Challenges in persistence homology and applications

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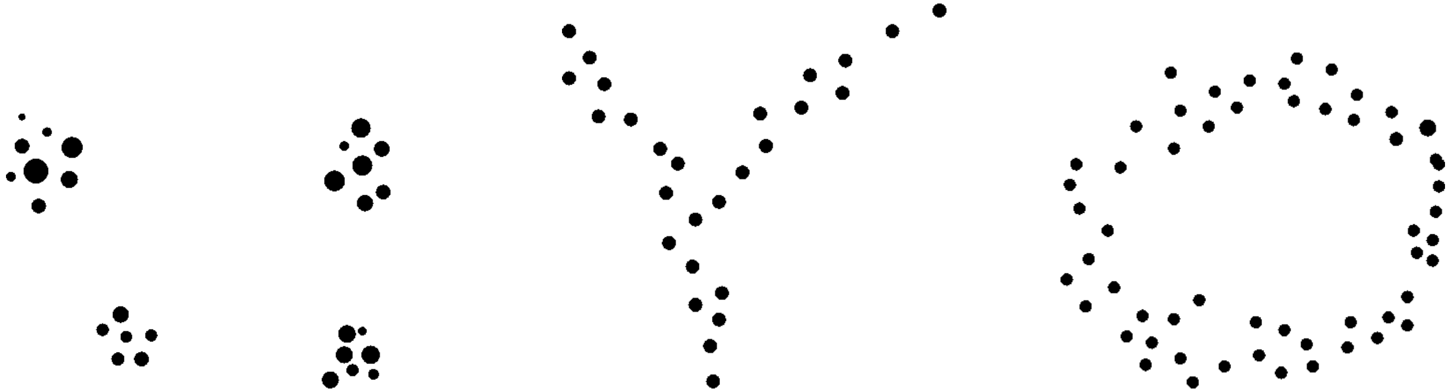
Topological Data Analysis TDA

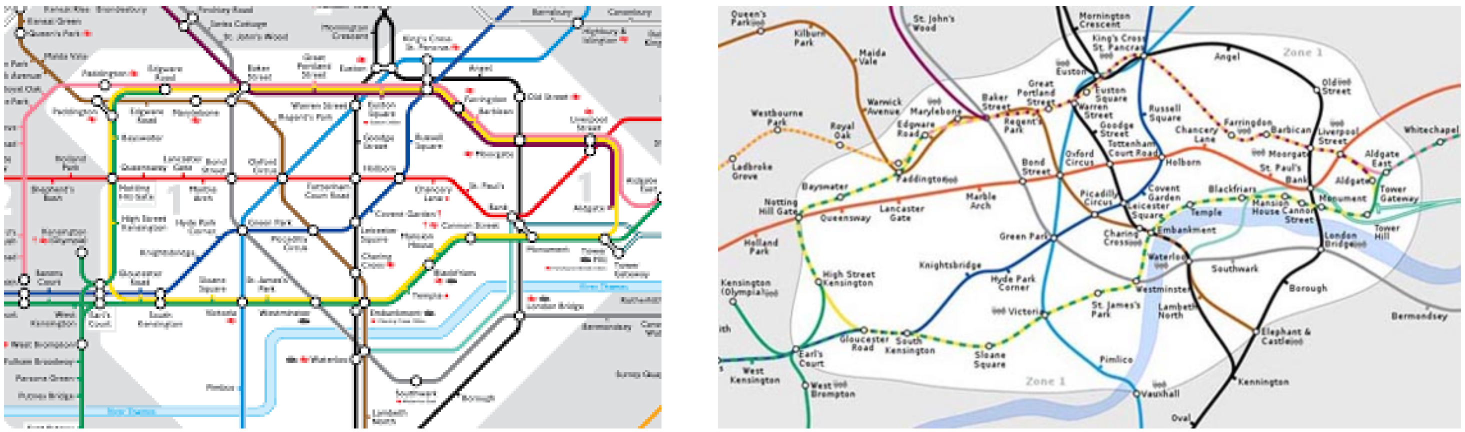
Shape and data



Clusters Y-formation Circle

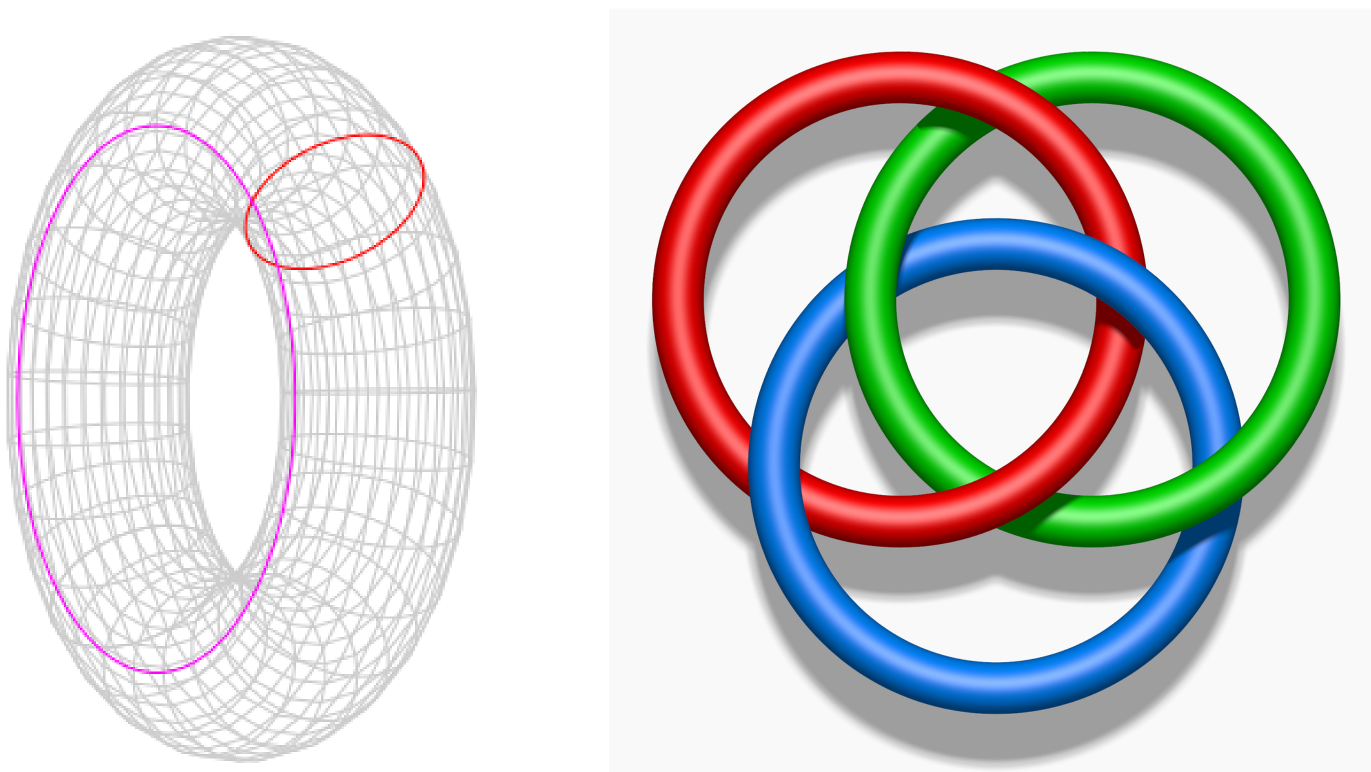
Why topology?

1. Suppressing information.



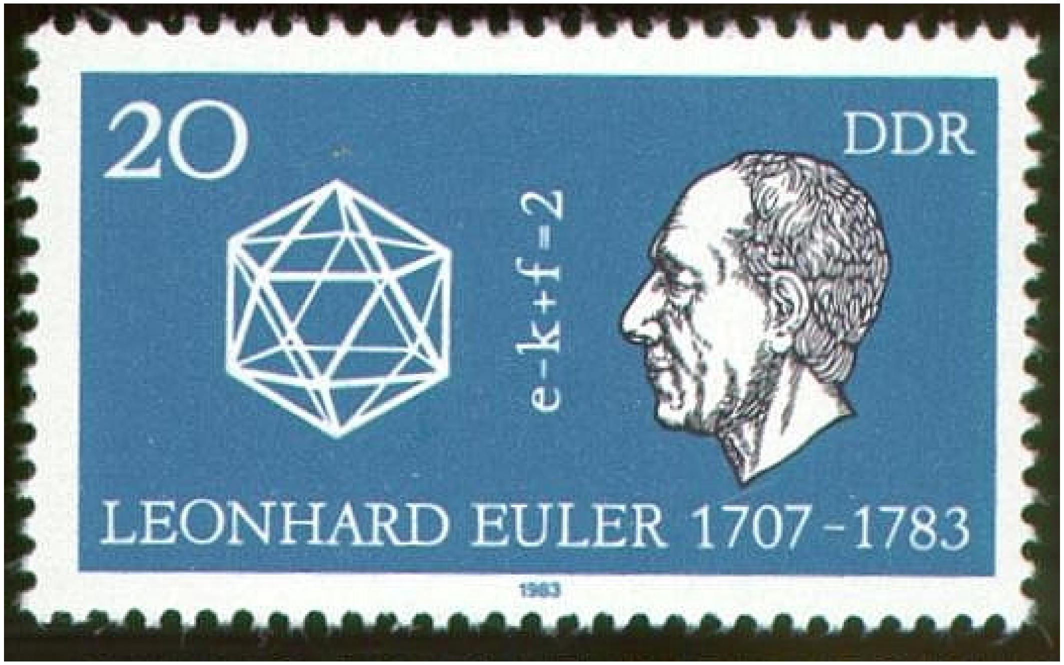
A topological and a geometric map

2. Higher dimensional information.



Torus and Borromean rings

3. Computable signatures.



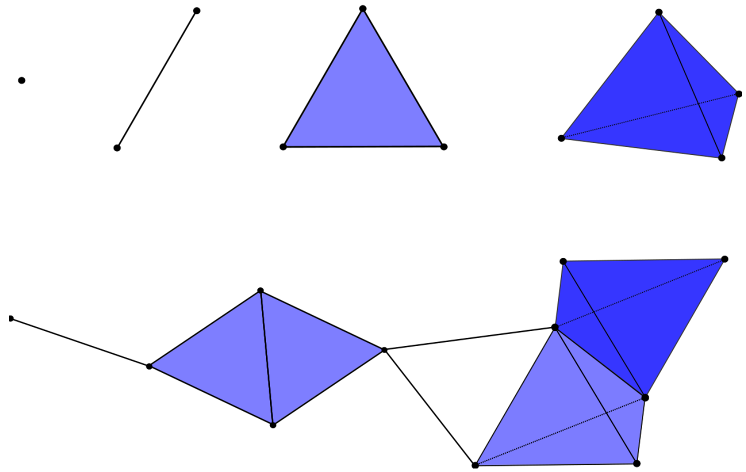
Euler characteristic:

$$\chi = \sharp\text{vertices} - \sharp\text{edges} + \sharp\text{faces}$$

Homology

(Persistent homology)

Combinatorial spaces K



Simplicial complex: K is a combinatorial representation of a topological space built from
$K_0 =$ vertices
$K_1 =$ edges
$K_2 =$ triangles
$K_3 =$ tetrahedra etc.

Homology of K

Chain complex (over a field $\mathbb{F}$)

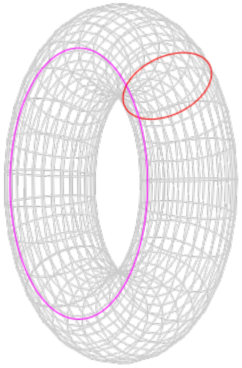
$$\ldots \mathbb{F}[K_{n+1}] \xrightarrow{d_{n+1}} \mathbb{F}[K_n] \xrightarrow{d_n} \mathbb{F}[K_{n-1}] \cdots \longrightarrow \mathbb{F}[K_1] \xrightarrow{d_1} \mathbb{F}[K_0]$$

Key-observation:
The boundary of a boundary is empty: $d_n \circ d_{n+1} = 0$

n-th **Homology group**: $H_n(K) := Ker(d_n)/Im(d_{n+1})$
n-th **Betti number**:

$$\beta_n := \text{dim} H_n(K) = \text{dim} Ker(d_n) - \text{dim} Im(d_{n+1})$$

$$\chi(K) = \beta_0 - \beta_1 + \beta_2 - \ldots$$



$\beta_0 = 1,\ \beta_1 = 2,\ \beta_2 = 1, \quad \chi = 0$

Functoriality: $L \subset K$ a subcomplex

induced map on chain complexes:

$$\mathbb{F}[L_n] \hookrightarrow \mathbb{F}[K_n]$$

induced map on homology:

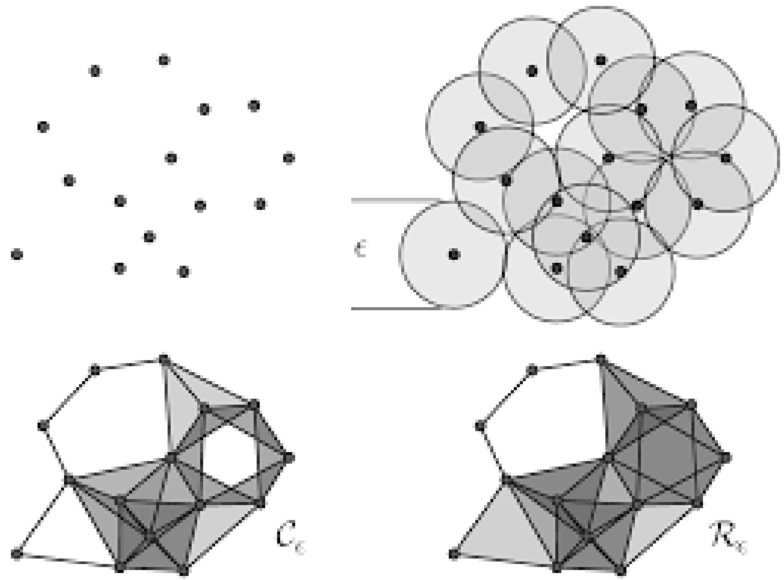
$$H_n(L) \longrightarrow H_n(K)$$

Henri Poincaré 1854–1912

Emmy Noether 1882–1935

From data to topological spaces

Topology of the ϵ-cloud



Čech-complex and Vietoris-Rips complex:

$K_0 =$ vertices
$K_1 =$ edges between point of distance $< \epsilon$
$K_2 = \ldots$

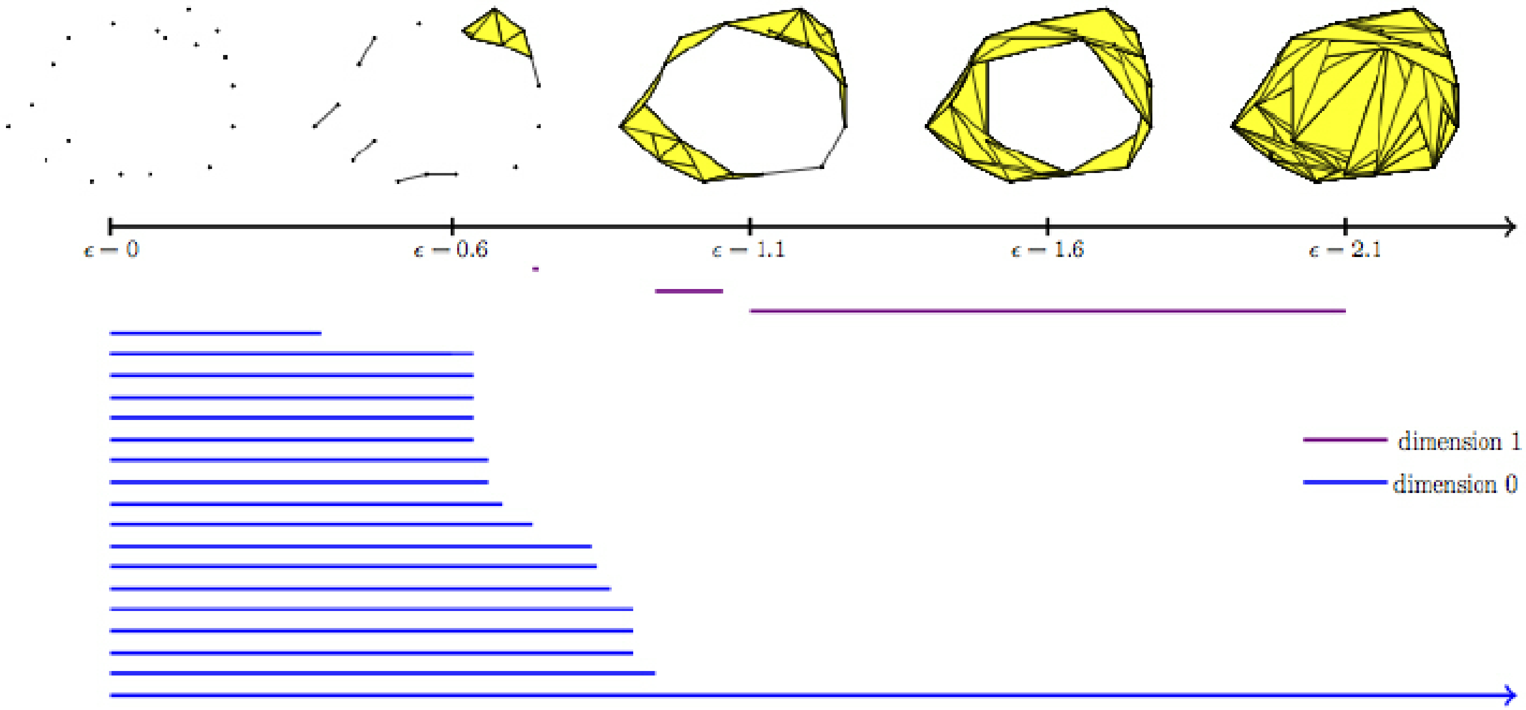
Nerve Theorem

For any point cloud in $\mathbb{R}^d$, its ϵ-cloud has the same homology as its ϵ-Čech-complex $\mathcal{C}_\epsilon$.

Also note:

$$\cdots \subset \mathcal{C}_\epsilon \subset \mathcal{R}_\epsilon \subset \mathcal{C}_{2\epsilon} \subset \ldots$$

Persistent homology



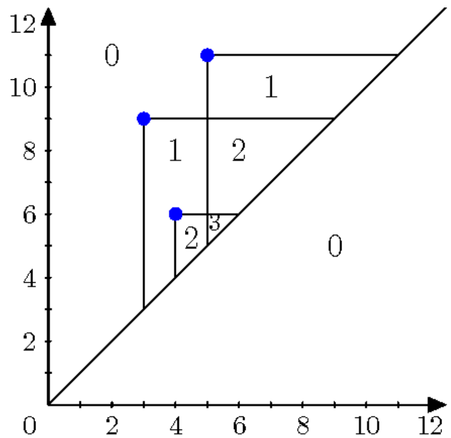
Source: Otter et al.

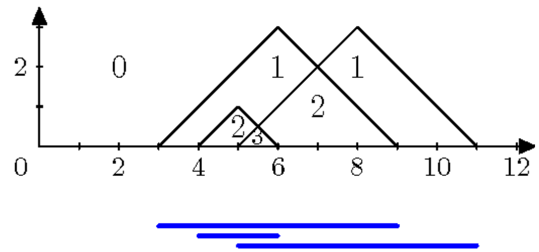
META THEOREMS

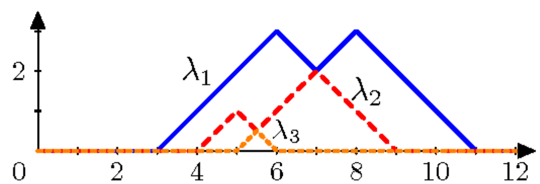
Existence of Barcodes [Carlsson-Zomorodian 2005]:
One can always find barcodes representing persistent homology (in a unique way upto permutation).

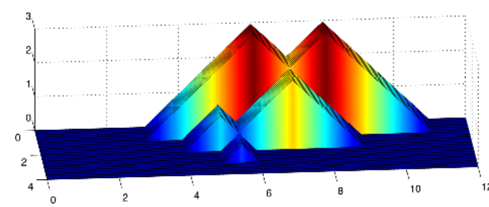
Stability Theorem [Cohen-Steiner-Edelsbrunner-Harer 2007]:
Small changes of input data result in small changes of output statistics.

Persistence diagrams and landscapes









Bubenik, JMLR 2015

- Equivalent information
- Advantages: (1) averages; (2) standard norms
- Properties: Law of Large Numbers, Central Limit Theorem

Computing PH

Survey

Otter, Porter, Tillmann, Grindrod, Harrington,
A Roadmap for the computation of persistent homology,
EPJ Data Science (2017),

Statistics

Fasy, Kim, Lecci, Maria, Rouvreau
TDA: Statistical Tools for Topological Data Analysis,
Introduction to the R package TDA (2019)

New improved software – inspired by Morse theory

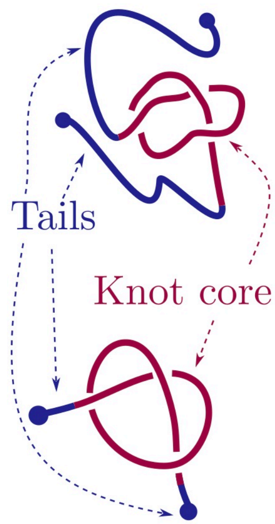
Bauer: Ripser (2019)

Identification of cycles

Henselman: Eirene

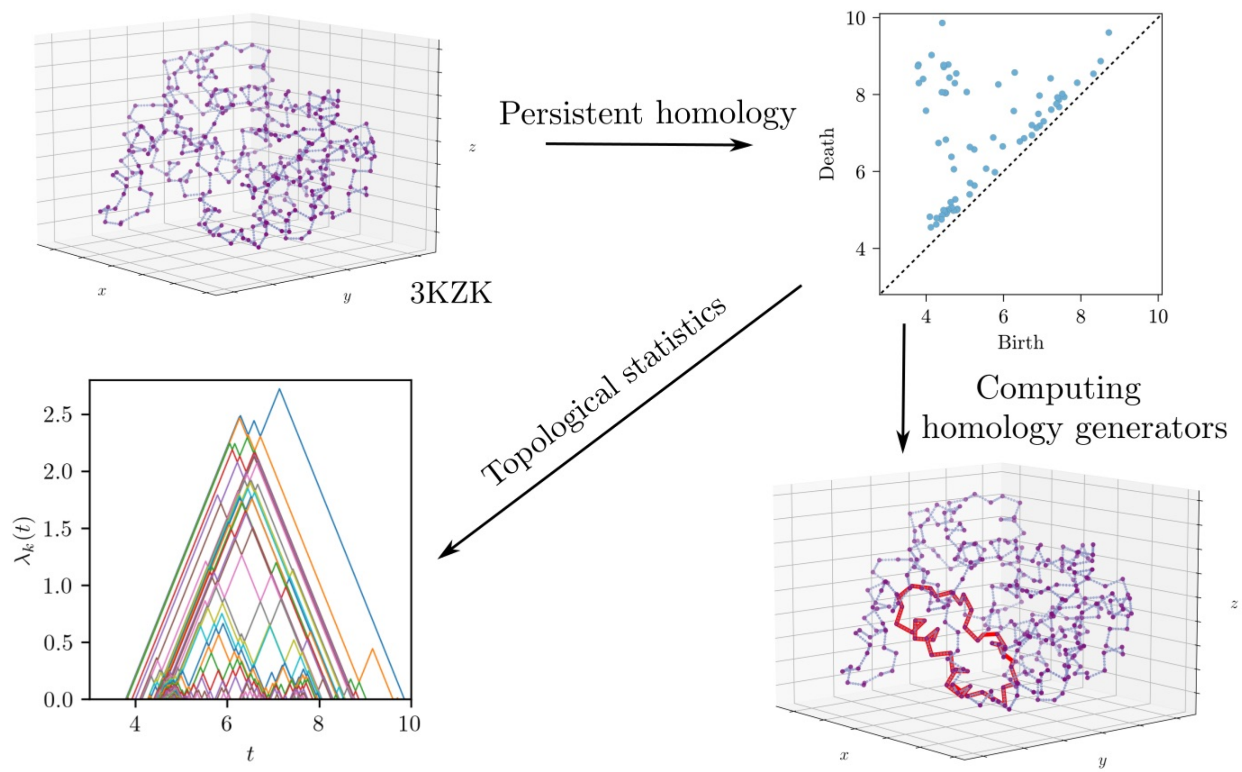
Applications of PH

1. Application: Knotted proteins



Knot depth: $D(Knot) = \frac{L(Tail_1)L(Tail_2)}{L(Knot)}$

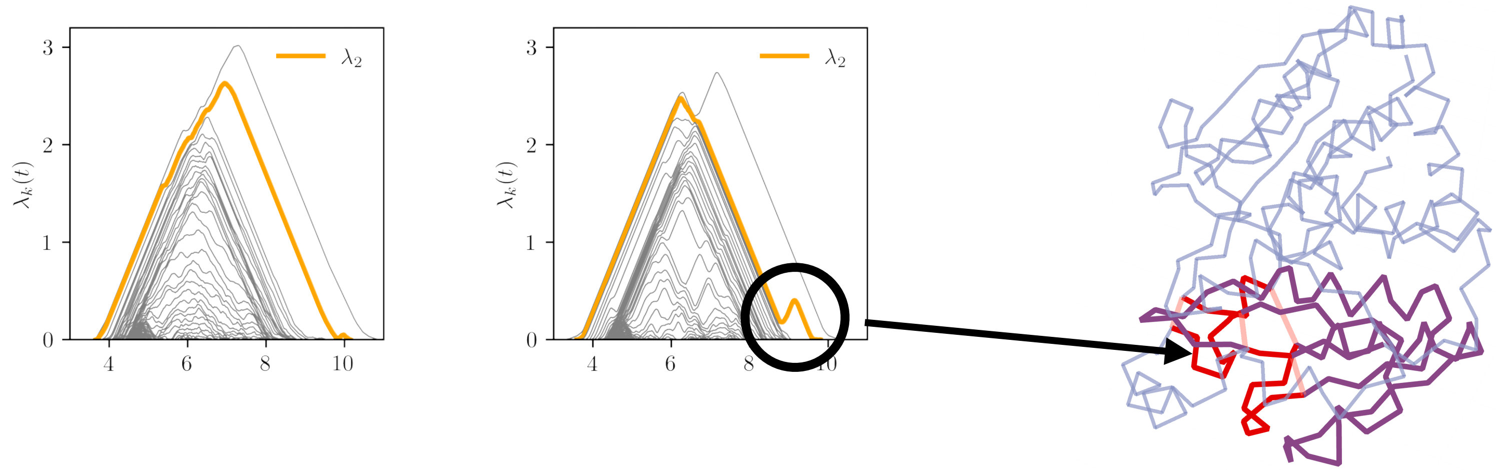
Protein-databank: PDB (aminoacid-chains: 3KZK, 4JQO, ..)
Knotted proteins: KnotProt (over 1000; structural stability (?))
Trefoil knots : most



TDA - Protein - Pipeline

Protein: C_α carbon point cloud — Persistence Diagram

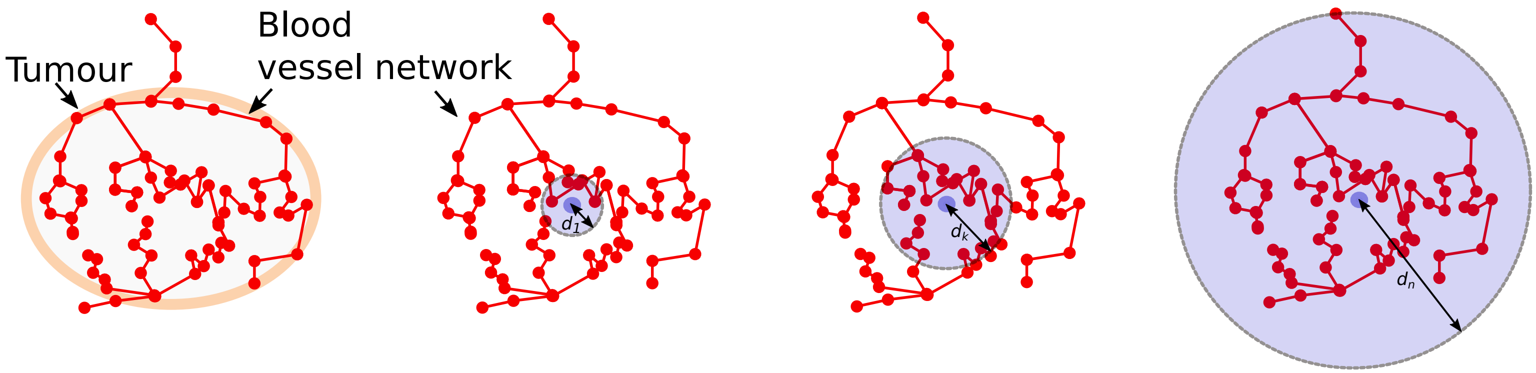
PH Landscape — Identification of cycles

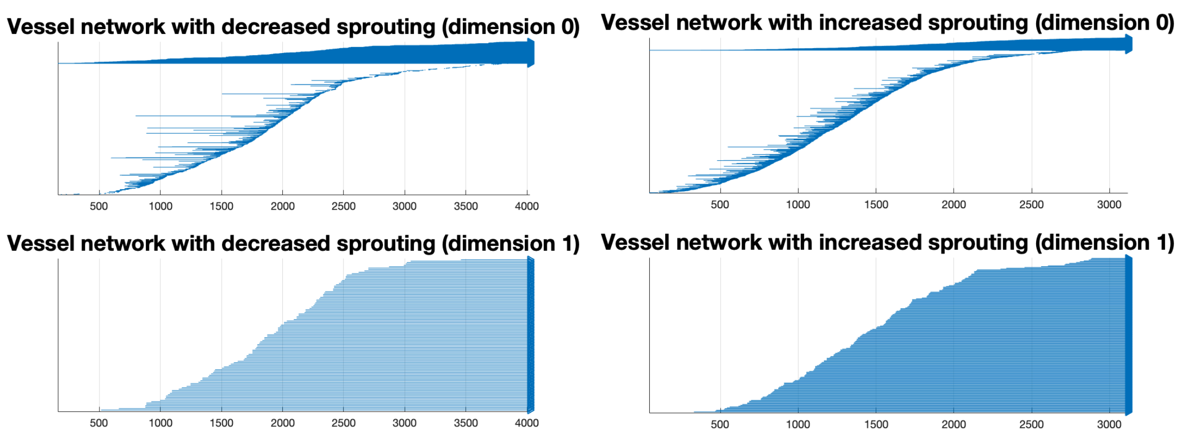


Right: knotted 3KZK overlaid with un-knotted 4JQO
Left: landscape averages: λ_2 in yellow
Right: extra cycle for 3KZK: contains critical crossing

Benjamin-Mukta-Moryoussef-Uren-Harrington-Tillmann-Barbensi,
J R Soc Interface 2023

2. Application: Blood vessels in tumors





Tortuosity (H_0) and cycles (H_1) Stolz et al. *IMA* 2019

An Introduction to Topological Data Analysis for Physicists: From LGM to FRBs

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Department of Mathematics and Applied Mathematics
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Topological data analysis for the string landscape

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Abstract: Persistent homology computes the multiscale topology of a data set by using a sequence of discrete complexes. In this paper, we propose that persistent homology may be a useful tool for studying the structure of the landscape of string vacua. As a scaled-down

Procedure to Reveal the Mechanism of Pattern Formation Process by Topological Data Analysis

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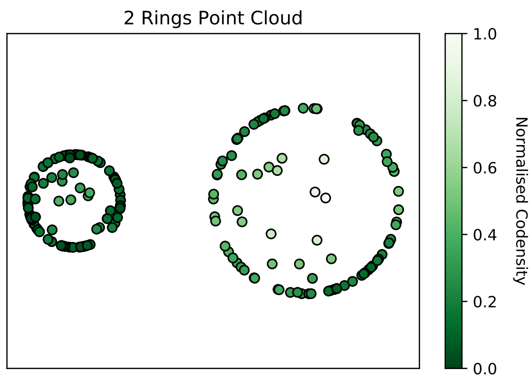
Hiraoka group at Kyoto University

Okinawa Institute of Science and Technology (24.06-08.08.2025)
TSVP Symposium: Representation Theory and TDA

Multiparameter PH and Noise



Multiparameter persistent homology:

$$MPH : (\mathbb{R}^n, \leq) \xrightarrow{K} \triangle - \text{Complex} \xrightarrow{H_p} \text{Vect}_{\mathbb{F}}$$

Multiparameter persistence modules = functors of categories:

$$V : (\mathbb{R}^n, \leq) \longrightarrow \text{Vect}_{\mathbb{F}}$$

- Interleaving distance well-defined (NP-hard for $n > 1$)
- Stability holds
- No analogue of barcodes (Gabriel's theorem)

Carlsson-Zomorodian, Lesnick-Wright, Miller, Harrington-Otter-Schenck-Tillmann, ...

Fix a multiparameter module $V : (\mathbb{R}^n, \leq) \longrightarrow \mathsf{Vect}_{\mathbb{F}}$

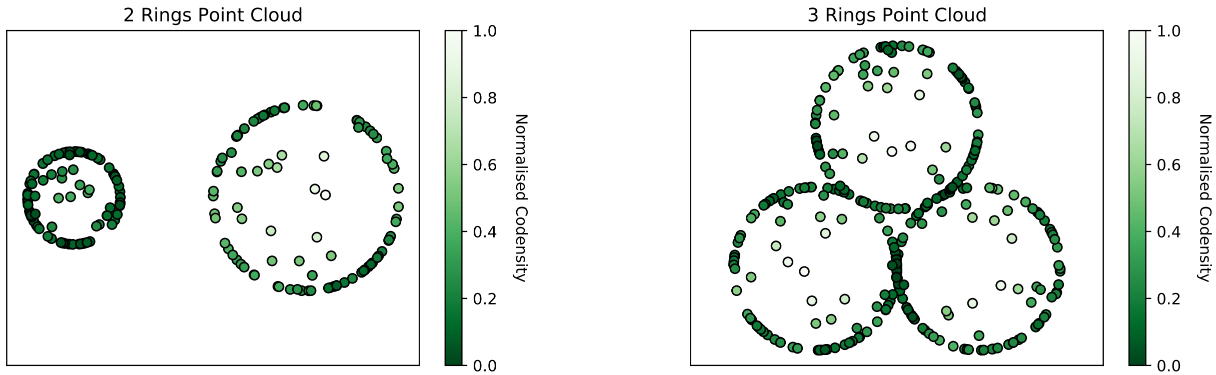
Rank invariant:
$\mathrm{rank}(\bar{s}, \bar{s} + \bar{t}) := \dim \mathrm{Im}(V(\bar{s}) \to V(\bar{s} + \bar{t}))$

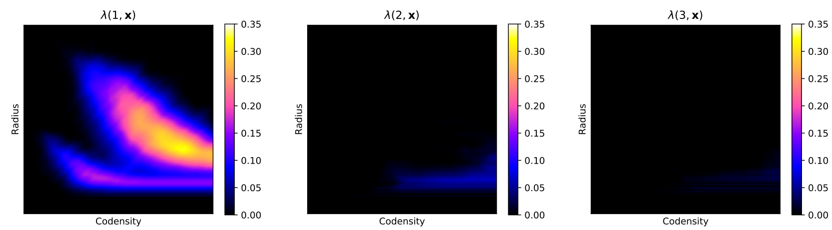
Multiparameter persistence landscape:
$\lambda : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by

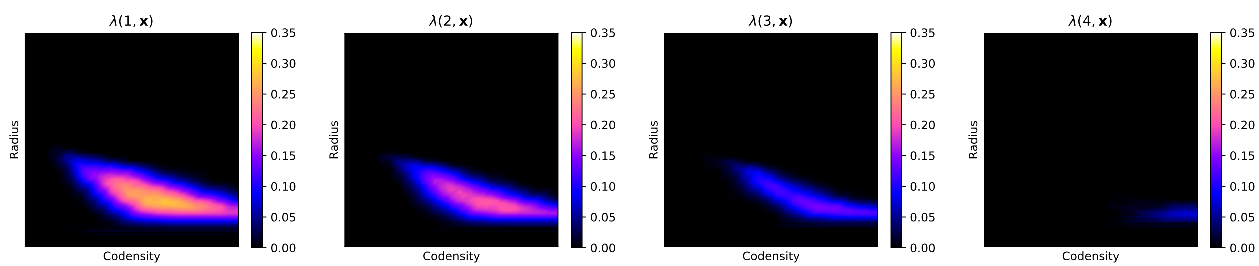
$$\lambda(k, \bar{x}) = \sup\{\epsilon \geq 0 : \mathrm{rank}(\bar{x} - \epsilon\bar{1}, \bar{x} + \epsilon\bar{1}) \geq k\}$$

- Stability
- Strong Law of Large Numbers
- Central Limit Theorem

$\rightsquigarrow$ approximate normal distributions and confidence intervals



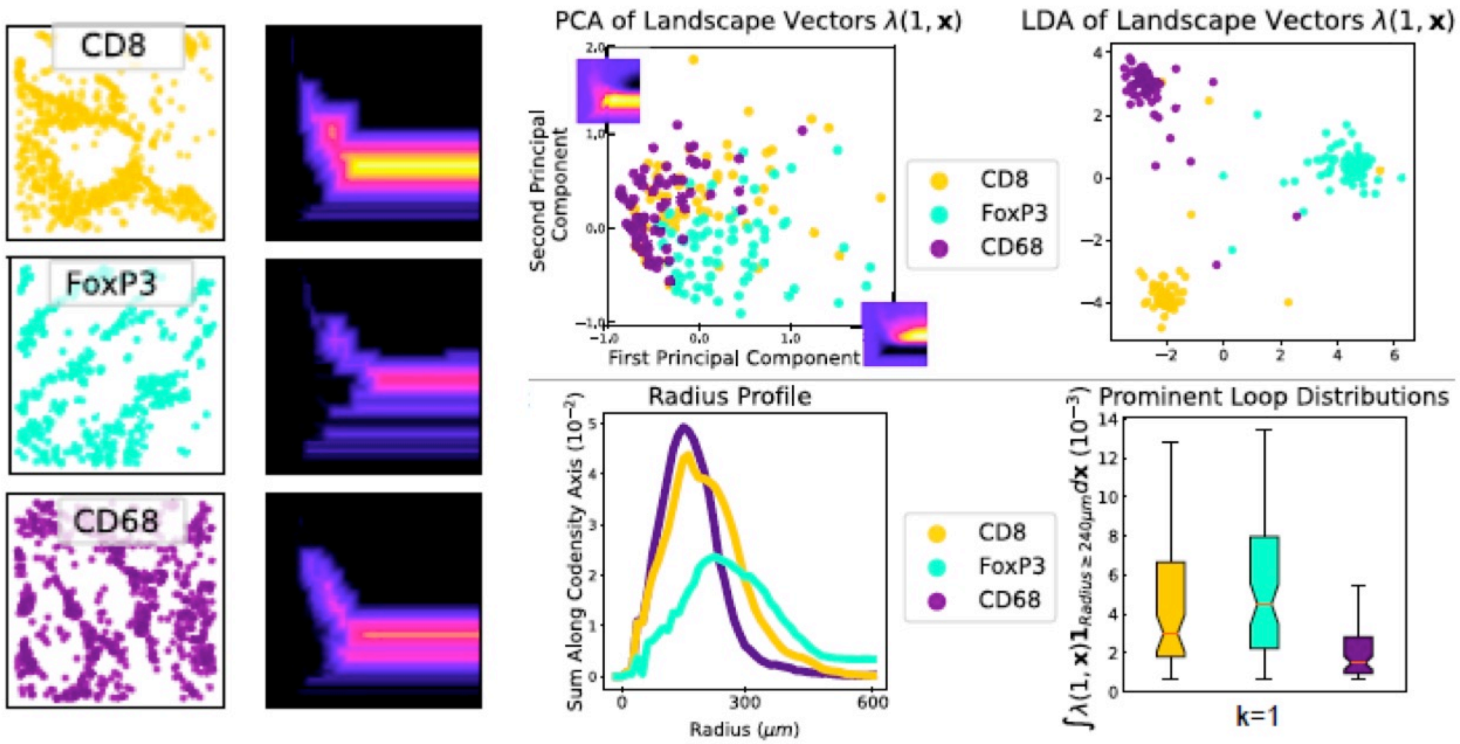




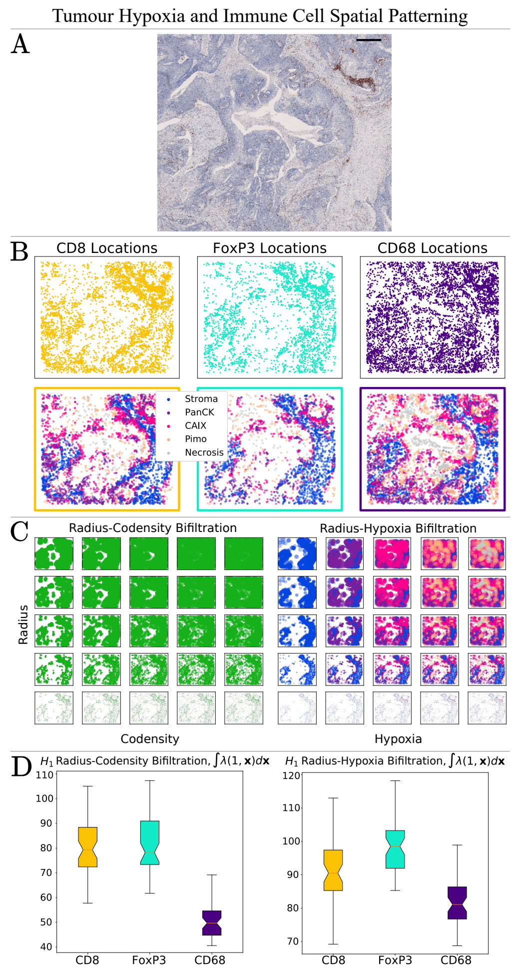
Source: Vipond et al

Applications of MPH

4. Application: immune cells in tumours



Head-Neck Tumor: T-cells (CD8, FoxP3), macrophages (CD68)
PCA: Principle Component Analysis
LDA: Linear Discriminant Analysis



Multi-parameter persistence: density/radius $\sim$ hypoxia/radius

References:

Vipond: Multiparameter Persistence Landscapes
. *J. Mach. Learn. Res 2020*

Vipond, Bull, Macklin, Tillmann, Pugh, Byrne, Harrington: Multiparameter persistence homology landscapes identify spatial patterns of immune cells in tumors *PNAS 2021*

Benjamin, ..., Tillmann, Harrington, Bull: Multiscale topology classifies cells in subcellular spatial transcriptomics *Nature 2024*

Theory

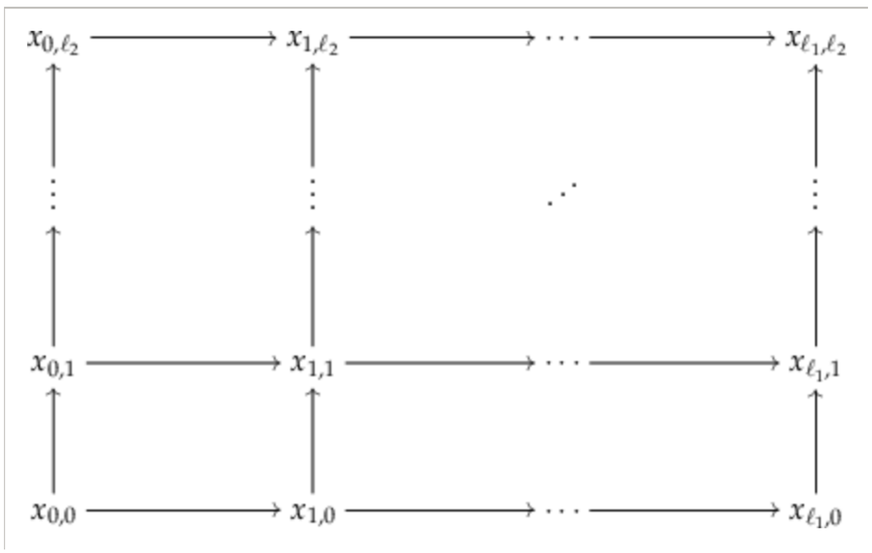
Quiver representations

$Q = (Q_0, Q_1)$ – a quiver with vertices Q_0 and arrows Q_1

Example: A_l ordinary and zig-zag

$$x_0 \overset{e_1}{\text{———}} x_1 \overset{e_2}{\text{———}} \cdots \overset{e_{\ell-1}}{\text{———}} x_{\ell-1} \overset{e_\ell}{\text{———}} x_\ell.$$

Example: multiparameter grid



$\mathcal{C}_Q$ associated category

Definition:
A quiver representation is a functor $V : \mathcal{C}_Q \to \mathsf{Vect}_{\mathbb{F}}$

V_x f.d. vector space for $x \in Q_0$
V_e linear map for $e \in Q_1$

Example: Identity representation on subquiver $Q' \subset Q$

$$I[Q']_x = \begin{cases} \mathbb{F} & \text{if } x \in Q'_0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad I[Q']_e = \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } e \in Q'_1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Example: Subrepresentation generated at x

$< V_x >$ smallest subrepresentation of V containing V_x

If Q is finite,
there is a unique decomposition into indecomposables:

$$V \simeq \bigoplus_{I \in \mathrm{Ind}_Q} d_V(I).I$$

Gabriel's Theorem:
Ind_Q is finite if and only if the underlying graph of Q is a finite union of simply laced Dynkin diagrams (type A, D, E).

Ordinary and zigzag PH are well behaved.
MPH is not!

$==== >$ search for good invariants for MPH

Harder-Narasimhan filtration

$\mathcal{C}$ abelian category

stability condition:
$Z : K(\mathcal{C}) \to \{z \in \mathbb{C} | \, Re(z) > 0\}$ group homomorphisms

slope:
$\mu_Z(V) := \frac{ImZ(V)}{ReZ(V)}$ for $V \neq 0$

V is Z-**semi-stable** if $\mu_Z(V') \leq \mu_Z(V)$ for all $V' \subset V$
V is Z-**stable** if $\mu_Z(V') < \mu_Z(V)$ for all $V' \subset V$

Theorem:
Harder-Narasimhan filtration of V along Z

If $\mathcal{C}$ is both Noetherian and Artinian, then every non-zero $V \in \mathcal{C}$ admits a unique filtration $V^\bullet$:

$$0 = V^0 \subset V^1 \subset \cdots \subset V^n = V$$

with semi-stable quotients $S^k = V^k / V^{k-1}$ such that

$$\mu_Z(S^1) > \mu_Z(S^2) > \cdots > \mu_Z(S^n)$$

$\mathcal{C} = \mathrm{Rep}(Q)$ satisfies hypothesis for finite Q

$\underline{\dim} : K(\text{Rep}(Q)) \to \mathbb{Z}^{Q_0}$

(This is an isomorphisms for acyclic quivers)

If Z factors through $\underline{\dim}$ then Z is determined by $\alpha, \beta : Q_0 \to \mathbb{R}$:

$$Z(V) = \Sigma_{x \in Q_0}(\beta(x) + i\alpha(x)). \dim V_x$$

may assume $\beta(x) = 1$ for all $x \in Q_0$;
so Z determined by **central charge** α;
slope simplifies to

$$\mu_\alpha(V) = \frac{\Sigma_{x \in Q_0}\alpha(x) \dim V_x}{\Sigma_{x \in Q_0} \dim V_x}$$

Definition: HN-type
$HN[V;\alpha] := (\underline{\dim} S^1, \underline{\dim} S^2, \ldots, \underline{\dim} S^n)$

α is **complete** if

$$HN[V;\alpha] = HN[W;\alpha] \iff V \simeq W$$

Proposition:
If α is complete then every indecomposable representation in Rep(Q) is α-stable

Example: Euler
$\epsilon : Q_0 \to \mathbb{R}$ with $\epsilon(x) = 1 - \sharp$(in-arrows)

Example: skyscraper
$\delta_y : Q_0 \to \mathbb{R}$ with $\delta_y(x) = \delta_{xy}$

Quivers of type $\mathbb{A}$

Ordinary persistence: $\mathbb{A}_n : x_0 \to \cdots \to x_{n-1}$

Euler $\epsilon = \delta_{x_0}$

Theorem:
Let V be a persistence module and let $j_1 < \ldots < j_\ell$ be the collection of all indices j in $\{0, \ldots, n-1\}$ satisfying $I[0, j] \in \mathrm{Ind}_Q$, ie $[0, j] \in \mathrm{Bar}(V)$.
Then $V^\bullet$ has length $\ell + 1$, and for each integer $0 < k \leq \ell$ the quotient $S^k := V^k / V^{k-1}$ satisfies

$$\dim(S^k)_{x_i} = \begin{cases} d_{0,j_k} & i \in [0, j_k], \\ 0 & \text{otherwise.} \end{cases}$$

δ_{x_0} is not complete!

Kinser (2022) determined all complete central charges

Corollary:
For persistence modules, a central charge

$$\alpha : Q_0 \to \mathbb{R}$$

is complete if and only if the inequality $\alpha(x_i) > \alpha(x_{i+1})$ holds for all $i \in \{0, 1, \ldots, \ell - 1\}$.

Complete charges have been computed for $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$ quivers.

Skyscraper invariant

Proposition:
Let $0 = V^0 \subsetneq \cdots \subsetneq V^n = V$ be the HN filtration of V **along** δ_x**.**
If j is the smallest index for which $V_x^j = V_x$, then:
(1) either $j = n$ or $j = n - 1$, and
(2) for every $1 \leq k \leq j$, we have $V^k = < V_x^k >$.

Proof: (1)

$$\mu_{\delta_x}(W) = \frac{\dim W_x}{\Sigma_{y \in Q_0} \dim W_y}$$

$V_x^j = V_x$. So $\mu_{\delta_x}(S^{j+1}) = \cdots = \mu_{\delta_x}(S^n) = 0$.
But these slopes are strictly decreasing.

(2) By induction.
Use that V^1 is semi-stable and $< V_x^1 >$ is a submodule.
Comparing slopes forces $V^1 = < V_x^1 >$.

Definition:

Skyscraper invariant $\delta_\bullet$ on Rep(Q) assigns to each representation V the collection of HN types along skyscraper central charges at all of the vertices:

$$\delta_V = \{HN[V; \delta_x] \mid x \in Q_0\}$$

Proposition
Skyscraper is complete on $\text{Rep}_{\mathbb{A}_n}$.

Rank invariant

Definition:
Rank invariant is the map $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ given by

$$\rho_V(x, y) = \dim < V_x >_y$$

This generalises Carlsson-Zomorodian's (2007) invariant for MPH to any quiver.

Theorem: The skyscraper invariant is strictly stronger than the rank invariant.

Proof: (1) $\rho(x, y) = \Sigma_{k=1}^{j} \dim S_y^k$ for j as in the Proposition. So the rank invariant can be recovered.
(2) By example (on a commutative square)

Generalised rank invariant

Definition:

Generalised rank Invariant is the map GRI that for each sub-quiver $Q' \subset Q$ assigns the rank of the canonical limit-to-colimit map,

$$\mathrm{rk}(\varinjlim V|_{Q'} \longrightarrow \varprojlim V|_{Q'})$$

This generalises Kim and Mémoli's (2021) invariant for MPH to any quiver.

Theorem:

The skyscraper and the GRI are not comparable.

Restricting the category

$\mathsf{Rep}_{id}(Q)$: the category generated by identity representations $I[Q']$ on any subquiver $Q' \subset Q$.

Theorem:
GRI is complete on $\mathsf{Rep}_{id}(Q)$

This generalises Kim and Mémoli's (2021) theorem for MPH to any quiver.

Problem:
Determine a maximal subcategory on which the skyscraper invariant is complete.

$Q \subset \mathbb{Z}^d$ grid quiver of shape $L = (\ell_1, \ell_2, \ldots, \ell_d)$

$R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ rectangle

$\text{Rep}_{rec}(Q)$: subcategory of direct sums of $I[R]$ for R and rectangle in Q

Theorem:
A central charge $\alpha \notin \mathcal{H}$ is complete on $\text{Rep}_{rec}(Q)$ if and only if it satisfies the inequality $\alpha \circ s(e) > \alpha \circ t(e)$ for each edge $e \in Q_1$.

Proof:
Uses lattice theory and max-flow/min-cut theorem.

References

Fersztand, Nanda, Tillmann : Harder-Narasimhan filtrations and zigzag persistence, *Advances in Applied Mathematics 2024*

Fersztand, Jacquard, Nanda, Tillmann : Harder–Narasimhan filtrations of persistence modules, *Transactions LMS 2024*

Fersztand : Harder-Narasimhan filtrations of persistence modules: metric stability, *arXiv 2025*

Fersztand, Jendrysiak : Computing the skyscraper invariant, *forth coming*

"Data has shape, and shape matters"

Gunnar Carlsson

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Engineering and Physical Sciences
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