# $\delta N$ formalism during ultra-slow-roll inflation

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PBH FOCUS WEEK IPMU – 15/11/2024

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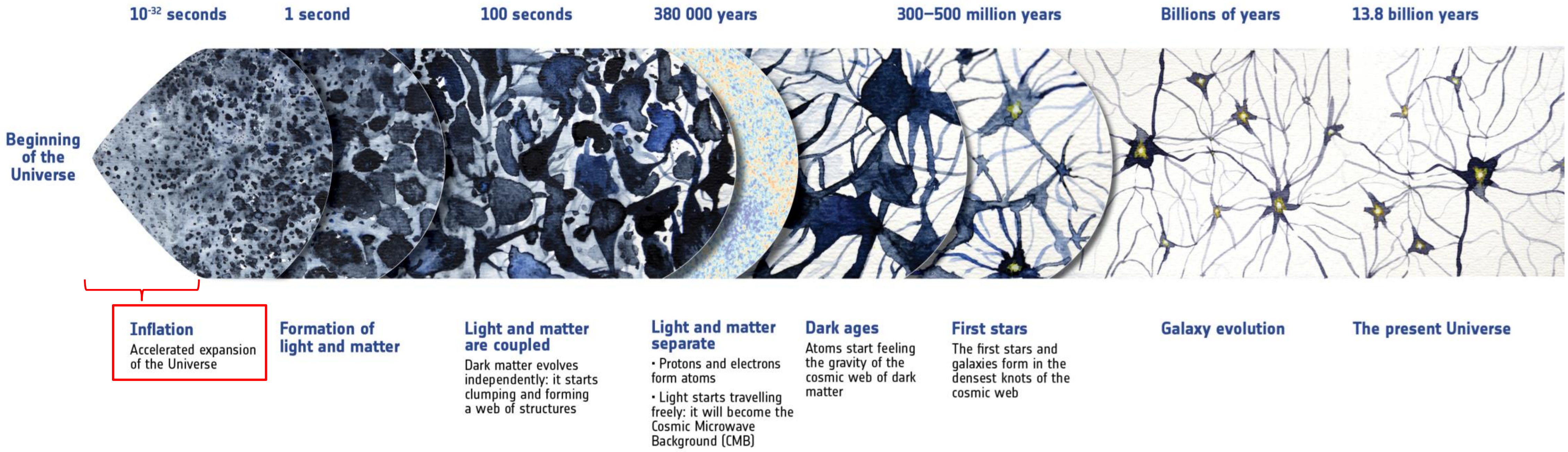
# Inflation




Tons
of
models

→

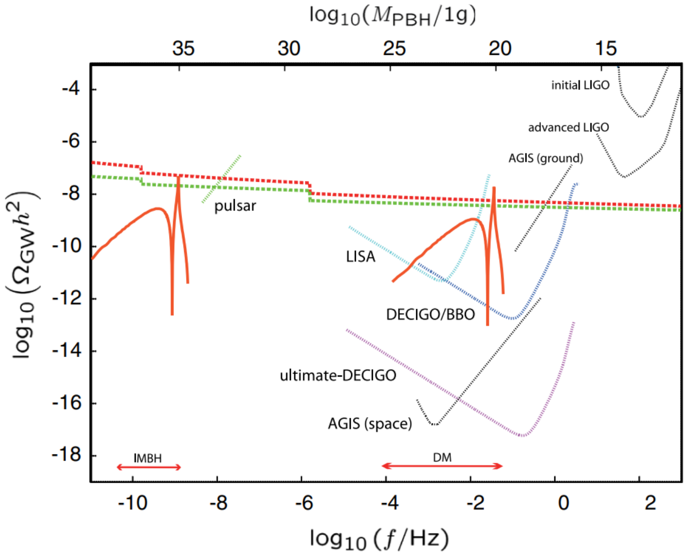
Tools to make
observational predictions

Observations
(PBHs, gravitational waves…)

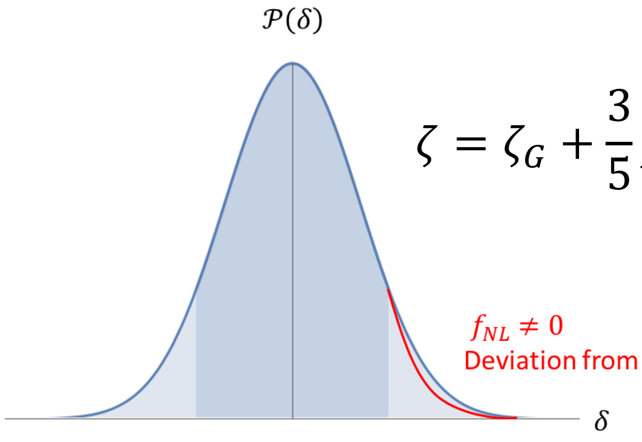
# Inflation

- Are we at the dawn of the observation of non-linear effects?

Induced gravitational waves



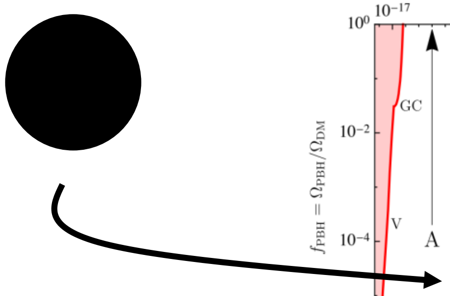

[Saito, Yokoyama (2008)]



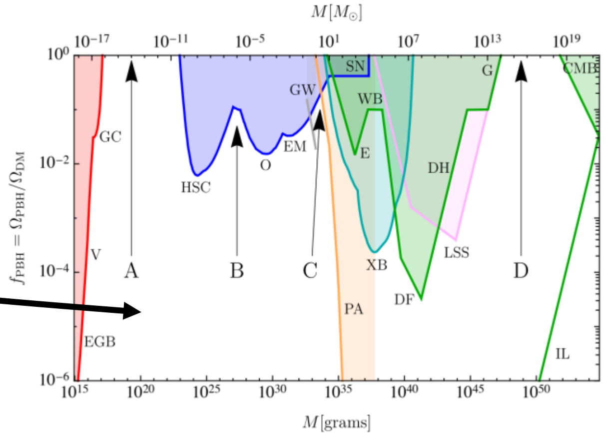

$$\zeta = \zeta_G + \frac{3}{5} f_{NL}\, \zeta_G^2$$

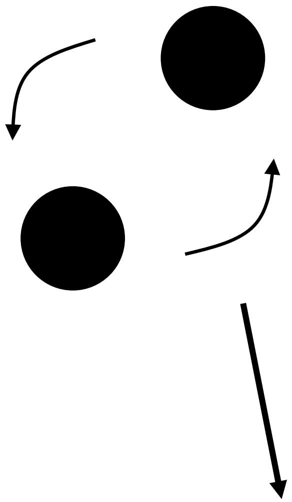
$f_{NL} \neq 0$

Deviation from Gaussianity ⟶ PBHs

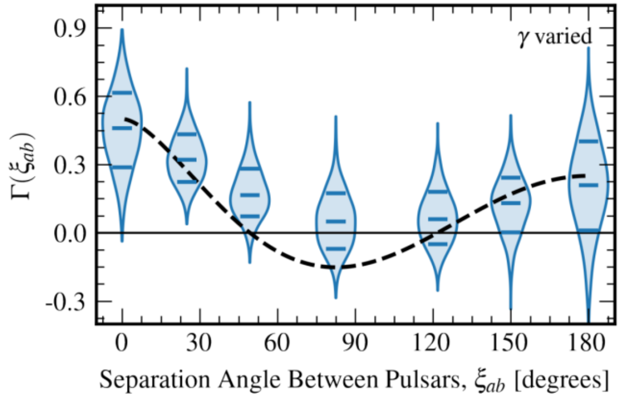


PBHs may represent dark matter



PBH-binaries contribute to the stochastic gravitational wave background




[Kuroyanagi, Chiba, Takahashi (2018)]
[NANOGrav collaboration (2023)]

[Carr, Hawking (1974)]
[Green, Kavanagh (2007)]
[Ezquiaga, Garcia-Bellido, Vennin (2019)]
[Escrivà, Kühnel, Tada (2022)]

# Inflation

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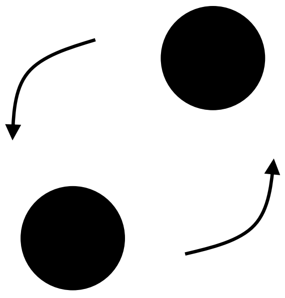

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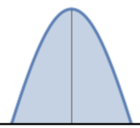
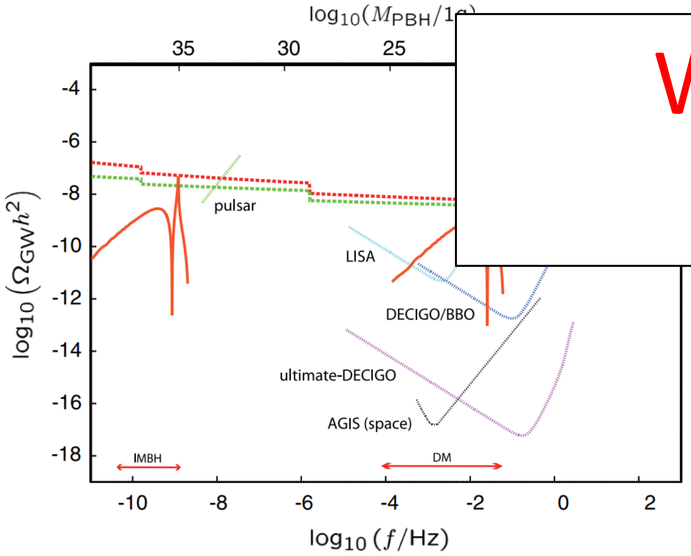
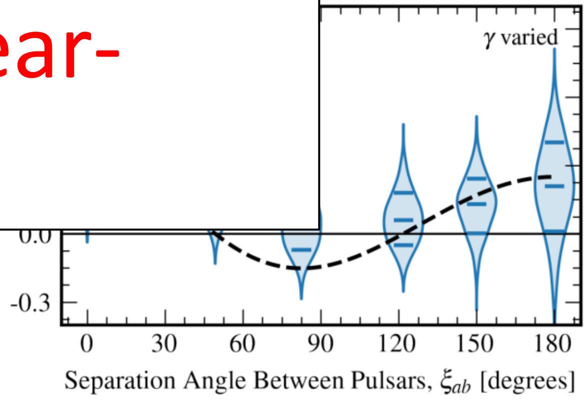



$$\zeta = \zeta_G + \frac{3}{5} f_{NL}\, \zeta_G^2$$

Induced gravitational waves




**We need to work beyond linear-perturbation theory**

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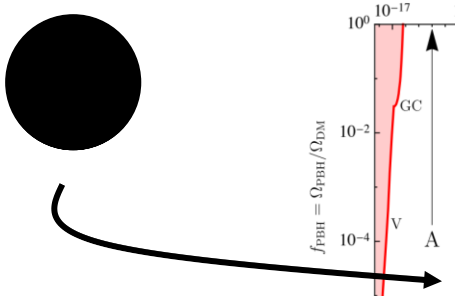
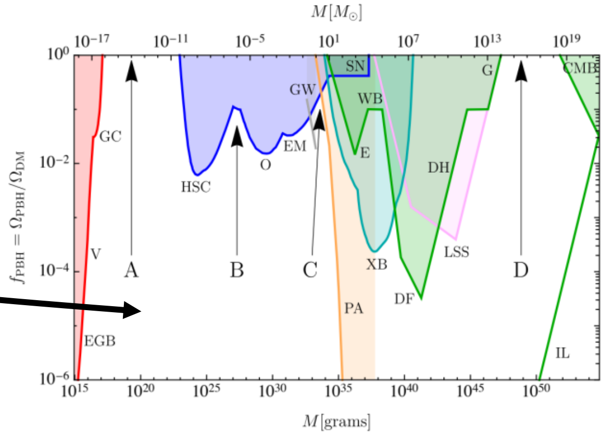





PBHs may represent dark matter

[Carr, Hawking (1974)]
[Green, Kavanagh (2007)]
[Ezquiaga, Garcia-Bellido, Vennin (2019)]
[Escrivà, Kühnel, Tada (2022)]

$$\epsilon_1(t) \coloneqq -\dot{H}(t)/H^2(t)$$

# Inflation

- The curvature perturbation is $\zeta_k = v_k/z$ where $z = \sqrt{2\epsilon_1(t)}\, a(t)$ and

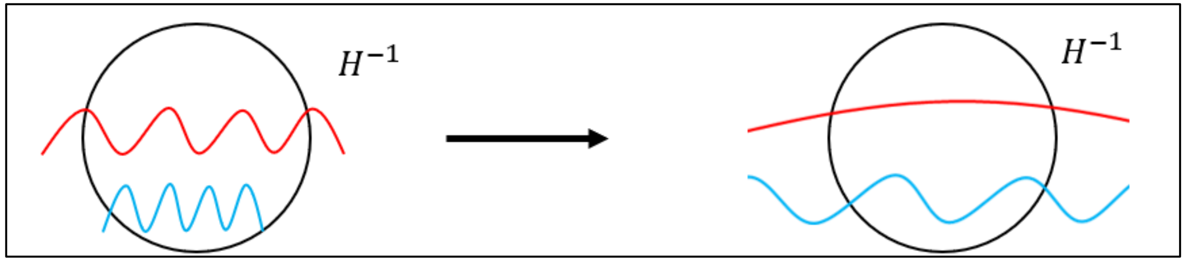
$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

High-frequency modes

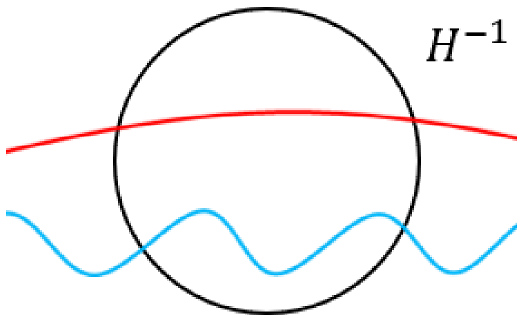
$$\zeta_k = \frac{B_1(k)}{a(t)} e^{-ikt} + \frac{B_2(k)}{a(t)} e^{ikt}$$

Low-frequency modes

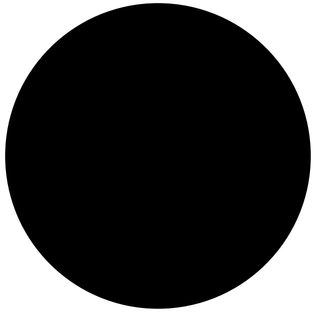
$$\zeta_k \approx C_1(k)$$



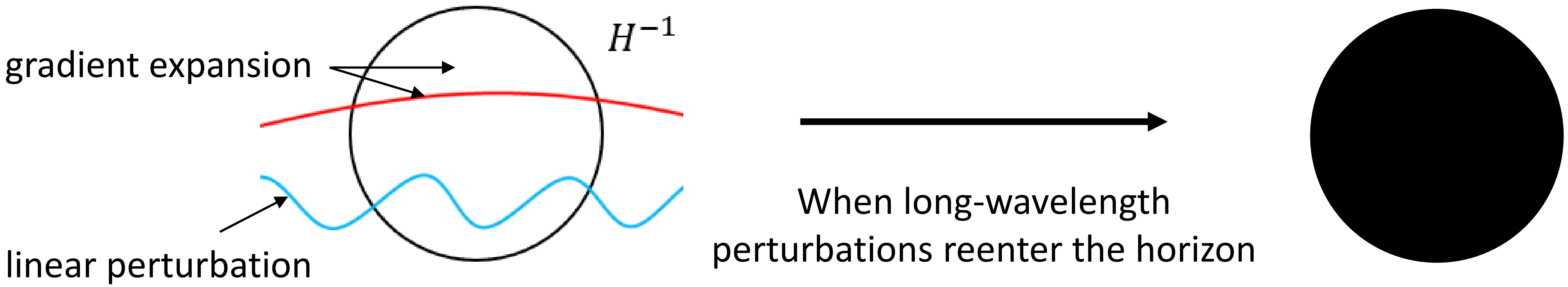
# Inflation




When long-wavelength
perturbations reenter the horizon



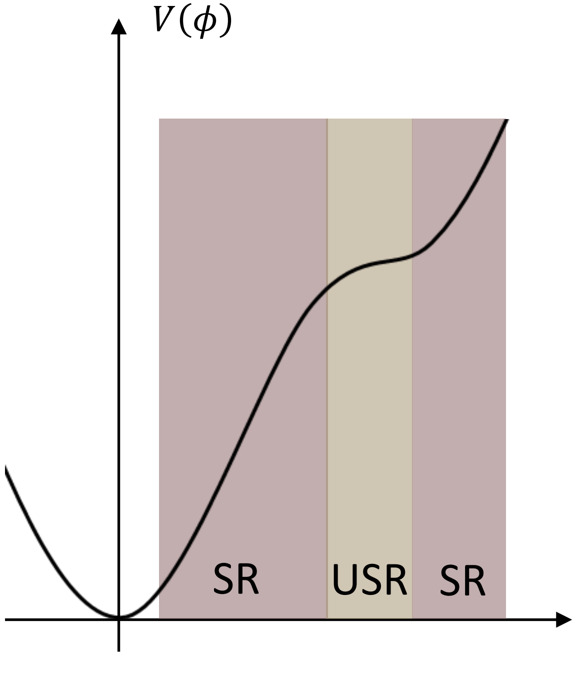
# Inflation




- Let’s describe non-linear effects of long-wavelength modes only: gradient expansion.

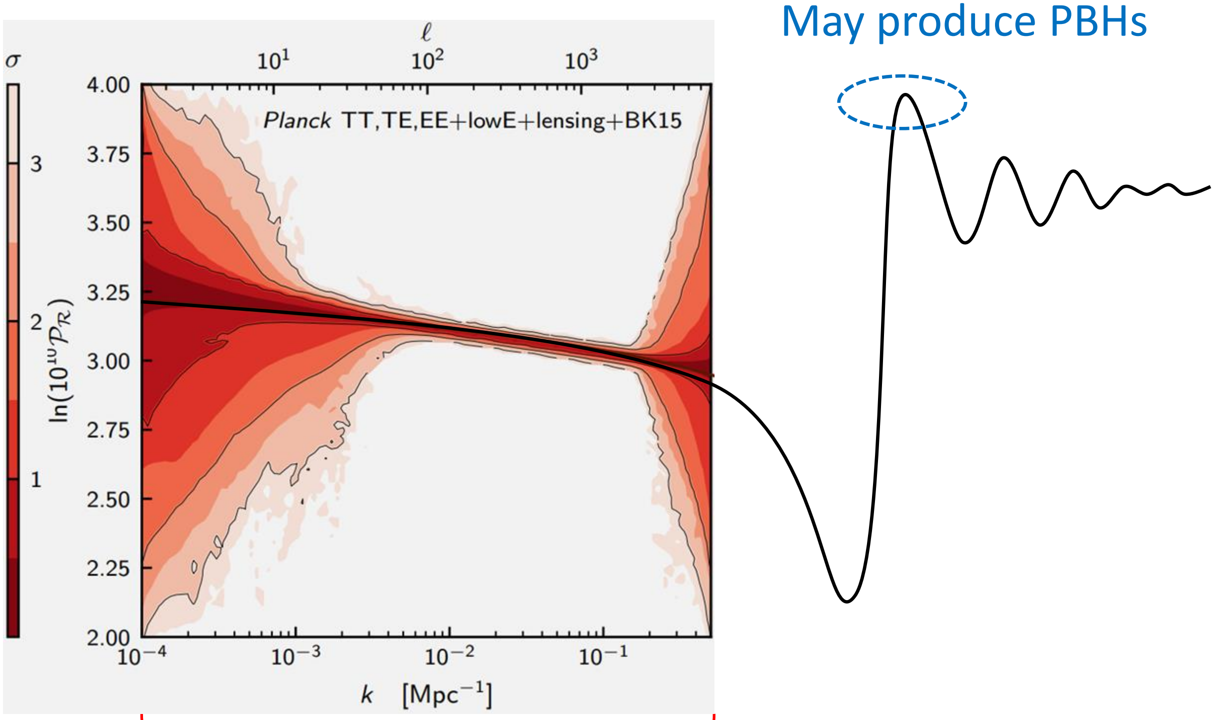
# Inflation

- For Bunch-Davies vacuum the power spectrum is scale invariant: $P_\zeta(k) \approx 10^{-9}$
- The PDF is Gaussian




[Starobinsky (1992)]
[Byrnes, Cole, Patil (2018)]

May produce PBHs




First $e$-folds of inflation

[Planck collaboration (2019)]

# Contents

$z = \sqrt{2\epsilon_1(\eta)}\, a(\eta)$

# Curvature perturbation

- The curvature perturbation obeys

$$\zeta_k'' + \frac{2\,z'}{z}\zeta_k' + k^2\zeta_k = 0$$

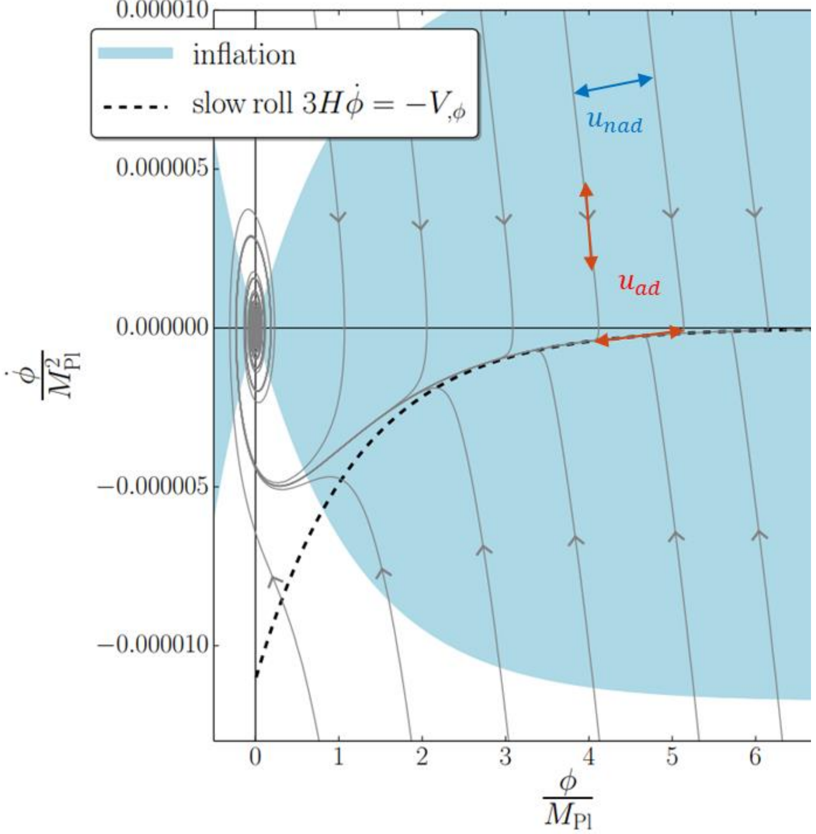
- Decompose the solution into adiabatic and non-adiabatic modes

$$\zeta(\eta) = \zeta_*\, u_{ad}(\eta) + \zeta_*'\, u_{nad}(\eta)$$

$$u_{ad}(\eta) = 1 - k^2 \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \int_{\eta_*}^{\tilde{\eta}} d\tilde{\tilde{\eta}}\, z^2(\tilde{\tilde{\eta}})$$

$$u_{nad}(\eta) = z^2(\eta_*)\left[\int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} - \mathcal{O}(k^2)\right]$$

Adapted from [Grain, Vennin (2017)]



# Gradient expansion

## Separate universe

- Perform a 1+3 splitting of the metric

$$g_{00} = -N^2 + \underbrace{\mathrm{N^i N}_i}_{=0} \quad , \quad g_{0i} = \underbrace{N_i}_{=0} \quad , \quad g_{ij} = \gamma_{ij}$$

- Define the integrated expansion rate

$$\mathcal{N} = \frac{1}{6}\int \gamma^{ij}\dot{\gamma}_{ij}\, d\tau$$

- At large scales $k \to 0$, the anisotropic part of the extrinsic curvature decays with the expansion

$$\dot{A}^i_j = -\frac{1}{2}\,(\gamma^{mn}\dot{\gamma}_{mn})A^i_j \quad \longrightarrow \quad A^i_j \propto \gamma^{-1/2}$$

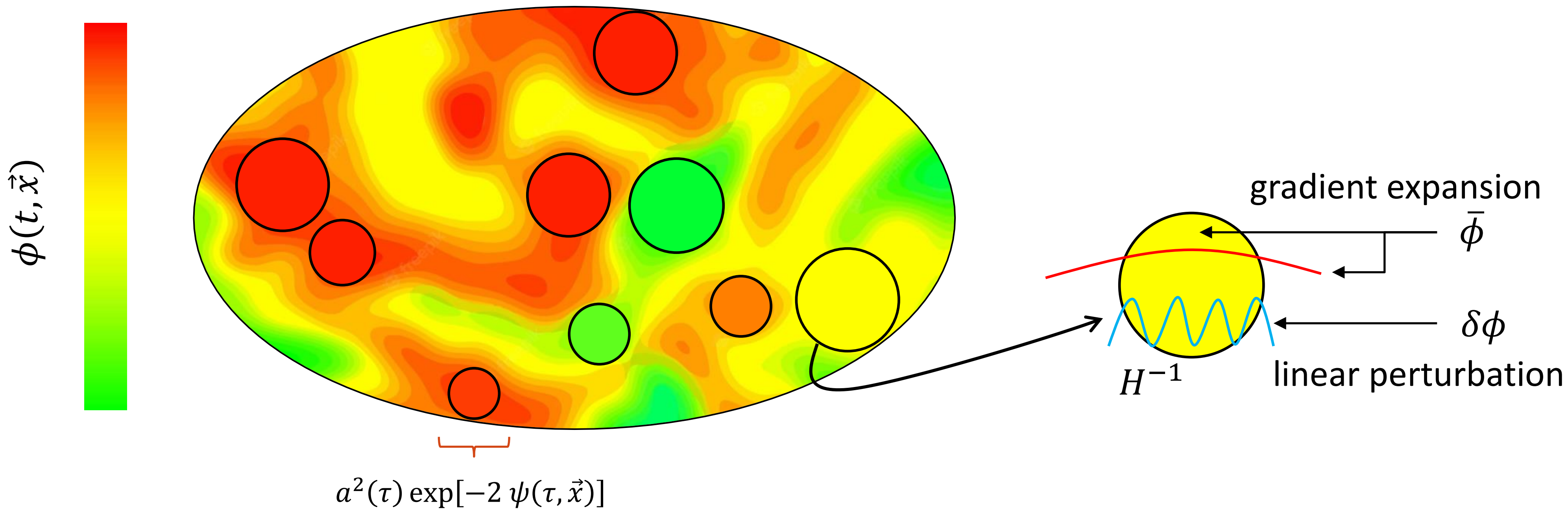
- The most general metric with vanishing anisotropy and $N^i$ is

$$\gamma_{ij}(\tau,\vec{x}) = a^2(\tau)\exp[-2\,\psi(\tau,\vec{x})]\,\underbrace{h_{ij}\,(\vec{x})}_{=\,\delta_{ij}\ \text{locally}}$$

[Salopek, Bond (1990)]

# Gradient expansion

## Separate universe





[Starobinsky (1983)]
[Salopek, Bond (1990)]
[Sasaki, Stewart (1996)]
[Sasaki, Tanaka (1998)]
[Wands, Malik, Lyth, Liddle (2000)]

# Gradient expansion

## Separate universe

- Nonlinearly:

$$\mathcal{N}(\tau_{in},\tau_f,\vec{x}) = \ln\left(\frac{a_f\, e^{-\psi_f}}{a_{in}\, e^{-\psi_{in}}}\right) = \bar{\mathcal{N}}(\tau_{in},\tau_f) + \ln\left(\frac{e^{-\psi_f}}{e^{-\psi_{in}}}\right)$$

$-\psi_f = \zeta_f$

$-\psi_{in} = 0$

$$\longrightarrow\ \delta\mathcal{N}(\tau_{in},\tau_f,\vec{x}) = \zeta(\tau_f,\vec{x})$$

- Take a set of FLRW universes

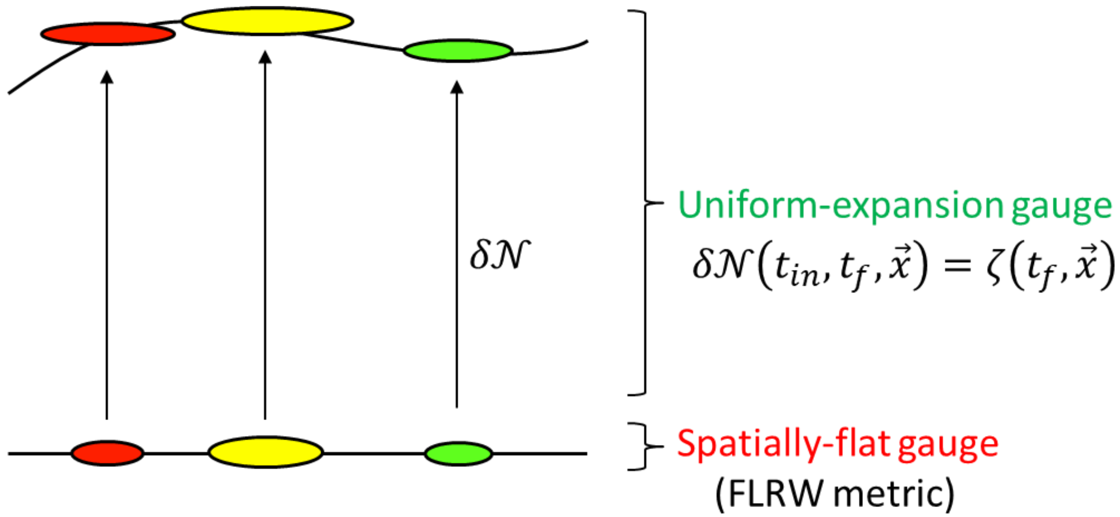
$$\gamma_{ij}(\tau,\vec{x}) = a^2(\tau)\,\delta_{ij}$$

- Perturb the FLRW equations

$$a+\delta a\ , \qquad H+\delta H\ ,$$
$$\phi+\delta\phi\ , \qquad \pi_\phi+\delta\pi_\phi$$

- The perturbed integrated expansion rate is

$$\delta\mathcal{N} = \delta a/a$$

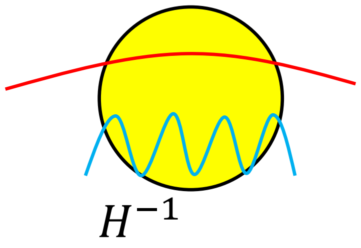


# Non-adiabatic counterpart

1. Gauges used in the gradient expansion (spatially flat) may be inconsistent with non-slow roll phases.

[**DA**, Grain, Vennin (2022)]
[**DA**, Grain, Vennin (2023)]
[**DA**, Frion, Miranda, Vennin, Wands (in prep.)]

2. The standard gradient expansion only captures the adiabatic mode.




This mode evolves at large scales depending on $u_{nad}$

$$\zeta(\eta) = \zeta_* \, u_{ad}(\eta) + \zeta_*' \, u_{nad}(\eta)$$

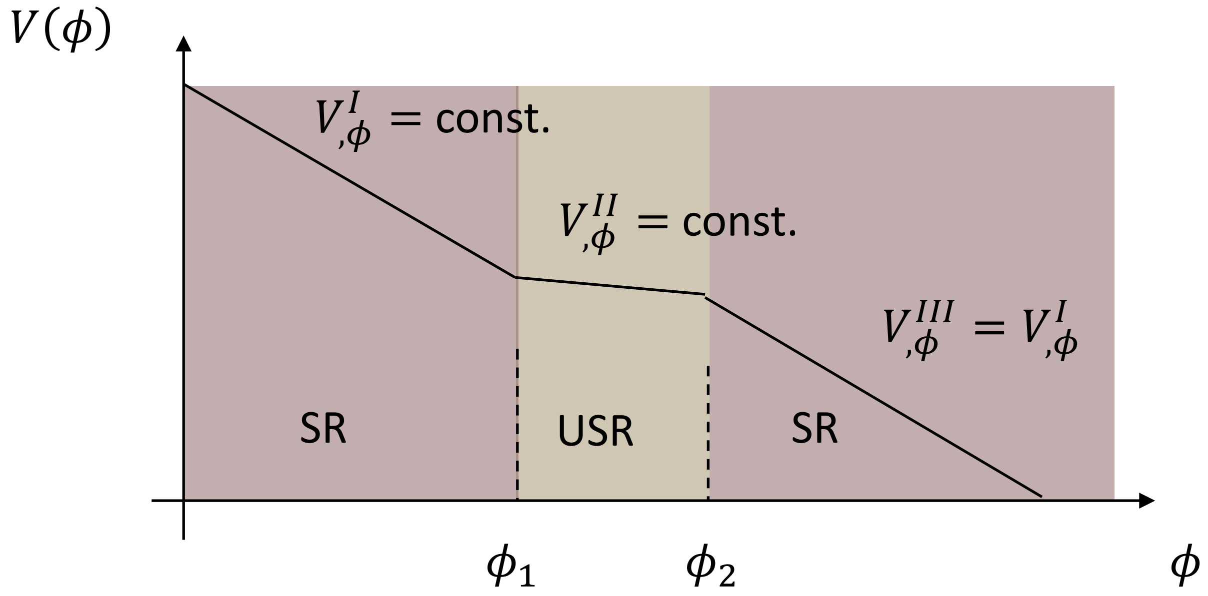
$$u_{ad}(\eta) = 1$$
$$u_{nad}(\eta) = z^2(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})}$$

But this term is relevant in non-slow-roll inflation (e.g. ultra-slow roll).

[Gordon, Wands, Bassett, Maartens (2000)]
[Takamizu et al. (2010)]
[Naruko, Takamizu, Sasaki (2012)]

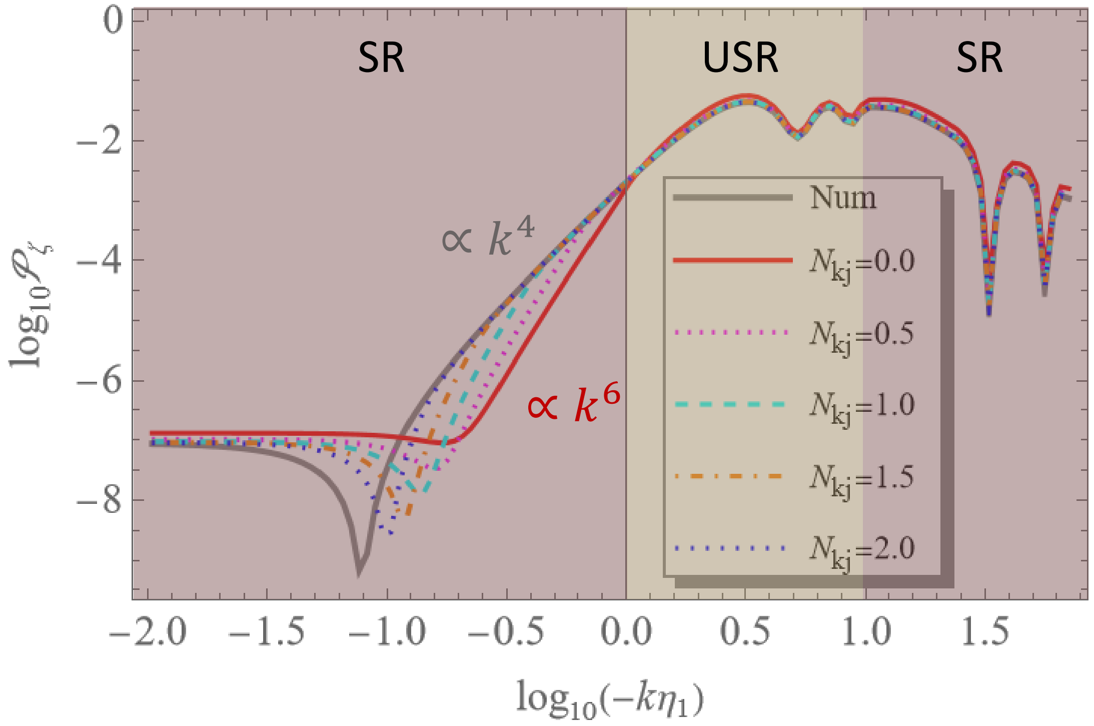
# Non-slow roll: Starobinsky model

- Consider the Starobinsky model




[Starobinsky (1992)]
[Pi, J. Wang (2022)]

# Non-slow roll: Starobinsky model




$N_k \equiv$ horizon-crossing time
$N_j \equiv$ start using gradient expansion

$$N_{kj} := N_j - N_k > 0$$

- If $N_{kj}$ is long, all trajectories align on the phase-space attractor: $u_{nad}$ is negligible and the usual separate-universe approach matches perturbation theory.
- If not, modes that exited the horizon during the first SR phase start evolving during USR.

[Leach, Sasaki, Wands, Liddle (2001)]
[Domenech, Vargas, Vargas (2023)]
[Jackson et al. (2023)]
[**DA**, Pi, Tanaka (2024)]

# Extended gradient expansion

$$\zeta(\eta) = \zeta_* \, u_{ad}(\eta) + \zeta_*' \, u_{nad}(\eta)$$

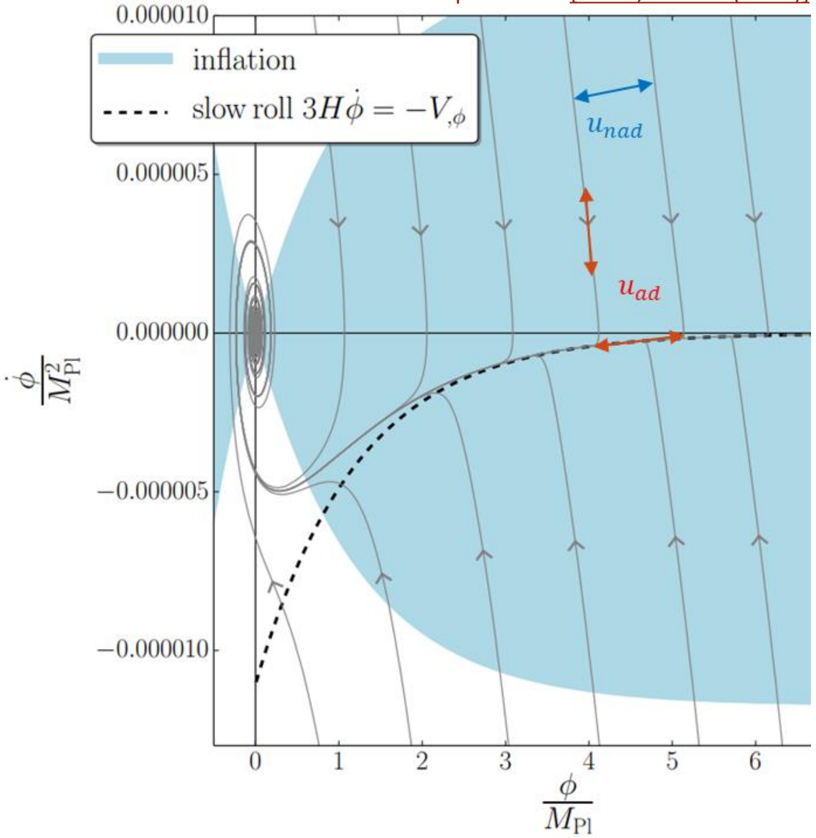
$$u_{ad}(\eta) = 1 - k^2 \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \int_{\eta_*}^{\tilde{\eta}} d\tilde{\tilde{\eta}} \, z^2(\tilde{\tilde{\eta}})$$

$$u_{nad}(\eta) = z^2(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})}$$

- The leading order of the non-adiabatic mode can be described as a $k^2$ correction to the adiabatic mode.

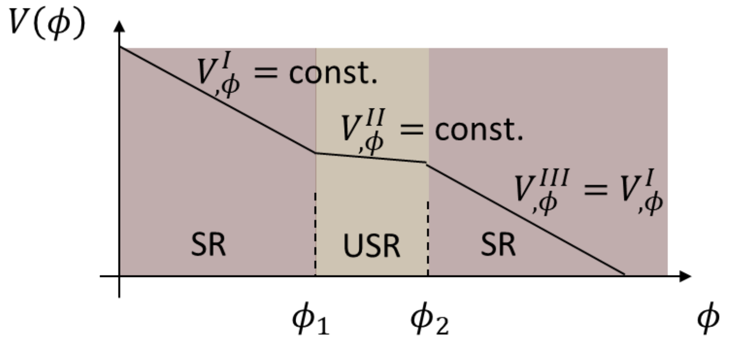
- Allow the gradient expansion to describe the $\mathcal{O}(k^2)$.

Adapted from [Grain, Vennin (2017)]




[Leach, Sasaki, Wands, Liddle (2001)]
[Takamizu, Mukohyama, Sasaki, Y. Tanaka (2010)]
[Naruko, Takamizu, Sasaki (2012)]
[Jackson, Assadullahi, Gow, Koyama, Vennin, Wands (2023)]

# Extended gradient expansion




- Perturb the Klein-Gordon equation $\phi \to \phi + \delta\phi$

$$\delta\phi_{NN} + 3\delta\phi_N + \underbrace{\frac{V_{,\phi\phi}}{H_0^2}\delta\phi}_{=\,0 \text{ on each segment}} + \frac{2V_{,\phi}}{H_0^2}A - \phi_N A_N + \frac{k^2 e^{-2N}}{H_0^2}\delta\phi = 0$$

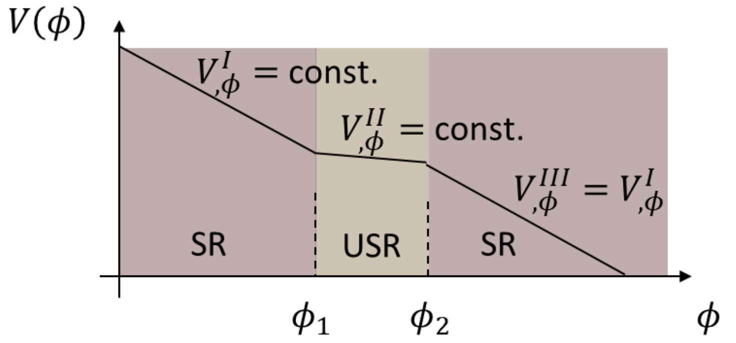
- Choose the initial conditions

$$\delta\phi(N_j) = 0$$ comoving gauge

$$\delta\phi_N(N) = \mathcal{O}(k^2)$$

[**DA**, Pi, Tanaka (2024)]

# Extended gradient expansion

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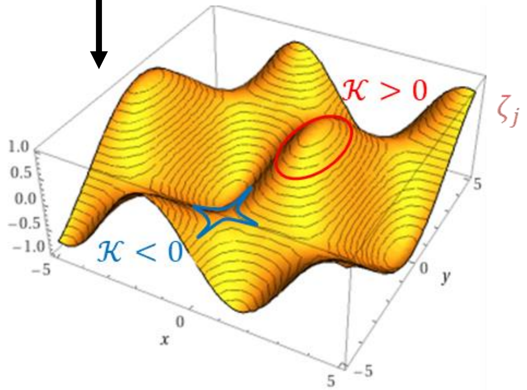
use the $\binom{0}{0}$-component of Einstein equations

- Consider curved FLRW patches

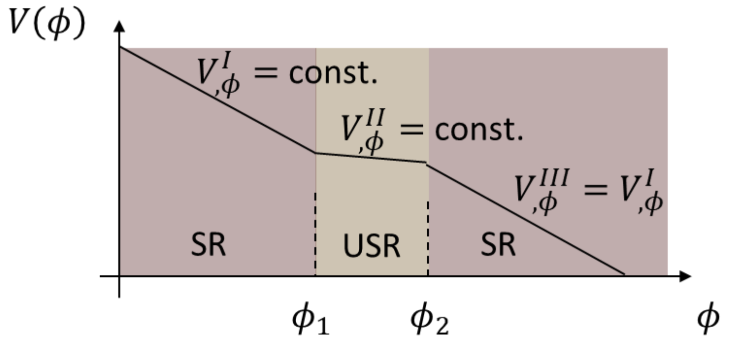
$$H^2 = H_0^2 - \mathcal{K}e^{-2(N-N_j)}$$

and initially $\mathcal{K}e^{2N_j} \equiv \frac{2k^2}{3}\zeta_j$




[DA, Pi, Tanaka (2024)]

# Extended gradient expansion

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use the $\binom{0}{0}$-component of Einstein equations

$$\left[\partial_N^2 + \left(3 + \frac{\mathcal{K}}{H_0^2}e^{-2\left(N-N_j\right)}\right)\partial_N\right]\phi + \frac{V_{,\phi}}{H_0^2}\left(1 + \frac{\mathcal{K}}{H_0^2}e^{-2\left(N-N_j\right)}\right) = \mathcal{O}(\mathcal{K}^2)$$

[**DA**, Pi, Tanaka (2024)]

# Extended gradient expansion

- Fix the $\delta N$ gauge: $N$ is equal to the background expansion rate.
- The scalar field obeys non-linearly to

$$\left[\partial_N^2 + \left(3 + \frac{\mathcal{K}}{H_0^2} e^{-2(N-N_j)}\right)\partial_N\right]\phi + \frac{V_{,\phi}}{H_0^2}\left(1 + \frac{\mathcal{K}}{H_0^2} e^{-2(N-N_j)}\right) = \mathcal{O}(\mathcal{K}^2)$$

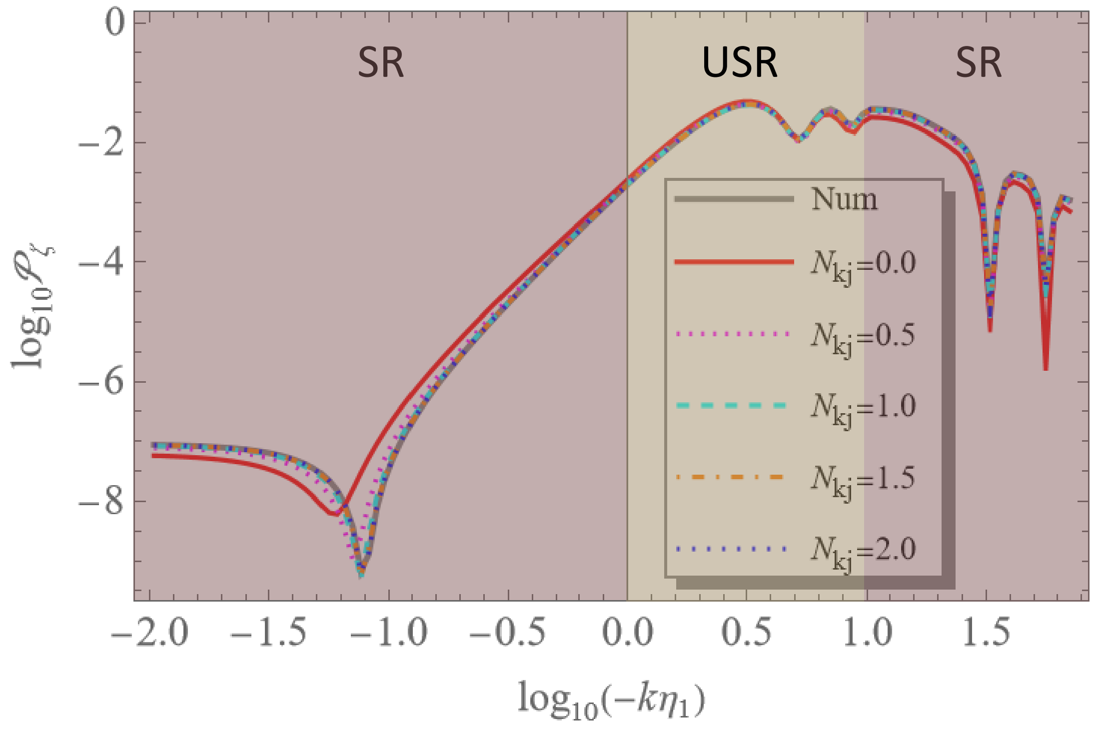
- It is easy to find an analytical solution for $\phi$ which can then be inverted to find the $e$-folding number for each phase.

$$N_{j1} = \underbrace{\mathcal{W}}_{\mathcal{O}(\mathcal{K})} e^{-2N_{j1}} + \underbrace{\mathcal{X}}_{\delta\phi_N(N_j)} e^{-3N_{j1}} + \underbrace{\mathcal{Z}}_{\delta\phi_N(N_j)}$$

Solutions: Lambert function.

[**DA**, Pi, Tanaka (2024)]

# Extended gradient expansion




$N_k \equiv$ horizon-crossing time
$N_j \equiv$ start using gradient expansion
$$N_{kj} := N_j - N_k > 0$$

- The generalised gradient expansion is consistent with linear perturbation theory during slow roll.
- We can use it to track non-linearities (such as $f_{NL}$) during the transition.

[**DA**, Pi, Tanaka (2024)]

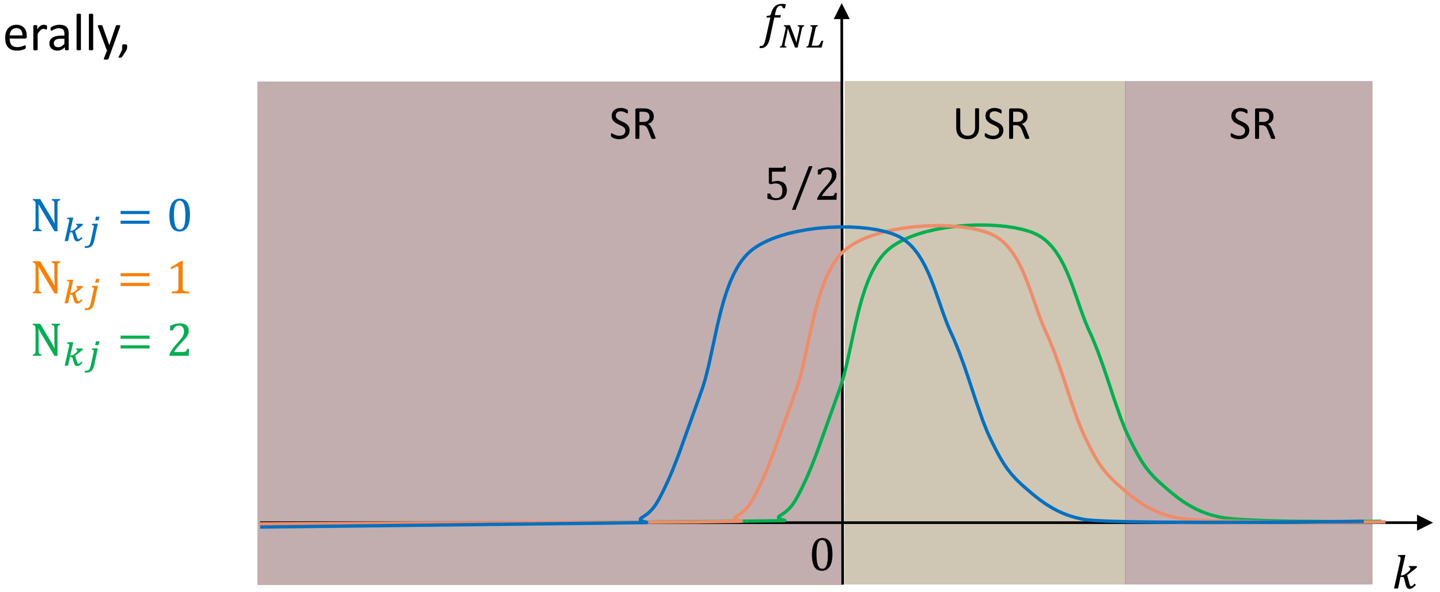
# Extended gradient expansion

- The $f_{NL}$ can be obtained from $N$. If N doesn't depend on $\phi_N$, then

[Maldacena (2002)]
[Bartolo et al. (2004)]
[Yokoyama, Suyama, Tanaka (2007)]

$$f_{NL} = \frac{5}{6}\frac{N_{\phi\phi}}{N_\phi^2} = \begin{cases} 0 \text{ in SR} \\ 5/2 \text{ in USR} \end{cases}$$

More generally,




- The $f_{NL}$ transits continuously from $0 \to 5/2 \to 0$ as expected.

# Conclusion

- To constrain inflationary models, non-linear effects may be important.

- The gradient expansion describes non-linear effects during inflation.

  Describe a set of flat FLRW patches. $\zeta = \delta N$.

  Well understood for the case of slow roll.

- Extended gradient expansion: curved FLRW patches.

  Captures the $k^2$-correction of $\zeta$. Relevant e.g. in ultra-slow roll.

- The $f_{NL}$ evolves continuously from slow roll to ultra-slow roll $0 \to 5/2$. PBHs may be created even from modes that exited the horizon during the slow-roll phase. [work in progress]

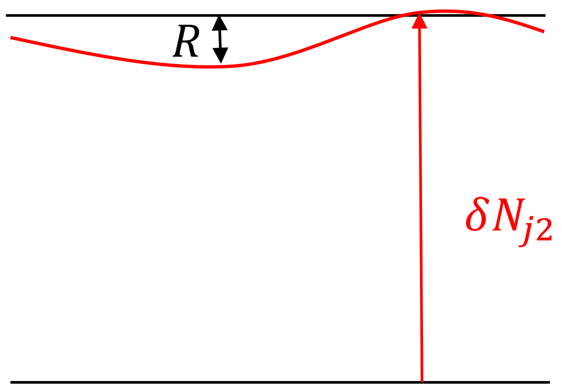
# Extended gradient expansion

- After a gauge transformation

$$\zeta(N_2) = R(N_2) + \delta N_{j2}$$

where

$$R(N) = \zeta(N_j) + \frac{k^2 \zeta(N_j)}{6\, H_0^2}\left(e^{-2N_j} - e^{-2N}\right)$$




$\delta\phi(N_2) = 0$

$\delta\phi(N_j) = 0$

[**DA**, Pi, Tanaka (2024)]

# 1. Gauges and the momentum constraint

- Since anisotropic degrees of freedom were neglected, the momentum constraint reads

$$\partial_i D = 0 = \underbrace{-\frac{2}{3}\partial_i K + \frac{1}{M_{Pl}^2}\frac{\phi'}{a}\partial_i\phi}_{\partial_i D_{iso}} \cancel{+\partial_j A_i^j}$$

- The Hamilton-Jacobi approach sets $D_{iso} = 0$ in the spatially-flat gauge.

[Salopek, Bond (1990)]
[Rigopoulos, Wilkins (2021)]
[Cruces (2022)]
[Launay, Rigopoulos, Shellard (2024)]

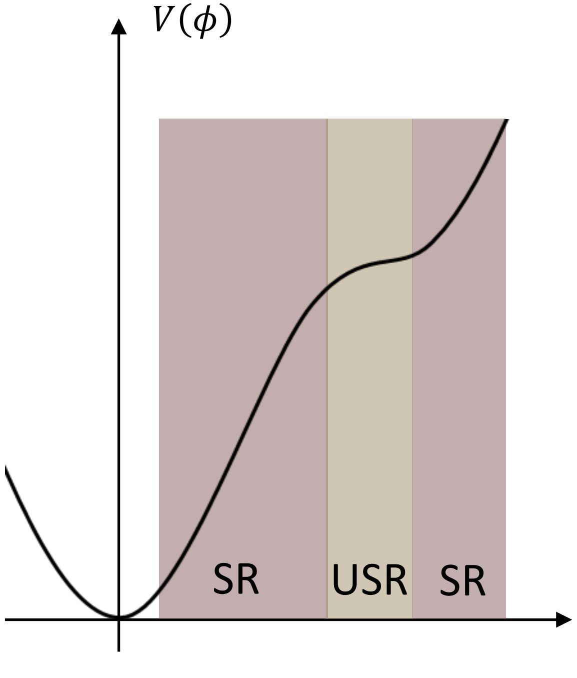
- But in this gauge $D_{iso} \propto u_{nad}$.

[**DA**, Grain, Vennin (2022)]
[**DA**, Frion, Miranda, Vennin, Wands (in prep.)]

- For more details about gauges check also [**DA**, Grain, Vennin (2023)]

# Inflation

- In slow roll, for Bunch-Davies vacuum the power spectrum is scale invariant: $P_\zeta(k) \approx 10^{-9}$




[Starobinsky (1992)]
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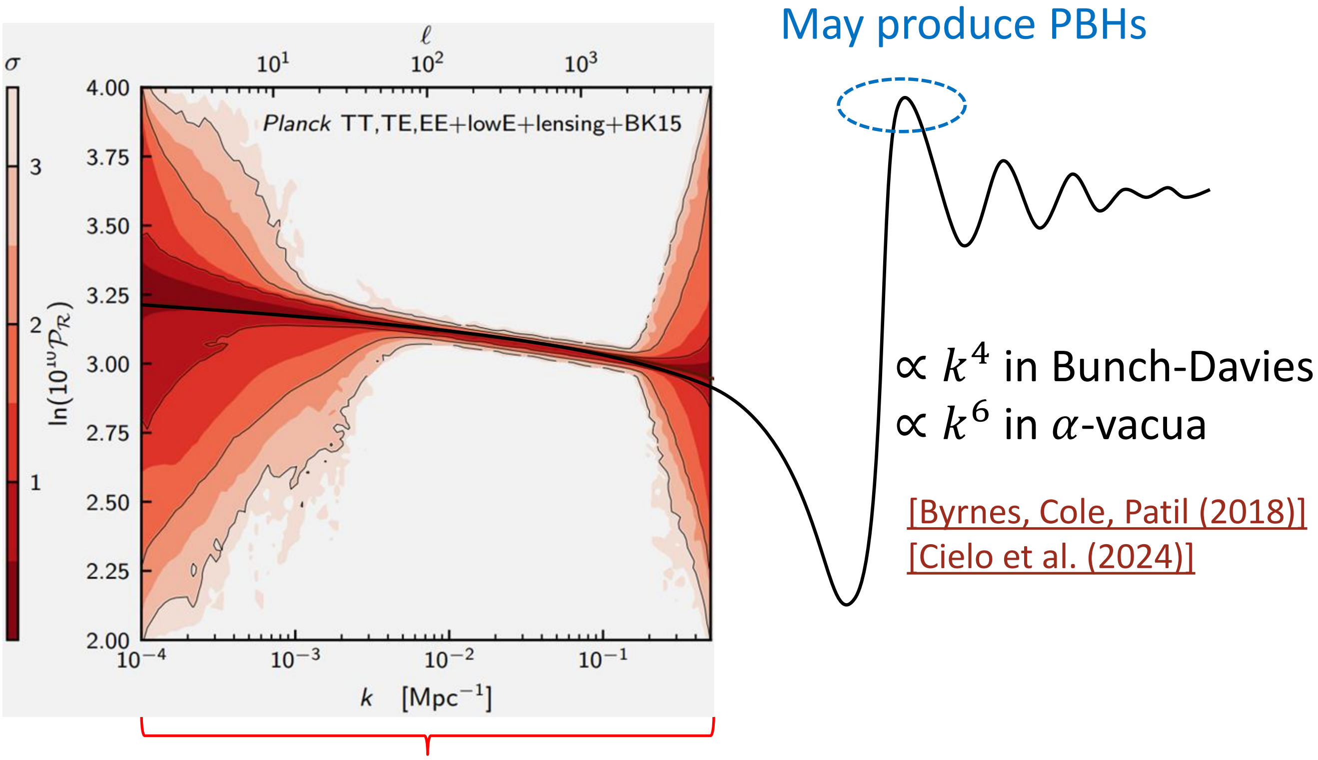


[Byrnes, Cole, Patil (2018)]
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