Spontaneous symmetry breaking & low-energy excitations in gapless frustration free systems

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Applied Physics department
University of Tokyo

- UC Berkeley PhD program in 2011-2015
- Official PhD advisor: Ashvin Vishwanath
  But worked a lot with Hitoshi
- Area: Condensed-Matter Theory

What I discuss today

Spontaneous Symmetry Breaking + **Frustration Free**

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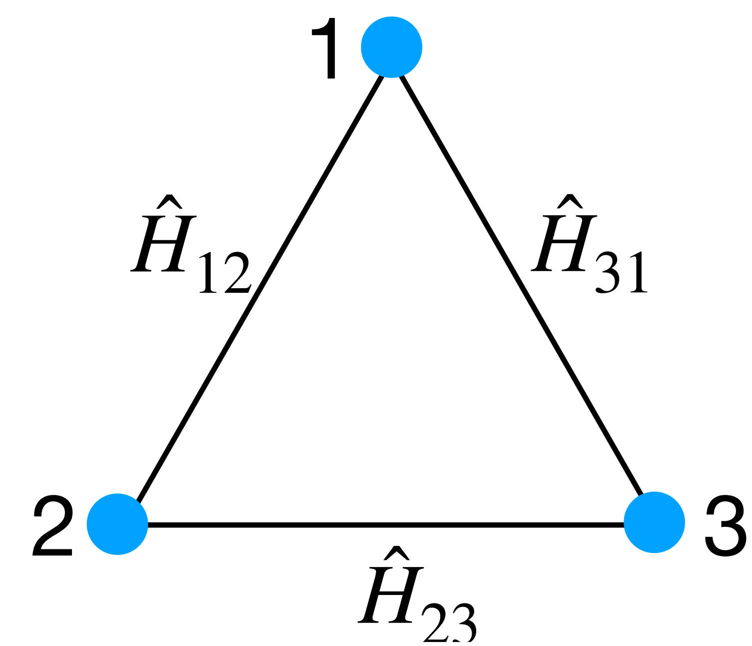
Frustration Free

Frustration in Quantum Many-Body Systems

- Antiferromagetic interaction among three spins 1,2,3.
  $\hat{H} = \hat{H}_{12} + \hat{H}_{23} + \hat{H}_{31}$



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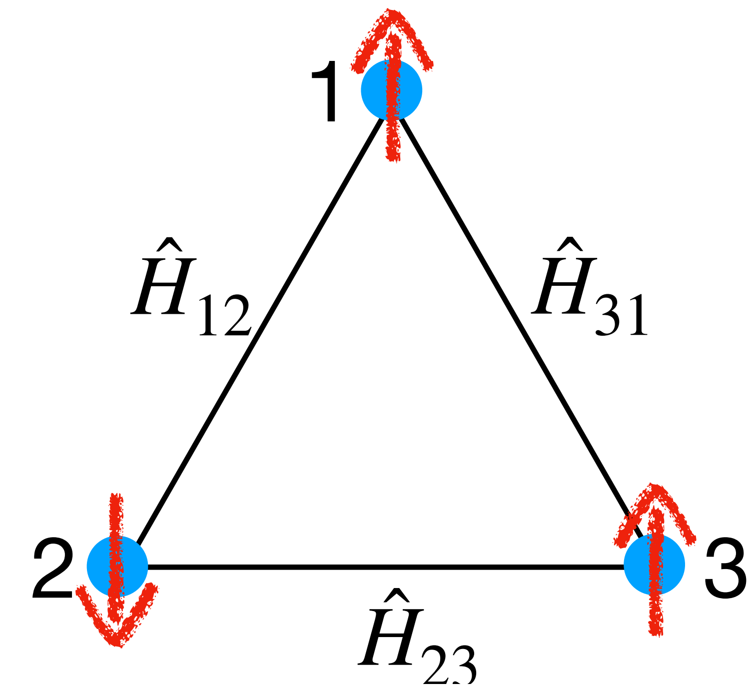

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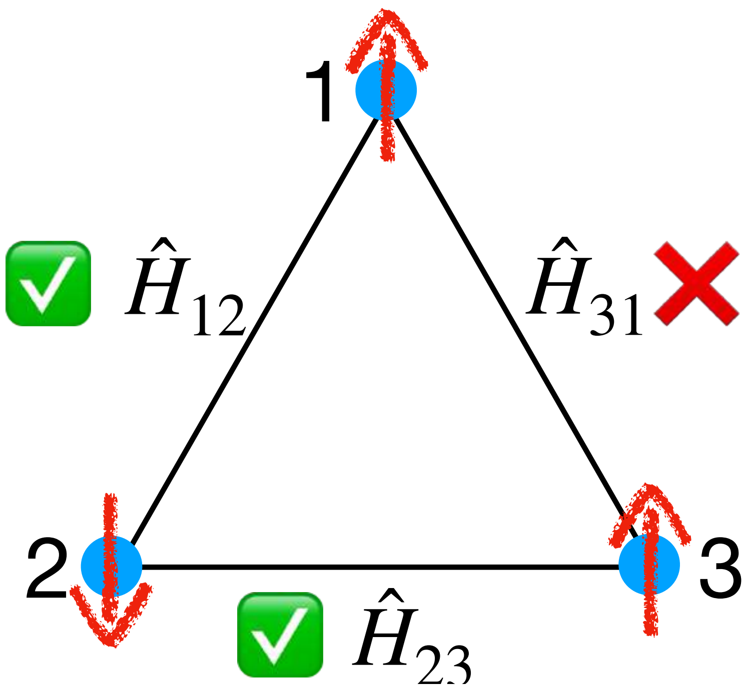
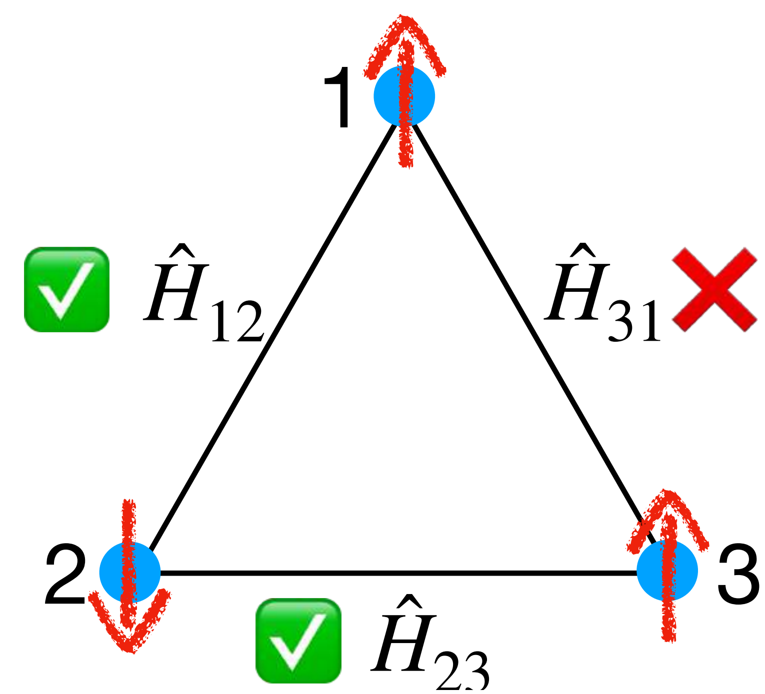


Frustration in Quantum Many-Body Systems

- Antiferromagetic interaction among three spins 1,2,3. $\hat{H} = \hat{H}_{12} + \hat{H}_{23} + \hat{H}_{31}$
- No way of making all terms simultaneously minimized. → *frustration*.



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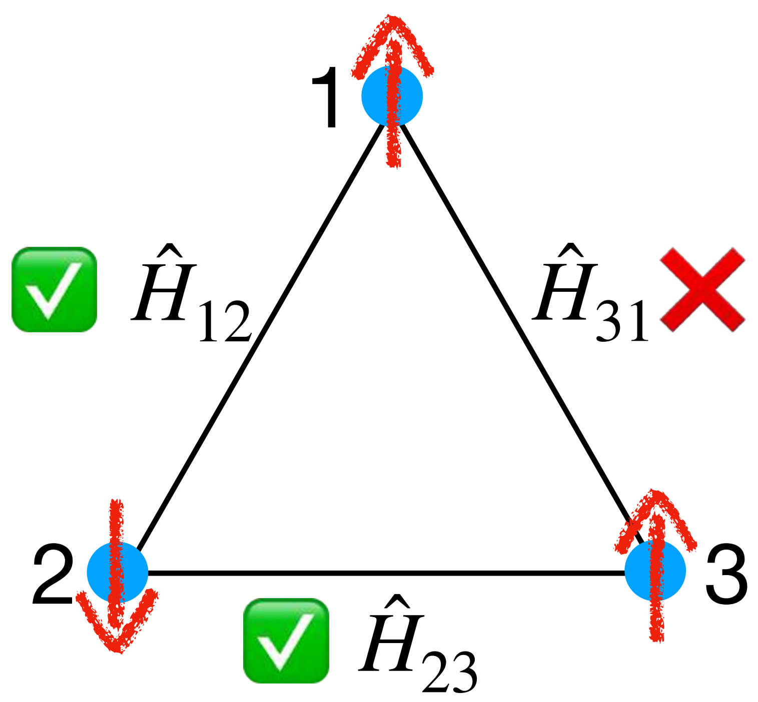
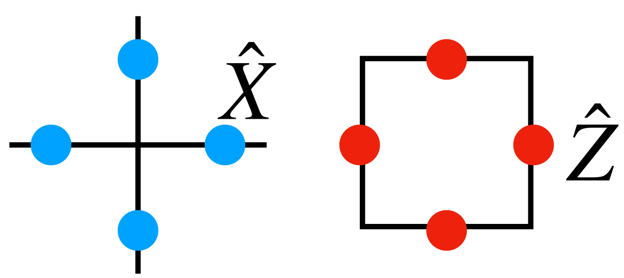


- More generally, $\hat{H}$ is *frustration free* if
  - There exists a decomposition $\hat{H} = \sum_i \hat{H}_i$ with following properties.
  - $\hat{H}_i$'s are finite ranged. $\hat{H}_i$'s *do not* have to commute with each other.
  - Ground state $|\Phi_{\mathrm{GS}}\rangle$ of $\hat{H}$ minimizes all $\hat{H}_i$ simultaneously.
    i.e., $\hat{H}_i|\Phi_{\mathrm{GS}}\rangle = E_{\mathrm{GS},i}|\Phi_{\mathrm{GS}}\rangle$ and $E_{\mathrm{GS},i}$ =0 is GS energy of $\hat{H}_i$.

Examples of FF spin models

- Paramagnet:
$$\hat{H}_i = -\hat{s}_i^z$$
- Majumdar-Ghosh model:
$$\hat{H}_i^{(S=1/2)} = \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1} + \hat{\vec{s}}_{i+1} \cdot \hat{\vec{s}}_{i+2} + \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+2}$$
- AKLT:
$$\hat{H}_i^{(S=1)} = \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1} + \frac{1}{3}(\hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1})^2$$
- Toric code (commuting projector):
$$\hat{H} = -\sum_{+} \hat{V}_{+} - \sum_{\square} \hat{P}_{\square}$$


- Fractons, …

Examples of FF spin models

- Paramagnet:
  $$\hat{H}_i = -\hat{s}_i^z$$
  Trivial
- Majumdar-Ghosh model:
  $$\hat{H}_i^{(S=1/2)} = \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1} + \hat{\vec{s}}_{i+1} \cdot \hat{\vec{s}}_{i+2} + \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+2}$$
  SSB of translation
- AKLT:
  $$\hat{H}_i^{(S=1)} = \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1} + \frac{1}{3}(\hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1})^2$$
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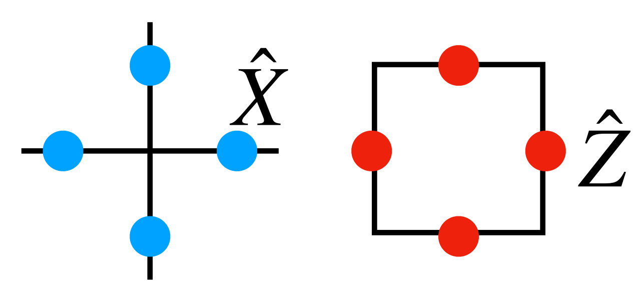
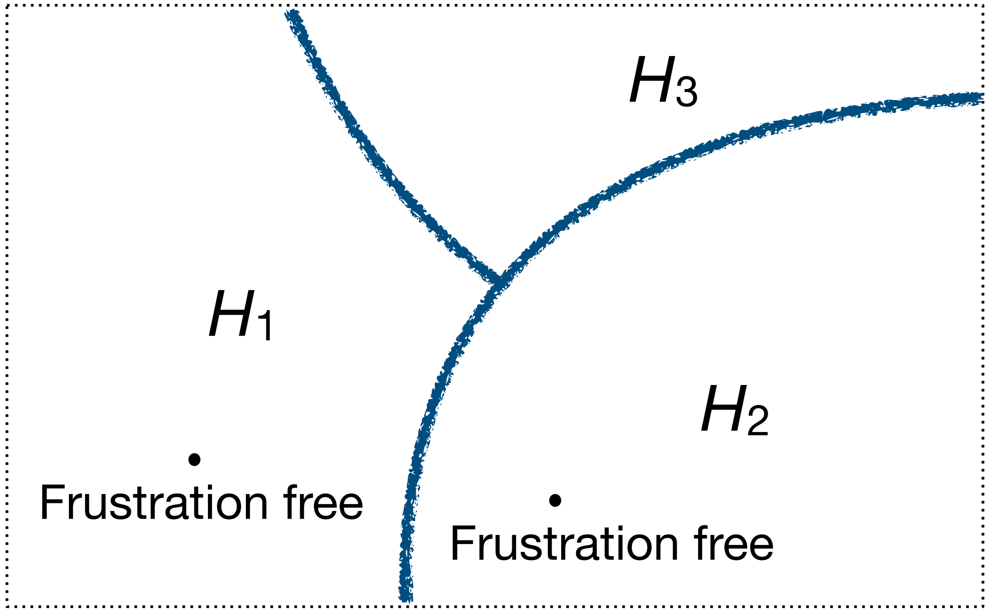
  

  Topological Order
- Fractons, ...

Goal

- We want to understand
  - General properties and limitations of FF Hamiltonians.
  - Which phase can be represented by FF Hamiltonians.



- We discuss several conjectures/new results on FF systems.
  - If gapless, excitation is quadratic or softer: $E_{\vec{k}} = O(|\vec{k} - \vec{k}_0|^2)$
  - If gapless, finite size gap is $\epsilon = O(L^{-2})$.
  - If gapped, finite size splitting of degeneracy is absent.

Markov Chain Monte Carlo (MCMC)

- Simulation method for classical statistical mechanical systems
- Boltzmann weight $w(C) = e^{-E(C)}$
  e.g. Ising model $E(C) = -J\sum_{(i,j)} \sigma_i \sigma_j$
- Master equation
  $$\frac{d}{dt}p(t,C) = \sum_{C' \in S} W_{C,C'} p(t,C')$$
- Local update rule
  $$W = \sum_i W_i$$
- Detailed balance condition
  $$W_{C,C'} w(C') = W_{C',C} w(C)$$

2D Ising model (ordered phase)

Critical Slowing Down

- As the system approaches to a critical point,
  the relaxation time τ becomes longer and longer.

- At the critical point, the relaxation time $\tau \propto L^z$ (z : dynamic critical exponent).

$$|\langle O e^{Wt} O\rangle - \langle O\rangle^2| \simeq C e^{-t/\tau} \text{ with } \tau = 1/\epsilon.$$

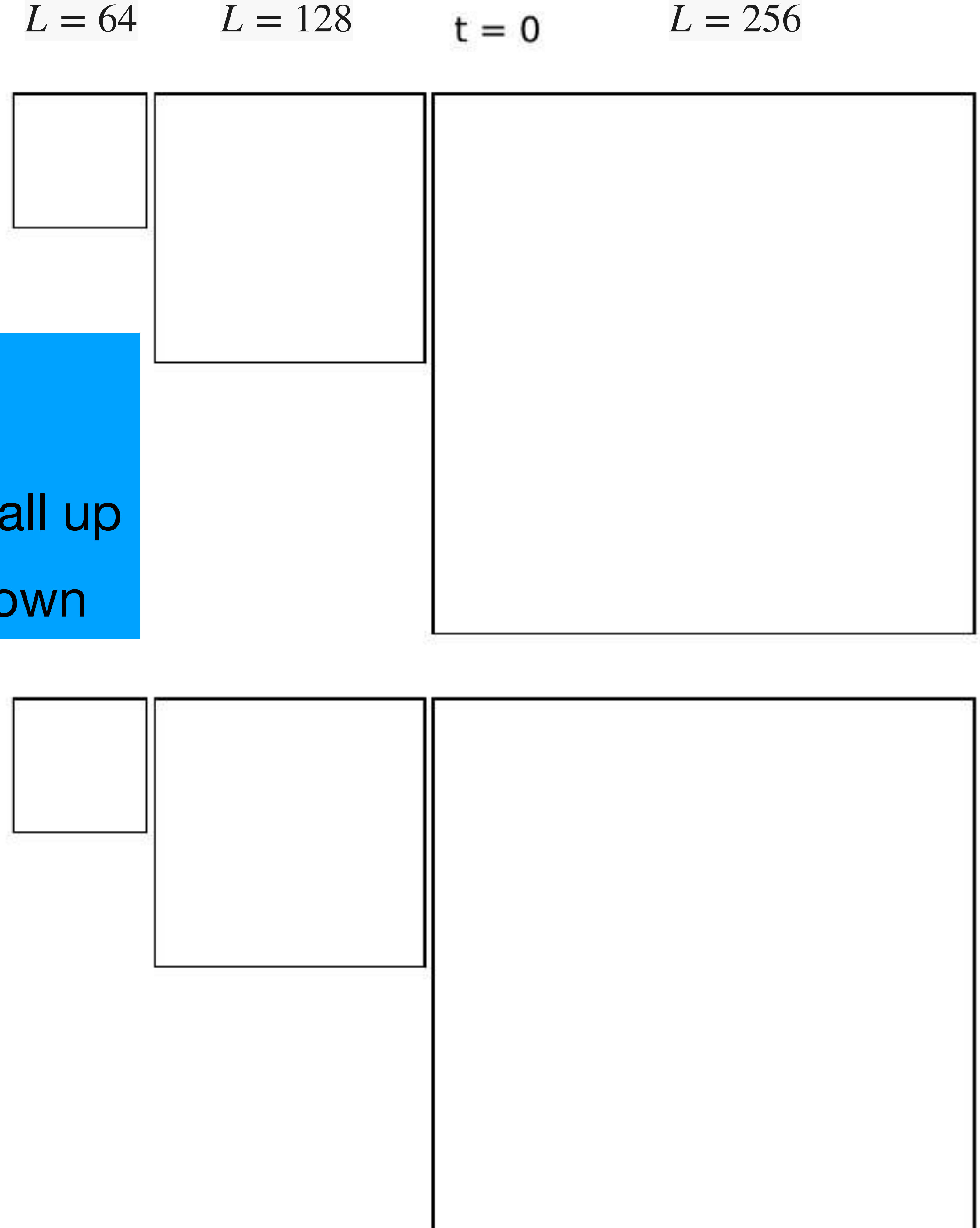
Models	Dynamic critical exponent z
Ising (2D)	2.1667(5) [14]
Ising (3D)	2.0245(15) [15]
Heisenberg (3D)	2.033(5) [16]
Three-state Potts (2D)	2.193(5) [17]
Four-state Potts (2D)	2.296(5) [18]

People believe $z \geq 2$ without a proof.

Obtained smaller z by giving up locality or detailed balance.

$J = 0.9 J_c$
(disordered phase)
$\xi < L$: no L dependence

2D Ising model
$L = 64,\ 128,\ 256$
Boundary condition：all up
Initial condition：all down



$J = 0.99 J_c$
(disordered phase)
$\xi > L$: clear L dependence

$$J = J_c = 0.440687\cdots$$

(critical point)

$$\tau \propto L^z, z = 2.1667(5)$$

$$J = 1.1346 J_c$$

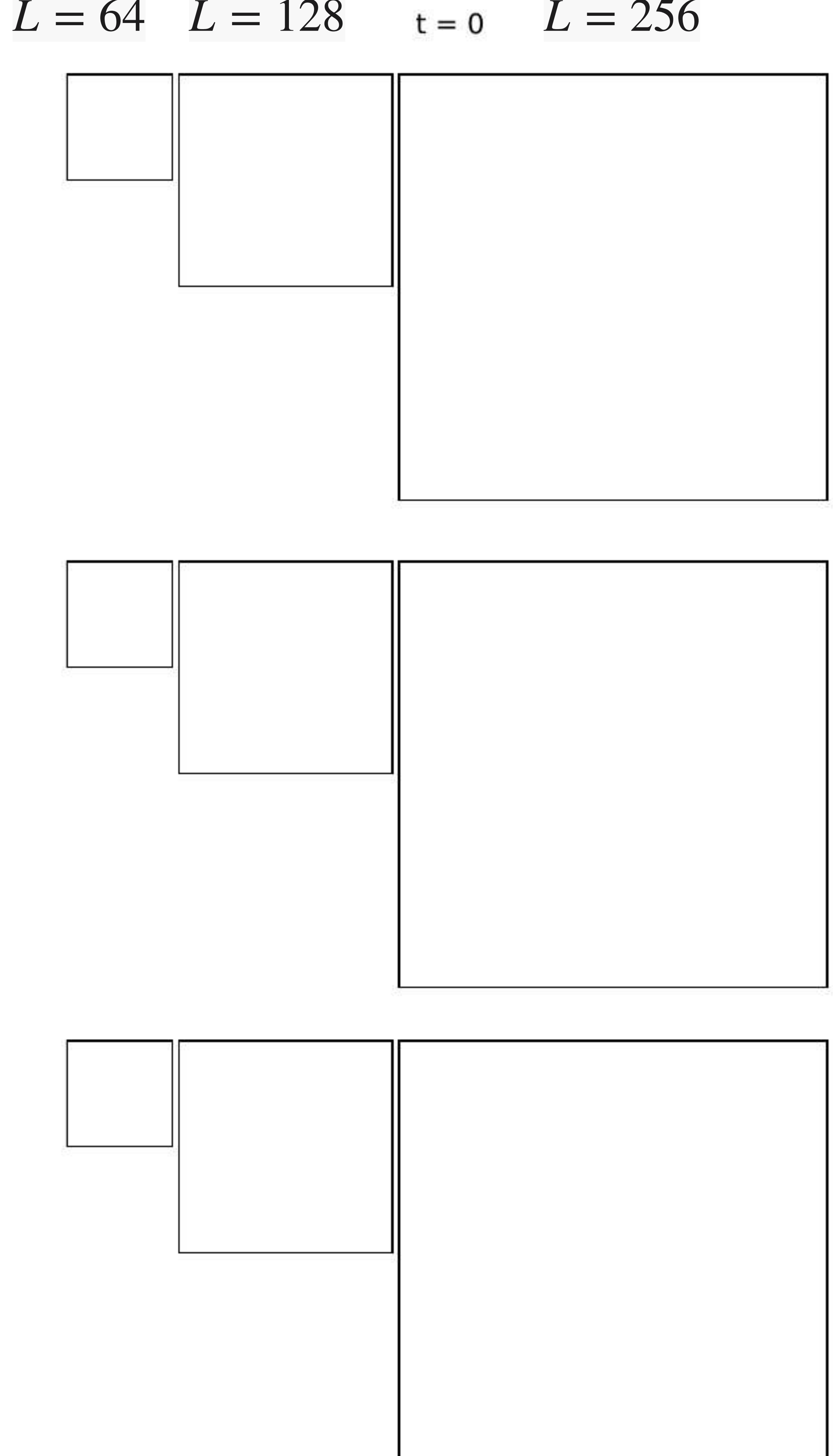
(ordered phase)

$$\tau \propto L^z, z = 2$$

$$J = 2.269 J_c$$

(ordered phase)

$$\tau \propto L^z, z = 2$$



$L = 64$ $L = 128$ t = 0 $L = 256$

$J = J_c = 0.440687\cdots$
(critical point)
$\tau \propto L^z, z = 2.1667(5)$

J
(o
τ

Our general results on *quantum* FF systems
gives the fist proof of
$z \geq 2$ in MCMC for *classical* critical systems.

$J = 2.269 J_c$
(ordered phase)
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Outline

- Nambu-Goldstone bosons in nonrelativistic systems
  HW, H. Murayama, PRL (2012) *Editors' Suggestions*

- Spontaneous breaking of U(1) symmetry in 1+1D
  HW, H. Katsura, J.Y. Lee, PRL (2024) *Editors' Suggestions*

- Low-energy excitations in frustration-free systems
  R. Masaoka, T. Soejima, HW, PRB (2024)
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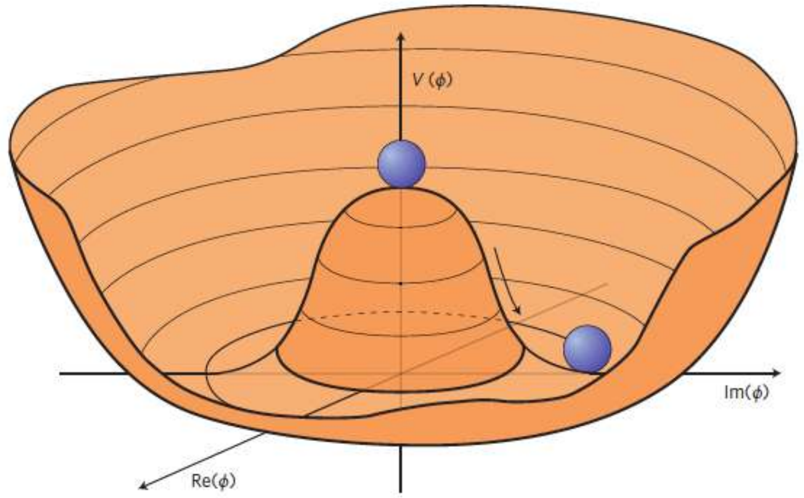
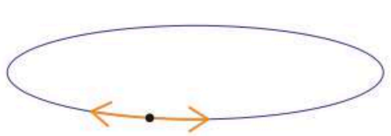
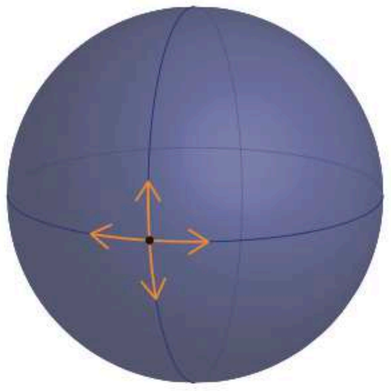
Nambu-Goldstone bosons in relativistic systems

- Suppose the symmetry of the system G is spontaneously broken to H.
- Coset space G/H is the space of degenerate ground states.
- Nambu-Goldstone bosons are low-energy fluctuations within the coset space.
- The number of broken generators $N_{\mathrm{BG}} = \dim(G/H) = \dim G - \dim H$.
- The number Nambu-Goldstone bosons N_{NGB} is always given by N_{BG}.
- Effective Lagrangian $\mathscr{L} = \frac{1}{2} g_{ab}(\pi)\partial_\mu \pi^a \partial^\mu \pi^b + \cdots = \frac{1}{2} g_{ab}(0)\partial_\mu \pi^a \partial^\mu \pi^b + \cdots$







Nambu-Goldstone bosons in nonrelativistic systems

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- Many examples
- Systematic understanding?

Examples	G/H	n_{BG}	n_{NGB}
QCD	$\mathrm{SU}(3) \times \mathrm{SU}(3)/\mathrm{SU}(3)$	8	8
Antiferromagnet	$\mathrm{SO}(3)/\mathrm{SO}(2)$	2	2
Ferromagnet	$\mathrm{SO}(3)/\mathrm{SO}(2)$	2	1
Ferrimagnet	$\mathrm{SO}(3)/\mathrm{SO}(2)$	2	1
Kaon ($\mu = 0$)	$\mathrm{U}(2)/\mathrm{U}(1)$	3	3
Kaon ($\mu > 0$)	$\mathrm{U}(2)/\mathrm{U}(1)$	3	2
BEC (planar)	$\mathrm{SO}(3) \times \mathrm{U}(1)/\mathrm{U}(1)$	3	3
BEC (ferro)	$\mathrm{SO}(3) \times \mathrm{U}(1)/\mathrm{U}(1)'$	3	2
Crystal (2+1D)	T^2	2	2
Wigner crystal	T^2	2	1
Skyrmion crystal	T^2	2	1

Effective Lagrangian

- Lorentz symmetry (+derivative expansion)

$$\mathscr{L} = \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b + \cdots = \frac{1}{2} g_{ab}(0) \partial_\mu \pi^a \partial^\mu \pi^b + \cdots$$

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- Spacial rotation symmetry (+derivative expansion)

$$\begin{aligned} \mathscr{L} &= c_a(\pi)\dot{\pi}^a + \frac{1}{2}\bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}\bar{g}_{ab}(\pi)\,\nabla\pi^a \cdot \nabla\pi^b + \cdots \\ &= \underline{\rho_{ab}\pi^b\dot{\pi}^a} + \frac{1}{2}\bar{g}_{ab}(0)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}\bar{g}_{ab}(0)\,\nabla\pi^a \cdot \nabla\pi^b + \cdots \end{aligned}$$

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- Skew matrix ρ_{ab} is related to the broken generators: $\rho_{ab} = \dfrac{i}{V}\langle[\hat{Q}_a, \hat{Q}_b]\rangle$

Block diagonal form: $$\rho = \begin{pmatrix} 0 & \lambda_1 & & & & & & & \\ -\lambda_1 & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & \lambda_m & & & & \\ & & & -\lambda_m & 0 & & & & \\ & & & & & 0 & \cdots & 0 \\ & & & & & \vdots & \ddots & \vdots \\ & & & & & 0 & \cdots & 0 \end{pmatrix}$$

type B

type A

m blocks

$$m = \frac{1}{2}\mathrm{rank}\rho$$

Counting formula

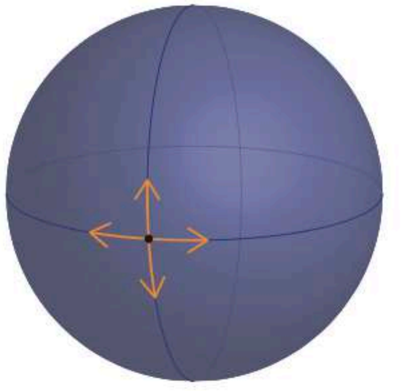
Counting formula

- Type A: $\mathscr{L} = \frac{1}{2}\bar{g}_{ab}(0)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}\bar{g}_{ab}(0)\,\nabla\pi^a\cdot\nabla\pi^b + \cdots$

  π^{2m+1}, …, $\pi^{N_{\mathrm{BG}}}$ are independent variables.

  $\rightarrow N_{\mathrm{A}} = N_{\mathrm{BG}} - 2m = N_{\mathrm{BG}} - \frac{1}{2}\mathrm{rank}\rho$

  Dispersion is generically linear: $\omega_{\vec{k}} \propto k$.



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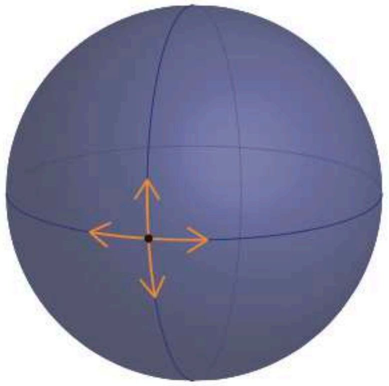
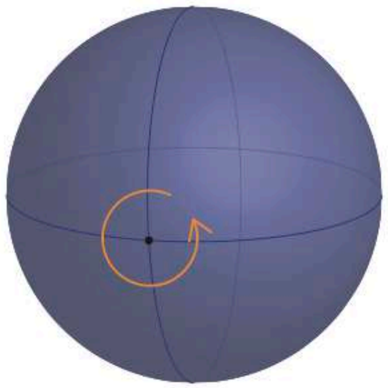


- Type B: $\mathscr{L} = \rho_{ab}\pi^b\dot{\pi}^a - \frac{1}{2}\bar{g}_{ab}(0)\,\nabla\pi^a\cdot\nabla\pi^b + \cdots$

  $\pi^{2\ell-1}$ and $\pi^{2\ell}$ ($\ell = 1,2,\cdots,m$) are canonically conjugate variables.

  $\to N_{\mathrm{B}} = m = \frac{1}{2}\mathrm{rank}\rho$

  Dispersion is generically quadratic: $\omega_{\vec{k}} \propto k^2$.



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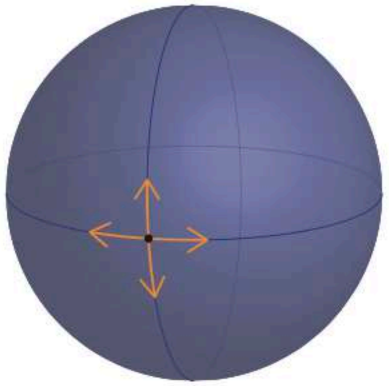

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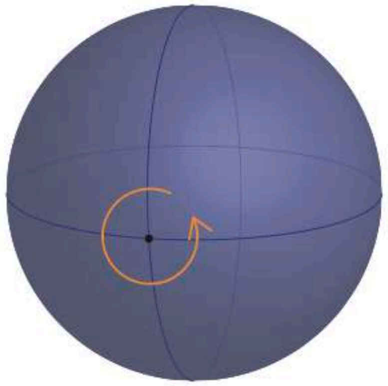

HW, T. Brauner, PRD (2011)
HW, H. Murayama, PRL (2012)
Y. Hidaka, PRL (2013)

- $N_{\mathrm{A}} + 2N_{\mathrm{B}} = N_{\mathrm{BG}}$
- $N_{\mathrm{NGB}} = N_{\mathrm{A}} + N_{\mathrm{B}} = N_{\mathrm{BG}} - \frac{1}{2}\mathrm{rank}\rho, \quad \rho_{ab} = \frac{i}{V}\langle[\hat{Q}_a, \hat{Q}_b]\rangle$

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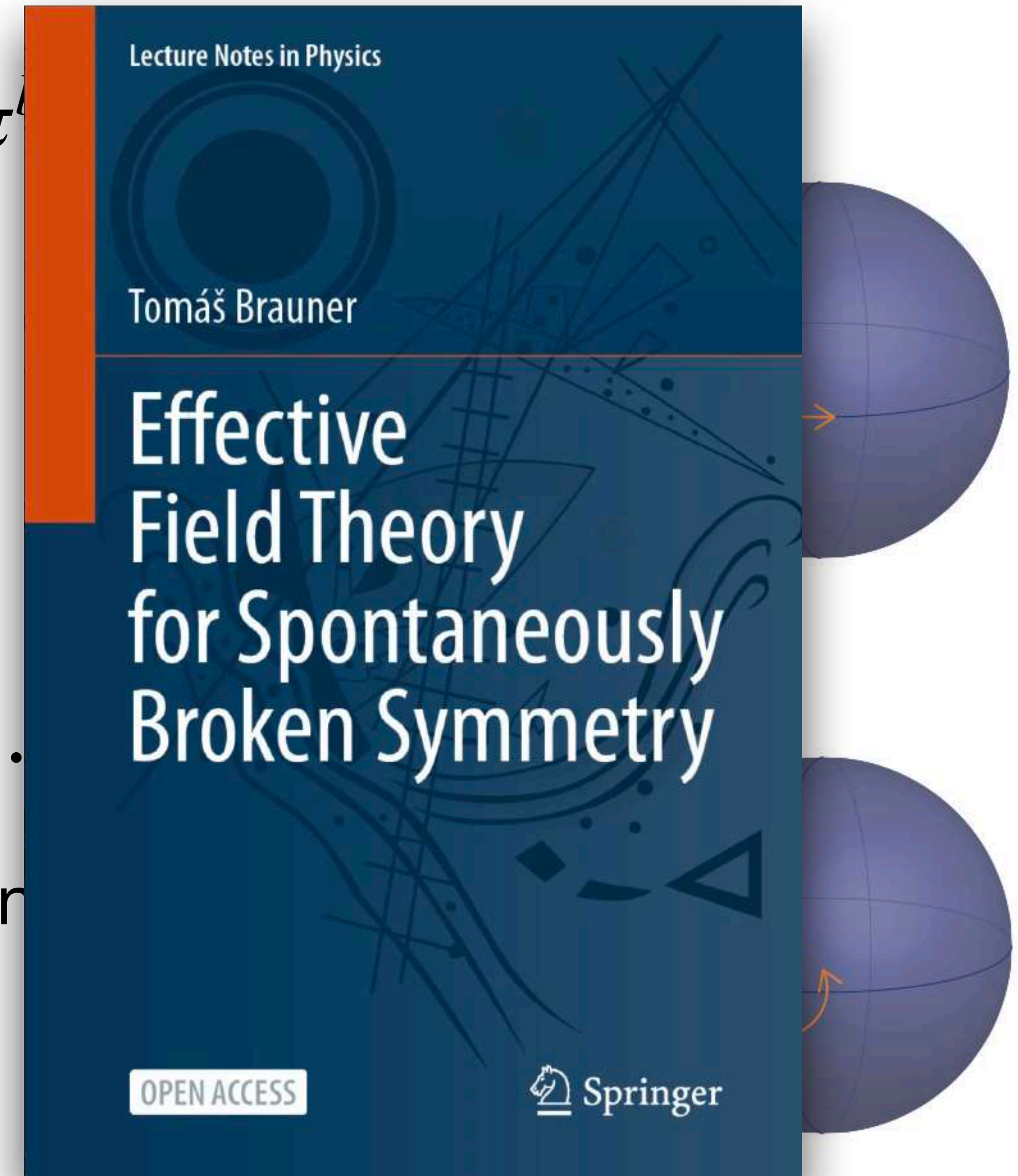
  Dispersion is generically quadratic: $\omega_{\vec{k}} \propto k^2$.



- $N_{\mathrm{A}} + 2N_{\mathrm{B}} = N_{\mathrm{BG}}$

- $N_{\mathrm{NGB}} = N_{\mathrm{A}} + N_{\mathrm{B}} = N_{\mathrm{BG}} - \frac{1}{2}\mathrm{rank}\rho, \quad \rho_{ab} = \frac{i}{V}\langle[\hat{Q}_a, \hat{Q}_b]\rangle$

HW, T. Brauner, PRD (2011)
HW, H. Murayama, PRL (2012)
Y. Hidaka, PRL (2013)

Nambu-Goldstone bosons in nonrelativistic systems

- Suppose the symmetry of the system G is spontaneously broken to H.
- Coset space G/H is the space of degenerate ground states.
- Nambu-Goldstone bosons are low-energy fluctuations within the coset space.
- The number of broken generators $N_{\mathrm{BG}} = \dim(G/H) = \dim G - \dim H$.
- The number Nambu-Goldstone bosons N_{NGB} is sometimes smaller than N_{GB}.
- Many examples
- Systematic understanding achieved!

Examples	G/H	n_{BG}	n_{NGB}
QCD	$\mathrm{SU}(3)\times\mathrm{SU}(3)/\mathrm{SU}(3)$	8	8
Antiferromagnet	$\mathrm{SO}(3)/\mathrm{SO}(2)$	2	2
Ferromagnet	$\mathrm{SO}(3)/\mathrm{SO}(2)$	2	1
Ferrimagnet	$\mathrm{SO}(3)/\mathrm{SO}(2)$	2	1
Kaon ($\mu = 0$)	$\mathrm{U}(2)/\mathrm{U}(1)$	3	3
Kaon ($\mu > 0$)	$\mathrm{U}(2)/\mathrm{U}(1)$	3	2
BEC (planar)	$\mathrm{SO}(3)\times\mathrm{U}(1)/\mathrm{U}(1)$	3	3
BEC (ferro)	$\mathrm{SO}(3)\times\mathrm{U}(1)/\mathrm{U}(1)'$	3	2
Crystal (2+1D)	T^2	2	2
Wigner crystal	T^2	2	1
Skyrmion crystal	T^2	2	1

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Examples	G/H	n_{BG}	$-\ n_{\mathrm{NGB}}$	$= (1/2)\,\mathrm{rank}\rho$
QCD	$SU(3)\times SU(3)/SU(3)$	8	8	0
Antiferromagnet	$SO(3)/SO(2)$	2	2	0
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My works with Hitoshi

- H. Watanabe and H. Murayama,
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  Physical Review Letters 108, 251602 (2012).
- H. Watanabe and H. Murayama,
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Outline

- Nambu-Goldstone bosons in nonrelativistic systems
  HW, H. Murayama, PRL (2012) *Editors' Suggestions*

- Spontaneous breaking of U(1) symmetry in 1+1D
  HW, H. Katsura, J.Y. Lee, PRL (2024) *Editors' Suggestions*

- Low-energy excitations in frustration-free systems
  R. Masaoka, T. Soejima, HW, PRB (2024)
  R. Masaoka, T. Soejima, HW, arXiv:2406.06415

Hohenberg-Mermin-Wagner theorem at $T > 0$

- Hohenberg-Mermin-Wagner (HMW) theorem:

Hohenberg (1967),
Mermin-Wagner (1966)

Continuous symmetries cannot be broken at finite T in $d \leq 2$.

spatial dimension
(not including time)

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  1. Suppose a continuous symmetry is broken.
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Another mechanism:
Spontaneous breaking of
multipole symmetries & generalized symmetries

Definition of
Spontaneous Symmetry Breaking

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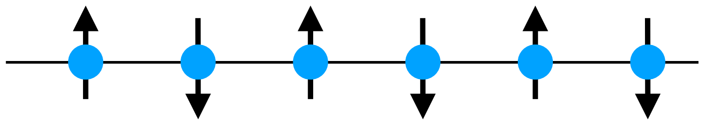

- Consider spin systems defined on d-dim lattice Λ.
- Suppose Hamiltonian $\hat{H} = \sum_{\vec{r}\in\Lambda} \hat{H}_{\vec{r}}$ has an **internal** continuous symmetry generated by $\hat{Q} = \sum_{\vec{r}\in\Lambda} \hat{Q}_{\vec{r}}$ i.e. $[\hat{H}, \hat{Q}] = 0$.
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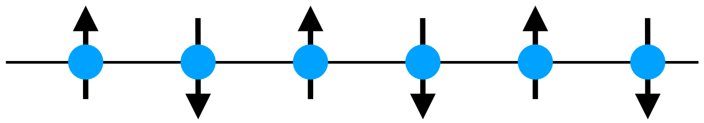

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- Order parameter operator $\hat{\mathcal{O}}$ takes the form $\hat{\mathcal{O}} = [i\hat{Q}, \hat{X}]$ with $\hat{X} = \sum_{\vec{r}\in\Lambda} \hat{X}_{\vec{r}}$.
- Apply a symmetry-breaking field h by $\hat{H}(h) = \hat{H} - h\hat{\mathcal{O}}$.
- Order parameter: $m(h) = \dfrac{\langle \hat{\mathcal{O}} \rangle}{V}$ for the ground state of $\hat{H}(h)$.
- Spontaneous symmetry breaking $\Leftrightarrow$ $\boxed{\lim_{h\to+0} \lim_{V\to\infty} m(h) \neq 0}$

Well-known counterexample to the $T = 0$ version of HMW theorem

- Heisenberg ferromagnet: $\hat{H} = -J \sum_{i=1}^{L} \left(\hat{s}_i^x \hat{s}_{i+1}^x + \hat{s}_i^y \hat{s}_{i+1}^y + \hat{s}_i^z \hat{s}_{i+1}^z \right)$.
- Spin rotation symmetry about z axis generated by $\hat{Q} = \sum_{i=1}^{L} \hat{s}_i^z$.
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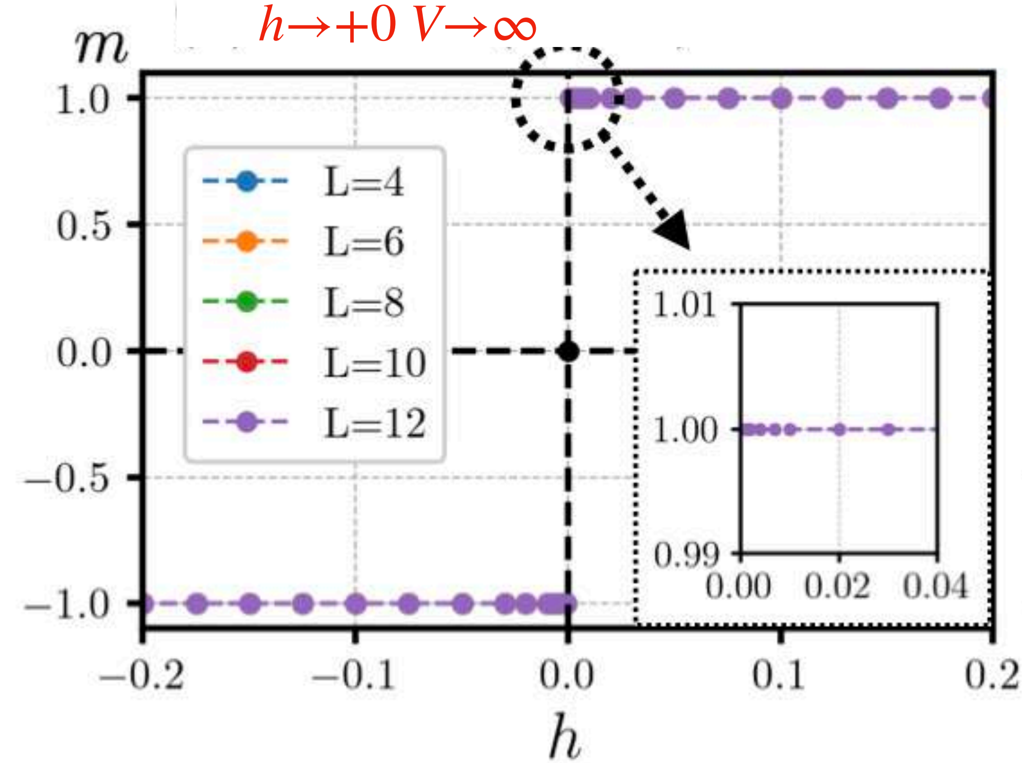
- $\hat{Q}' = \sum_{i=1}^{L}\hat{s}_i^y$ is also broken.

  $G = \mathrm{SO}(3), H = \mathrm{SO}(2) \to G/H = S^2$
- $[\hat{H}, \hat{\mathcal{O}}] = 0 \Rightarrow$ No quantum fluctuations.
- The dispersion for NGB: $\omega_{\vec{k}} \propto k^2$ (type B)

$$\lim_{h\to+0}\lim_{V\to\infty} m(h) \neq 0$$



One of new counterexamples at $T = 0$

HW, H. Katsura, J.Y. Lee, PRL (2024)
Inspired by O. Ogunnaike, J. Feldmeier, J.Y. Lee, PRL (2023)

- $s = 1$ XXZ spin chain with four-spin interaction. $\Delta = 1 \Rightarrow$ Heisenberg ferromagnet.
  $$\hat{H}_i = -J(\hat{s}_i^x \hat{s}_{i+1}^x + \hat{s}_i^y \hat{s}_{i+1}^y + \Delta \hat{s}_i^z \hat{s}_{i+1}^z) + \frac{J}{\Delta}\left[1 - (1-\Delta)(\hat{s}_i^z)^2\right]\left[1 - (1-\Delta)(\hat{s}_{i+1}^z)^2\right]$$
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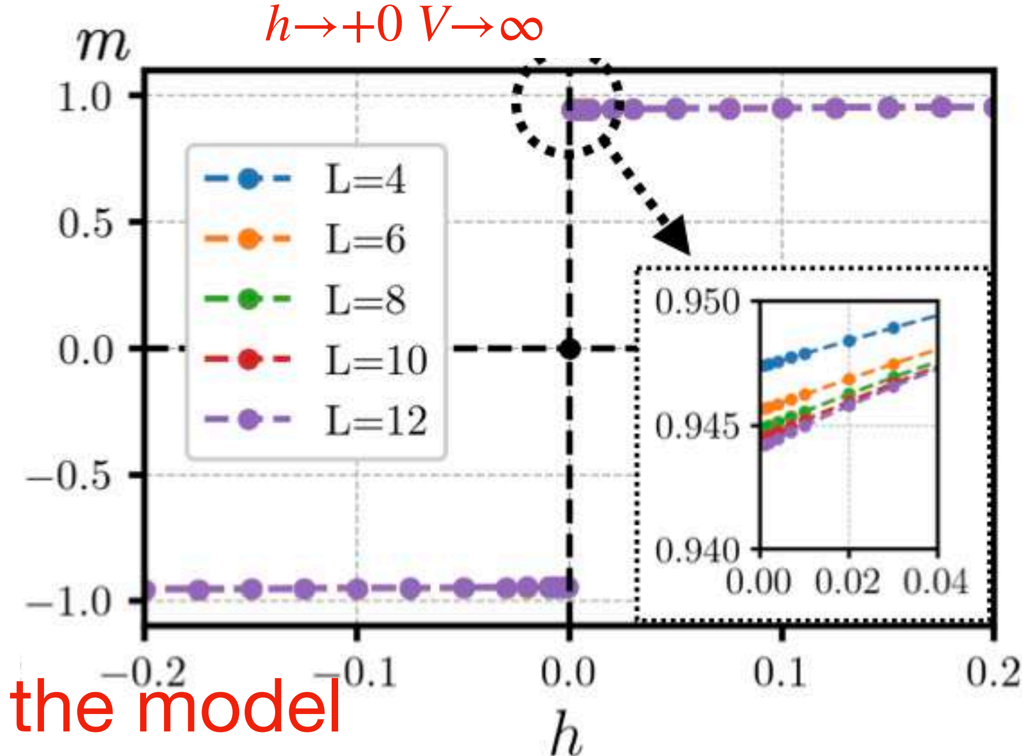
- No other symmetry in this model.
$G = \mathrm{SO}(2), H = e. \to G/H = S^1$
- $[\hat{H}, \hat{\mathcal{O}}] \neq 0$ when $\Delta \neq 1$.
- The dispersion for NGBs: $\omega_{\vec{k}} \propto k^2$
Can be explained by "frustration-free" nature of the model

$$\lim_{h \to +0} \lim_{V \to \infty} m(h) \neq 0$$



Proof of HMW theorem for a finite T via Bogoliubov inequality

Hohenberg (1967), Mermin-Wagner (1966)

- Fourier transformation: $\hat{Q}_{\vec{k}} = \sum_{\vec{r} \in \Lambda} \hat{Q}_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}$ and $\hat{X}_{\vec{k}} = \sum_{\vec{r} \in \Lambda} \hat{X}_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}$.

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$O(1)$

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$O(1)$ $\qquad\qquad \simeq Ck^2$

$$O(1) \geq \int \frac{d^d k}{(2\pi)^d} \frac{2T\,|m|^2}{Ck^2}$$

IR divergence in $d \leq 2 \;\; \Rightarrow \;\; m = 0$ (i.e., no SSB)

Proof of HMW theorem for $T = 0$ via Bogoliubov inequality

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In our example $n = 2$.
SSB is allowed
in $d > 0$.

$\omega_{\vec{k}} \simeq v k^n$

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$d \leq 2 - n$

IR divergence in $\cancel{d \leq 2}$ $\Rightarrow$ $m = 0$ (i.e., no SSB)

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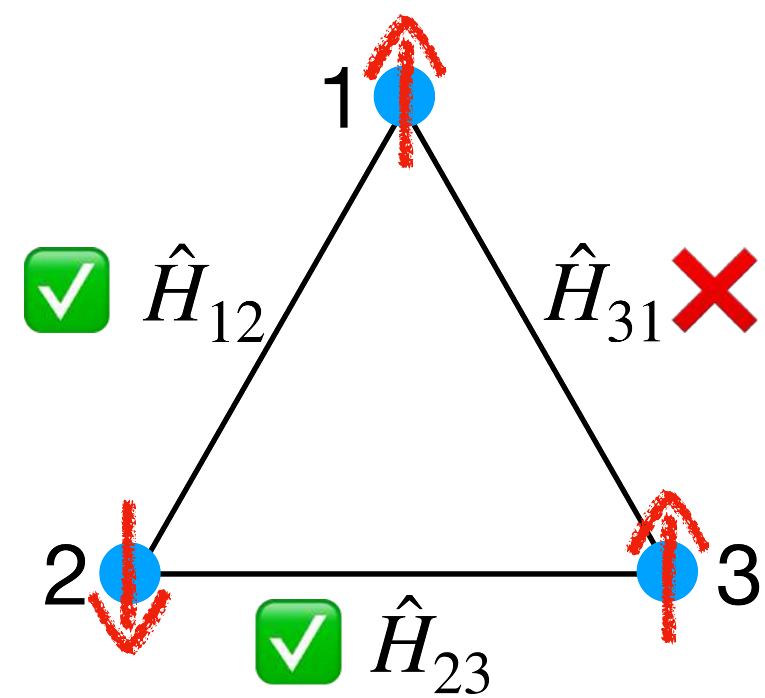
  HW, H. Murayama, PRL (2012) *Editors' Suggestions*

- Spontaneous breaking of U(1) symmetry in 1+1D
  HW, H. Katsura, J.Y. Lee, PRL (2024) *Editors' Suggestions*

- Low-energy excitations in frustration-free systems
  R. Masaoka, T. Soejima, HW, PRB (2024)
  R. Masaoka, T. Soejima, HW, arXiv:2406.06415

(Recap) Frustration in Quantum Many-Body Systems

- Antiferromagetic interaction among three spins 1,2,3. $\hat{H} = \hat{H}_{12} + \hat{H}_{23} + \hat{H}_{31}$



- No way of making all terms simultaneously minimized. → *frustration*.

- More generally, $\hat{H}$ is *frustration free* if
  - There exists a decomposition $\hat{H} = \sum_i \hat{H}_i$ with following properties.
  - $\hat{H}_i$'s are finite ranged. $\hat{H}_i$'s *do not* have to commute with each other.
  - Ground state $|\Phi_{\mathrm{GS}}\rangle$ of $\hat{H}$ minimizes all $\hat{H}_i$ simultaneously. i.e., $\hat{H}_i|\Phi_{\mathrm{GS}}\rangle = E_{\mathrm{GS},i}|\Phi_{\mathrm{GS}}\rangle$ and $E_{\mathrm{GS},i}$ =0 is GS energy of $\hat{H}_i$.

Conjecture 1: Quadratic Dispersion

If $\hat{H}$ is

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- Translation eigenstate: $\hat{T}_{\vec{a}}|\Psi_{\vec{k}}\rangle = e^{-i\vec{k}\cdot\vec{a}}|\Psi_{\vec{k}}\rangle$
- Soft dispersion: $\langle\Psi_{\vec{k}}|\hat{H}|\Psi_{\vec{k}}\rangle - E_0 = O(|\vec{k}-\vec{k}_0|^2)$

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Gapless phases with linear dispersion cannot be realized by FF H.

Conjecture 2:
Quadratic Dispersion

If $\hat{H}$ is

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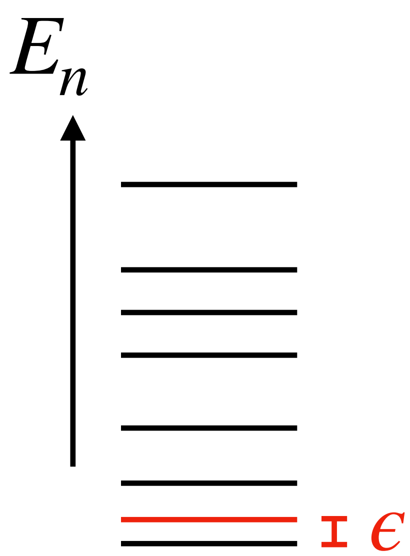

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If $\hat{H}$ is

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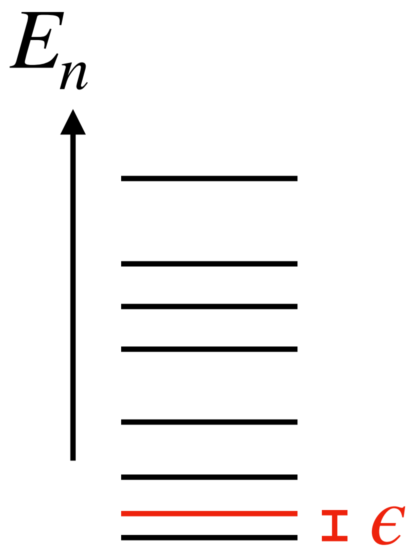
the finite size gap of $\hat{H}$ is $\epsilon = O(L^{-2})$.



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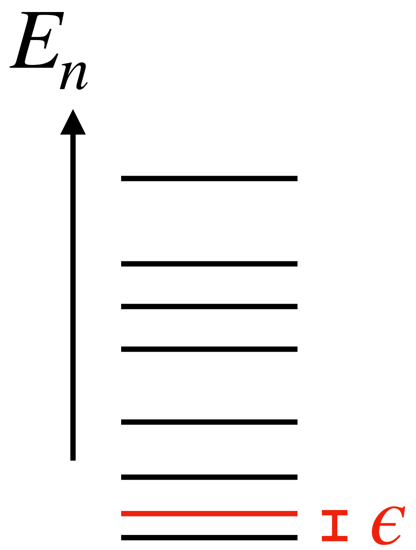


Low-energy effective field theory of critical FF system cannot be Lorentz invariant.

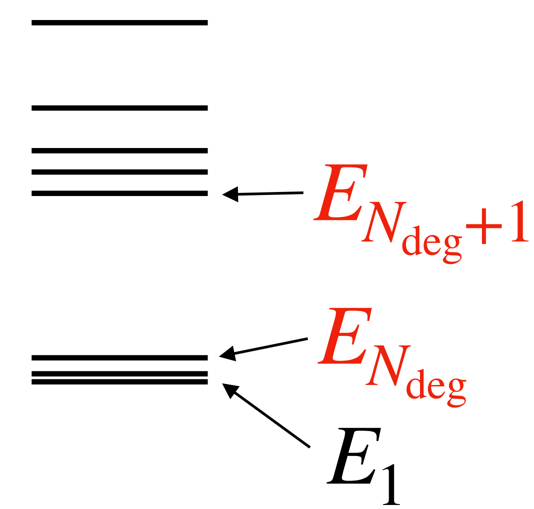
More basic conjecture: Absence of finite size splitting

- Arrange all eigenvalues of $\hat{H}$ as $E_1 \leq E_2 \leq \cdots \leq E_D$ with $E_1 = 0$.
- In general (regardless of frustration),
  $\hat{H}$ is gapped $\Leftrightarrow$ there exists $N_{\rm deg}(L)$ such that

$$\lim_{L\to\infty} E_{N_{\rm deg}} = 0 \text{ and } \lim_{L\to\infty} E_{N_{\rm deg}+1} \neq 0.$$

finite size splitting

energy gap



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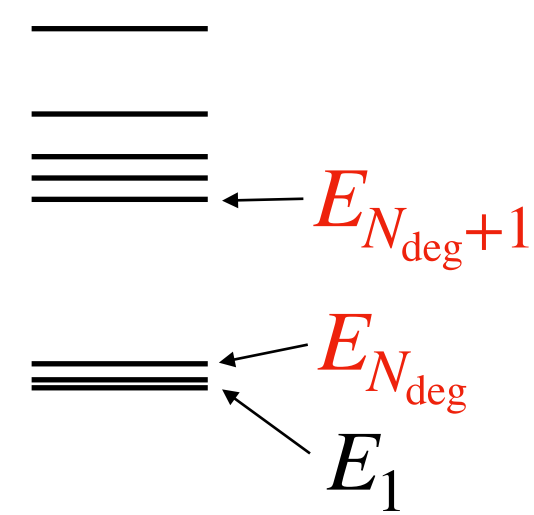
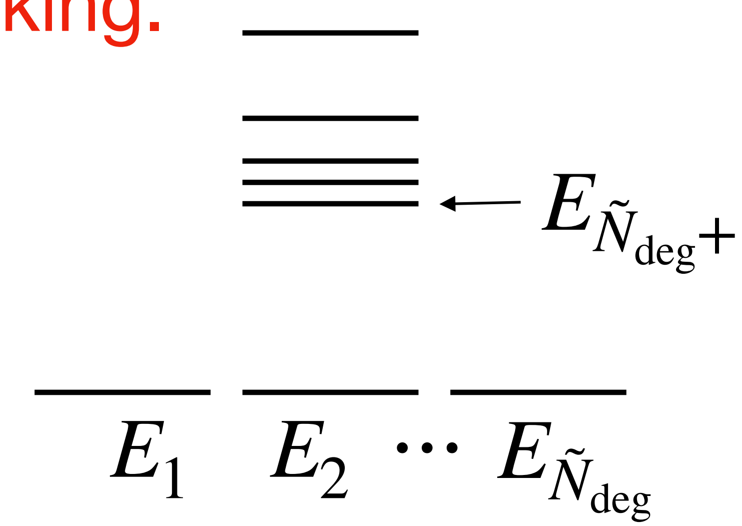
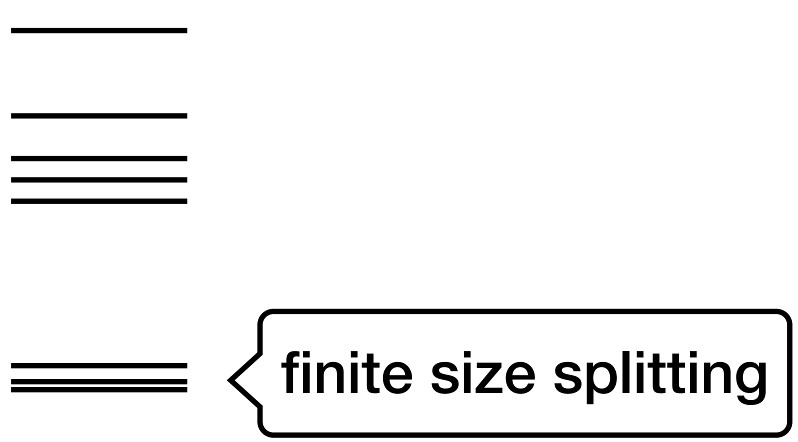


- Conjecture: If $\hat{H}$ is frustration-free, $E_{N_{\rm deg}} = 0$ even in finite L.
  i.e., no Anderson tower for continuous symmetry breaking.
- Let $\tilde{N}_{\rm deg}$ be the largest integer n with $E_n = 0$.
  Then define $\epsilon \equiv E_{\tilde{N}_{\rm deg}+1} \neq 0$.
  $\hat{H}$ is gapped $\Leftrightarrow \lim_{L\to\infty} \epsilon \neq 0$.



Examples

- Frustrated models
  - Transverse-field Ising model
  - perturbed MG model
  - Haldane model
  - perturbed toric code



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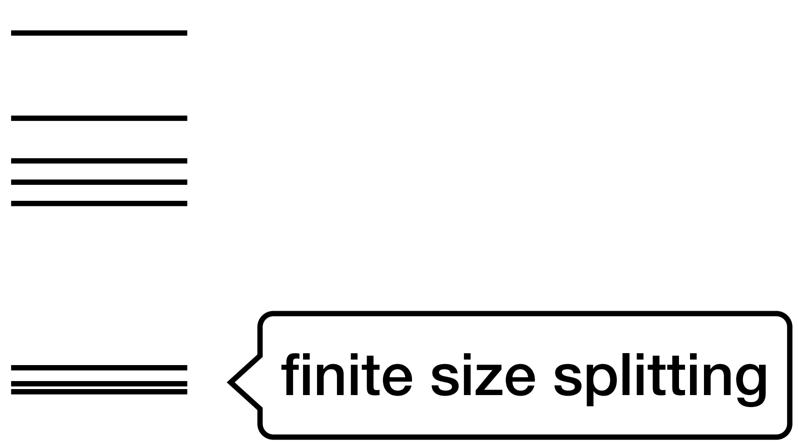
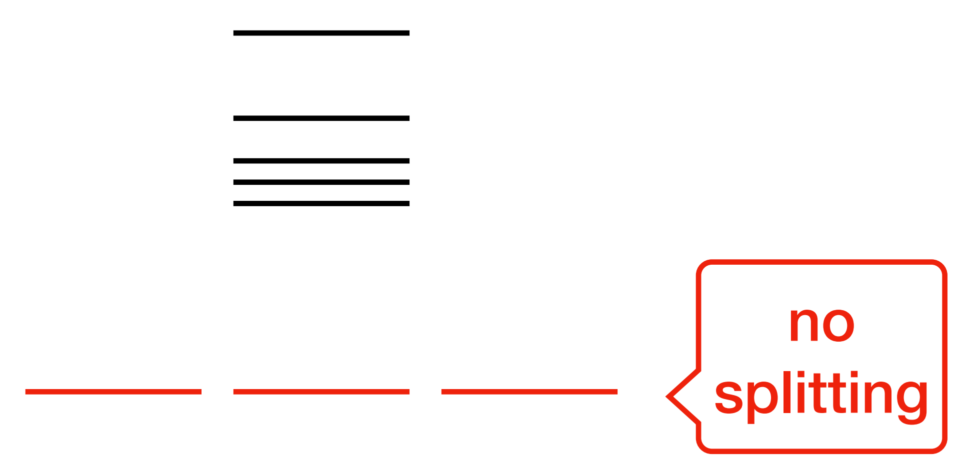
- Frustration Free
  - Ising model
  - MG model
  - ALKT model
  - toric code





New theorem

- Suppose $\hat{H}$ is frustration free.
- Consider an equal time correlation function $\langle \Phi_{\mathrm{GS}} | \hat{\mathcal{O}}^{\dagger}_{\vec{x}}(\hat{1} - \hat{G})\hat{\mathcal{O}}'_{\vec{y}} | \Phi_{\mathrm{GS}} \rangle$ for some operators $\hat{\mathcal{O}}_{\vec{x}}, \hat{\mathcal{O}}_{\vec{y}}$
  $\hat{G}$ is the projector onto GS manifold
- If it shows power-law decay
  $|\langle \Phi_{\mathrm{GS}} | \hat{\mathcal{O}}^{\dagger}_{\vec{x}}(\hat{1} - \hat{G})\hat{\mathcal{O}}'_{\vec{y}} | \Phi_{\mathrm{GS}} \rangle| \sim CL^{-p}$ for $|\vec{x} - \vec{y}| \sim L$

then, the finite size gap of $\hat{H}$ is $\epsilon = O(L^{-2})$.

Proof by Gosset-Huang inequality

- Hastings-Koma (2006): In general, in systems with spectral gap ϵ
  $|\langle\Phi_{\mathrm{GS}}|\hat{\mathcal{O}}^{\dagger}_{\vec{x}}(\hat{1}-\hat{G})\hat{\mathcal{O}}'_{\vec{y}}|\Phi_{\mathrm{GS}}\rangle| \leq Ce^{-g'|\vec{x}-\vec{y}|\epsilon}$
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- Gosset-Huang (2016): If H is frustration-free,
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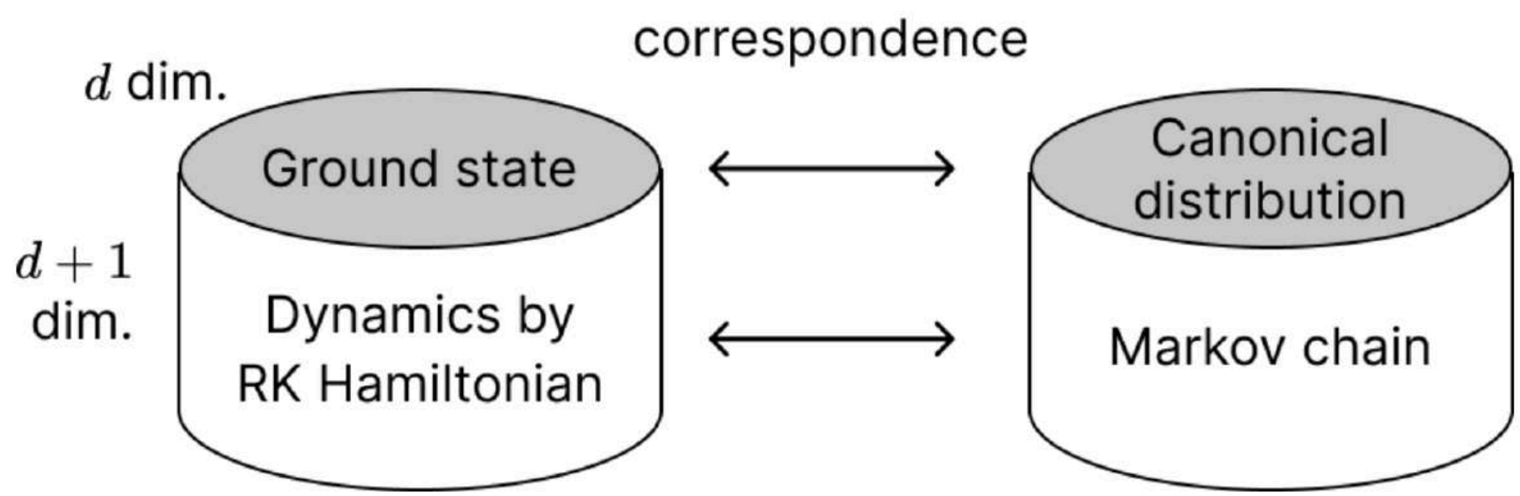
  $\rightarrow$ Correlation length $\xi \sim \dfrac{1}{\sqrt{\epsilon}}$
- Consistent with $|\langle \Phi_{\mathrm{GS}} | \hat{\mathcal{O}}^{\dagger}_{\vec{x}}(\hat{1} - \hat{G})\hat{\mathcal{O}}'_{\vec{y}} | \Phi_{\mathrm{GS}} \rangle| \sim C L^{-p}$ ($|\vec{x}-\vec{y}| \sim L$)

  only when $\epsilon = O(L^{-2})$.

Mapping to frustration-free Hamiltonian

- Markov chain with (i) local update rule & (ii) detailed balance condition

can be mapped to FF Hamiltonian by $H_{C,C'} = -\sqrt{\frac{w(C')}{w(C)}} W_{C,C'}$.



$$w(C) = e^{J\sum_{(i,j)} \sigma_i \sigma_j}$$

$$\hat{H}_{\vec{r}} = \frac{1}{2\cosh(J\sum_{\vec{r}'\in B_{\vec{r}'}} \hat{\sigma}^z_{\vec{r}'})} \left(e^{-J\hat{\sigma}^z_{\vec{r}} \sum_{\vec{r}'\in B_{\vec{r}}} \hat{\sigma}^z_{\vec{r}'}} - \hat{\sigma}^x_{\vec{r}} \right)$$ for Ising model

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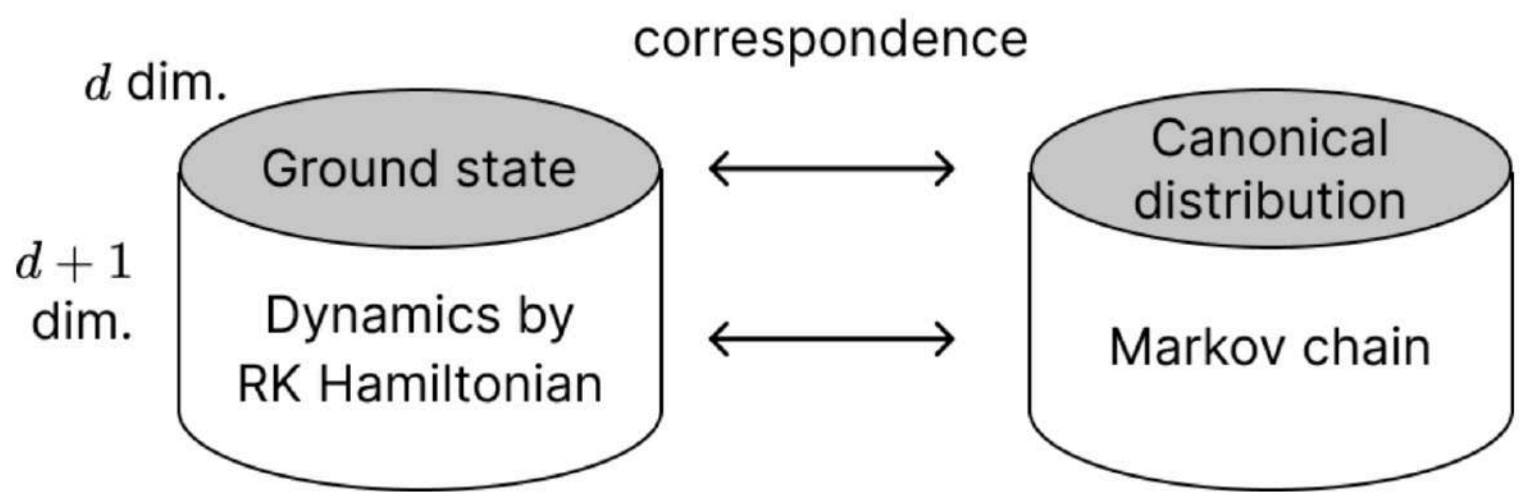

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- Then our result on FF Hamiltonian $\epsilon = O(L^{-2})$ immediately implies $z \geq 2$!!

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Field theoretic understanding?

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- Type A: $\mathscr{L} = \frac{1}{2}\bar{g}_{ab}(0)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}\bar{g}_{ab}(0)\,\nabla\pi^a\cdot\nabla\pi^b + \cdots$

  Usually linear dispersion.

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- Type B: $\mathscr{L} = \rho_{ab}\pi^b\dot{\pi}^a - \frac{1}{2}\bar{g}_{ab}(0)\,\nabla\pi^a \cdot \nabla\pi^b + \cdots$

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- Lifshitz type field $\mathscr{L} = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}(\nabla^2\theta)^2$ / Free boson $\mathscr{L} = i\psi^*\dot{\psi} - \frac{1}{2}\nabla\psi^*\nabla\psi$

Stronger fluctuation?
Finite-size splitting?

Symmetry is larger than U(1)

Examples	Generators $\hat{Q}$	Order Parameter $\hat{\mathcal{O}}$	Symmetry	$[\hat{H},\hat{\mathcal{O}}]$	Frustration-Free	Anderson Tower
$\hat{H}^{(\mathrm{FM})}$	$\sum_{i=1}^L \hat{s}_i^a$ $(a=x,y)$	$\sum_{i=1}^L \hat{s}_i^z$	Broken	$=0$	✓	Absent
$\hat{H}^{(\mathrm{chain})}$ $(s\geq 1)$	$\sum_{i=1}^L \hat{s}_{i,1}^z$	$\sum_{i=1}^L \hat{s}_i^x$	Broken	$\neq 0$	✓	Absent
$\hat{H}^{(\mathrm{ladder})}$	$\sum_{i=1}^L(\hat{s}_{i,1}^z+\hat{s}_{i,2}^z)$	$\sum_{i=1}^L(\hat{s}_{i,1}^x\hat{s}_{i,2}^x-\hat{s}_{i,1}^y\hat{s}_{i,2}^y)$	Broken	$\neq 0$	✓	Absent
$\hat{H}^{(\mathrm{ladder})}+\hat{V}$	$\sum_{i=1}^L(\hat{s}_{i,1}^z+\hat{s}_{i,2}^z)$	$\sum_{i=1}^L(\hat{s}_{i,1}^x\hat{s}_{i,2}^x-\hat{s}_{i,1}^y\hat{s}_{i,2}^y)$	Unbroken	$\neq 0$	—	—

Summary

- I thought I've done everything I can do for SSB / NGBs during my PhD program with Hitoshi.
- Turns out there are still many things to do.
- Surprising connection between seemingly unrelated puzzles.
- Let's work together again for possible field theoretic understanding!
  … I gave a similar talk at Berkeley but we didn't have much time to chat that time.
  Maybe this time…

- Congratulations again!