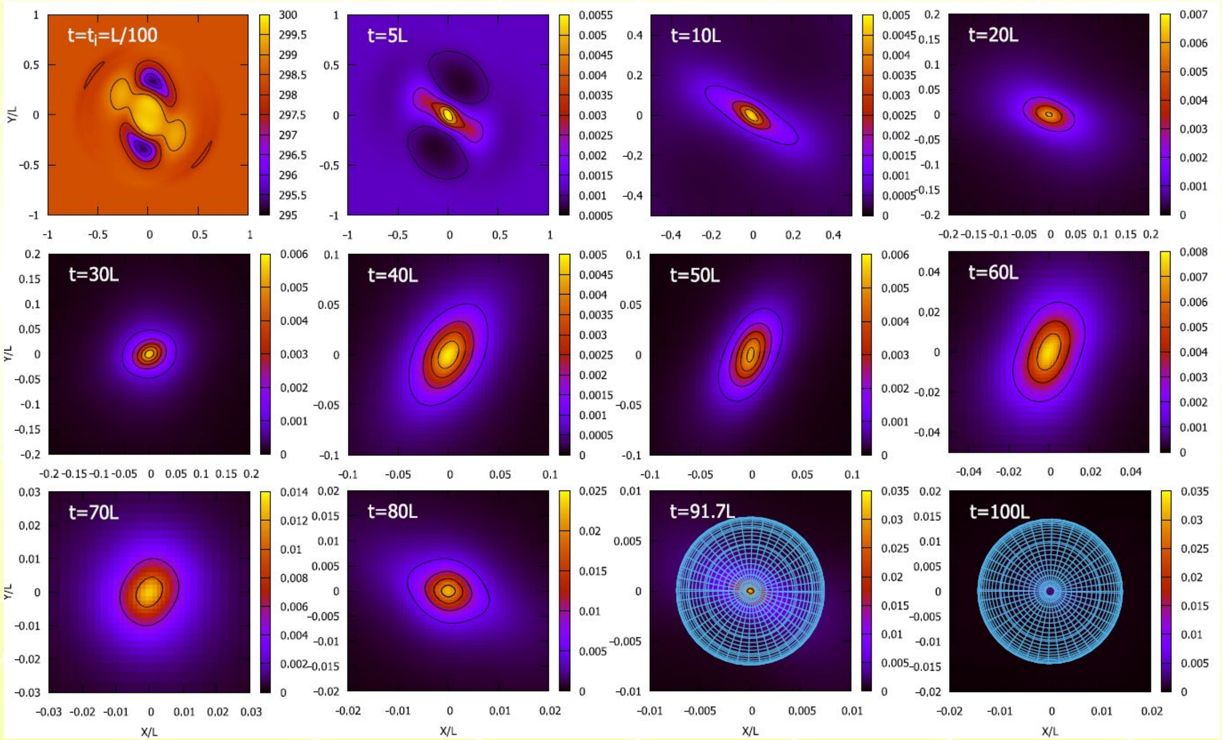
$\xi_2 = 10/L^2$
$\tilde{\xi}_2 = 15/L^2$

Too large to be statistically expected

⇒Too much idealized for spin generation

Snapshots

◎w=p/ρ=1/3, amplitude μ=0.92 case



Effective spin parameter

◎Kerr black hole case

$$\text{area}: \; A_{\rm Kerr} = 8\pi\left(M^2 + \sqrt{M^4 - a^2 M^2}\right)$$

$$\text{equatorial circumference}: \; d_{\rm Kerr} = 4\pi M$$

$$\Rightarrow s^2_{\rm Kerr} := \frac{a^2}{M^2} = \frac{4\pi A_{\rm Kerr}\left(d^2_{\rm Kerr} - \pi A_{\rm Kerr}\right)}{d^4_{\rm Kerr}}$$

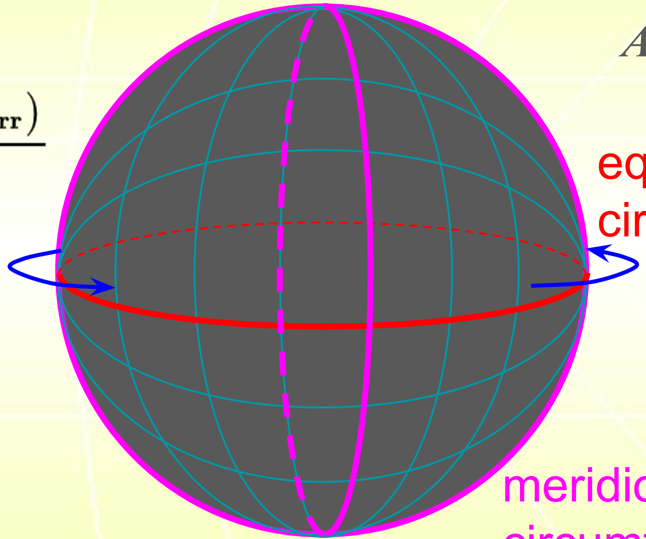
◎Effective dimensionless spin

$$s^2 := \frac{4\pi A\left(d^2 - \pi A\right)}{d^4}$$

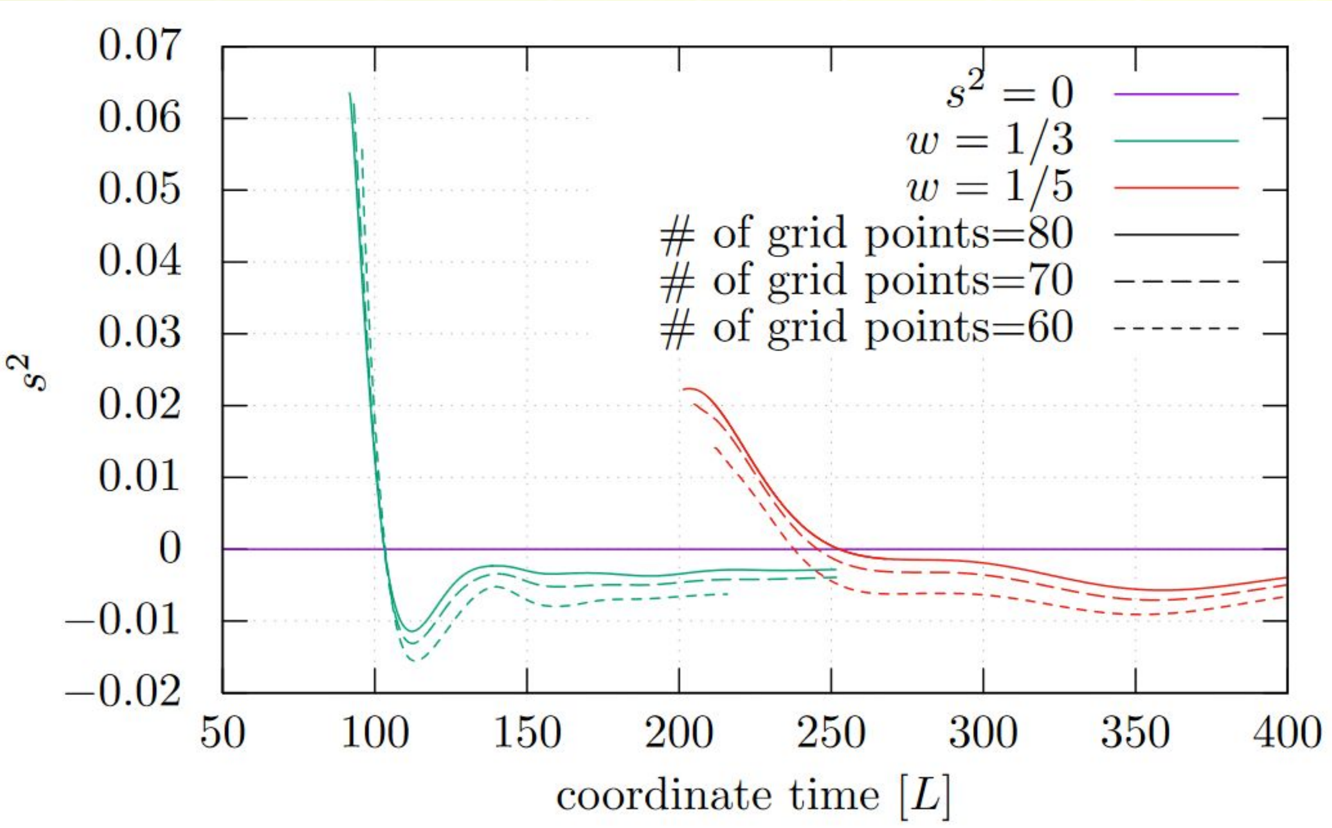
◎Asphericity parameters

$$\text{meridional circumference}: \; \ell_{x=0}, \ell_{y=0}$$

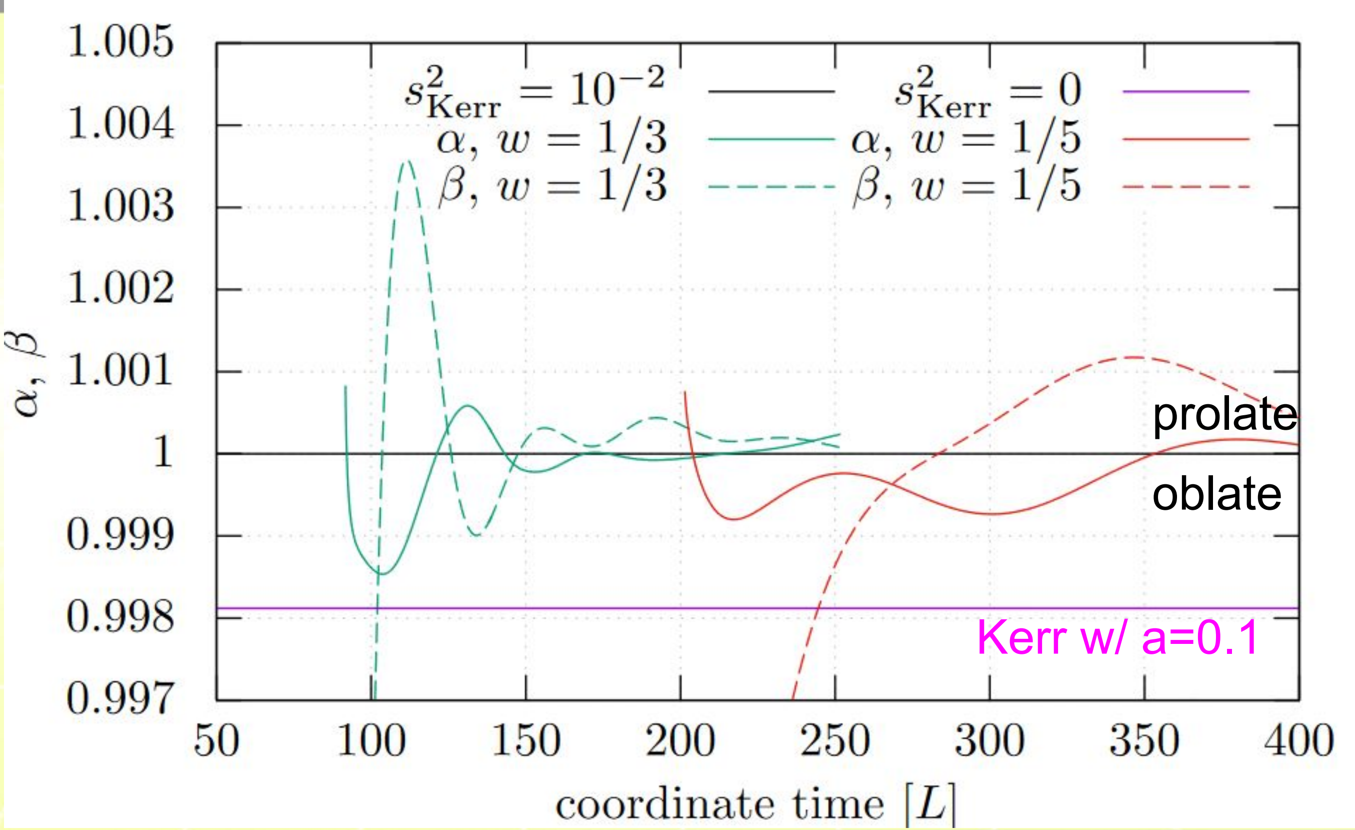
$$\alpha := \frac{\ell_{x=0}}{d} \qquad \beta := \frac{\ell_{y=0}}{d}$$



Effective spin parameter



Non-sphericity



Spin of PBH is very small

for the equation of states $p=w\rho$ with $w>1/5$

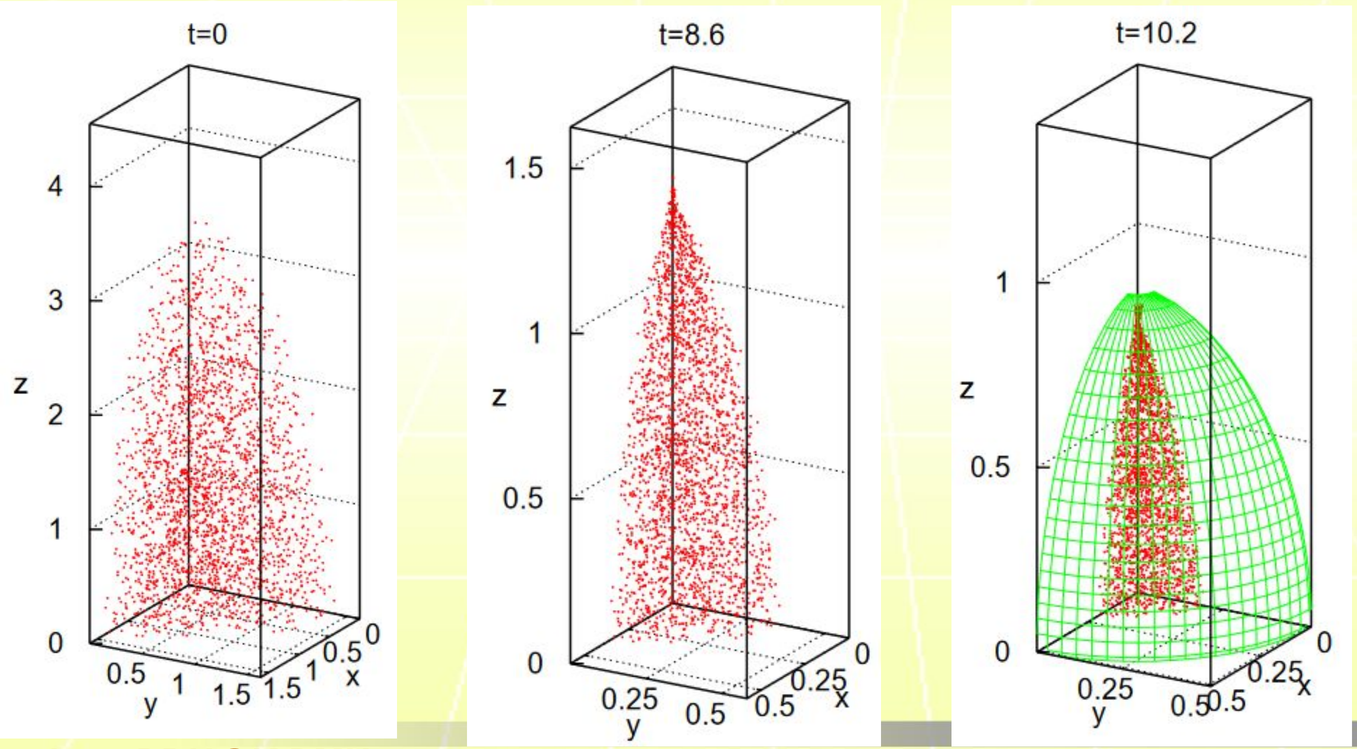
◎Caveats

- Results only for a specific initial profile
- The case in which the amplitude is very close to the critical value
- etc.

PBH formation in MD(?)

GR N body simulation

◎Spindle collapse (asymptotically flat) [CY, Harada, Okawa arXiv:2004.01042]



Code structure

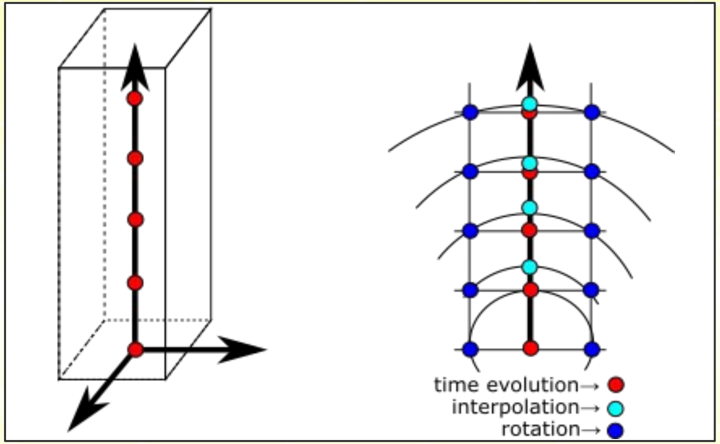
Codes

Should we make it public…? Do you want?

◎COSMOS

- 3+1 dim, 4th order Rungekutta
- Geometry+fluid(P=wρ)+Real Scalar
- Non-Cartesian coord.
- Mesh-refinement
- Apparent horizon finder
- Excision(cubic domain)
- Curvature invariants calculation
- Constraints solver(SOR method, not so reliable)
- Elliptic eq. solver for Maximal slice and minimal distortion(SOR, not so reliable)

CARTOON method



◎COSMOS_S

- Spherical sym. with CARTOON method with spline interpolation
- Asymptotically FLRW boundary cond.
- Non-Cartesian + mesh refinement

Thank you for your attention

Thank you for your attention

Should we make it public…? Do you want?