Challenges in “physical and mathematical” cosmology (and data analysis)

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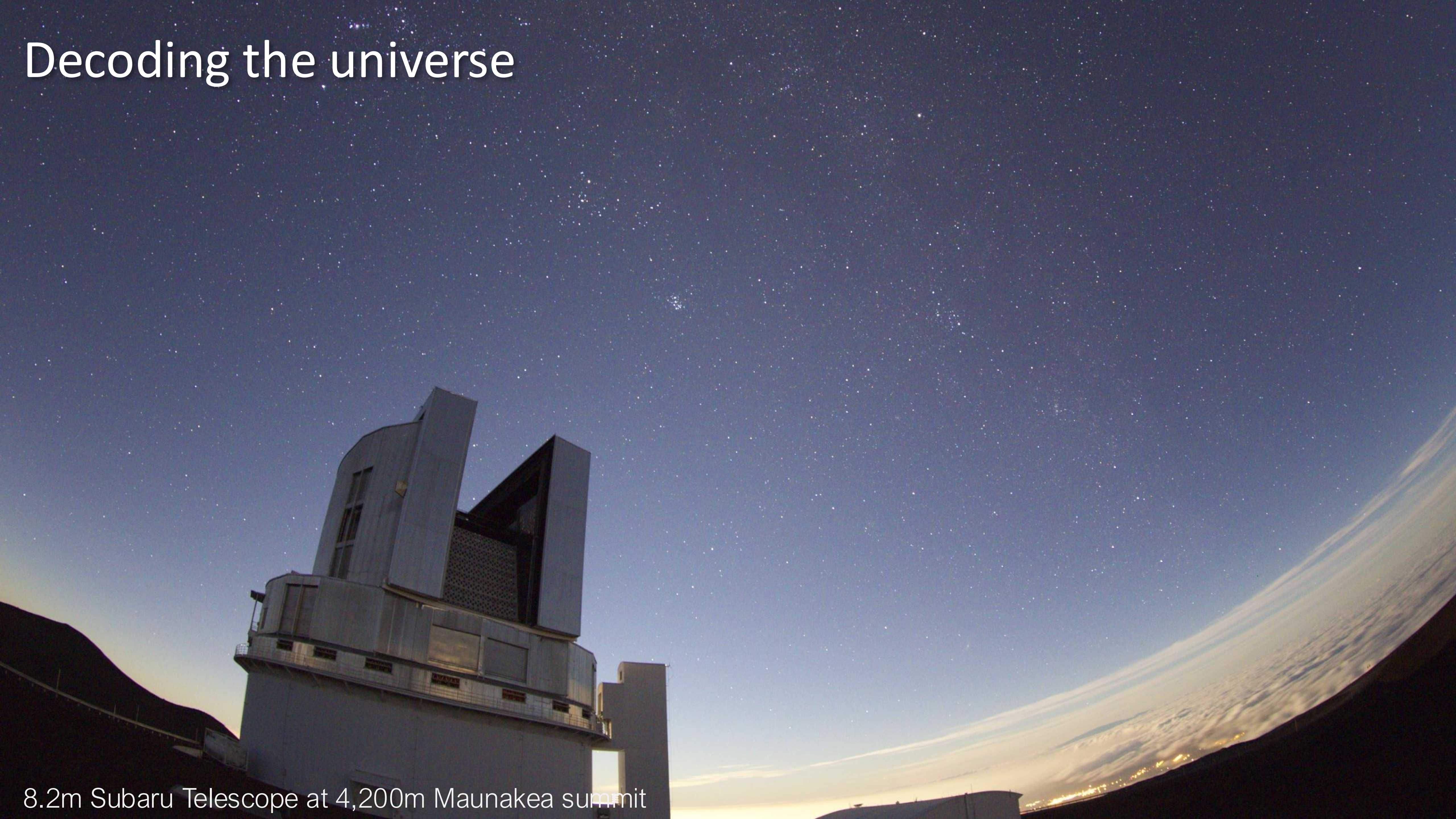
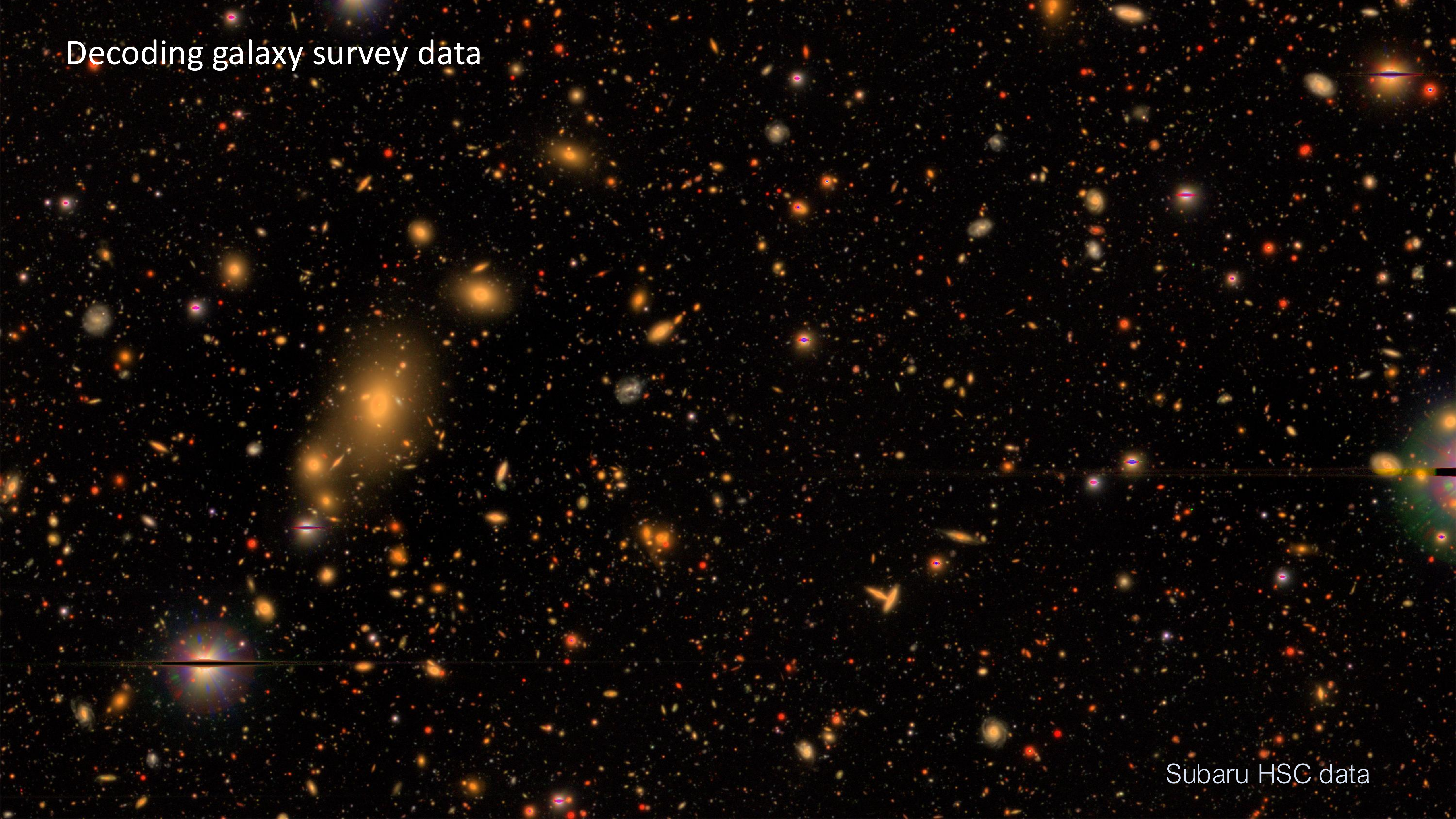
@W(I)MPU, Kavli IPMU, July 2025

Decoding the universe



8.2m Subaru Telescope at 4,200m Maunakea summit

Decoding galaxy survey data



Subaru HSC data

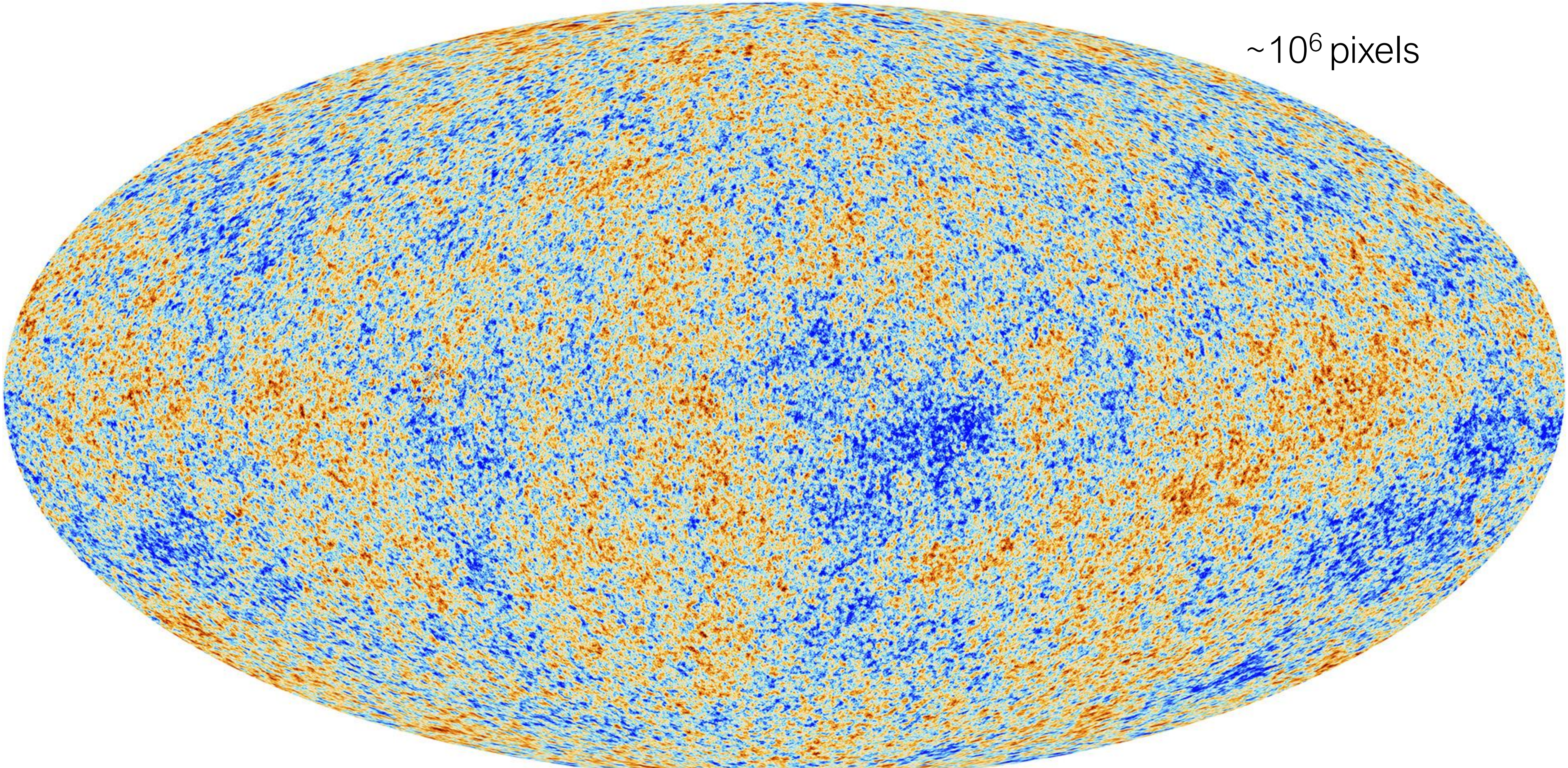
Vera Rubin Observatory (2,672nm in Chile)

the first-light image (June 2025)

$\sim 10^9 \text{galaxies}$

each galaxy contains the information (2D or 3D position, shapes, color, rotation, age, …)

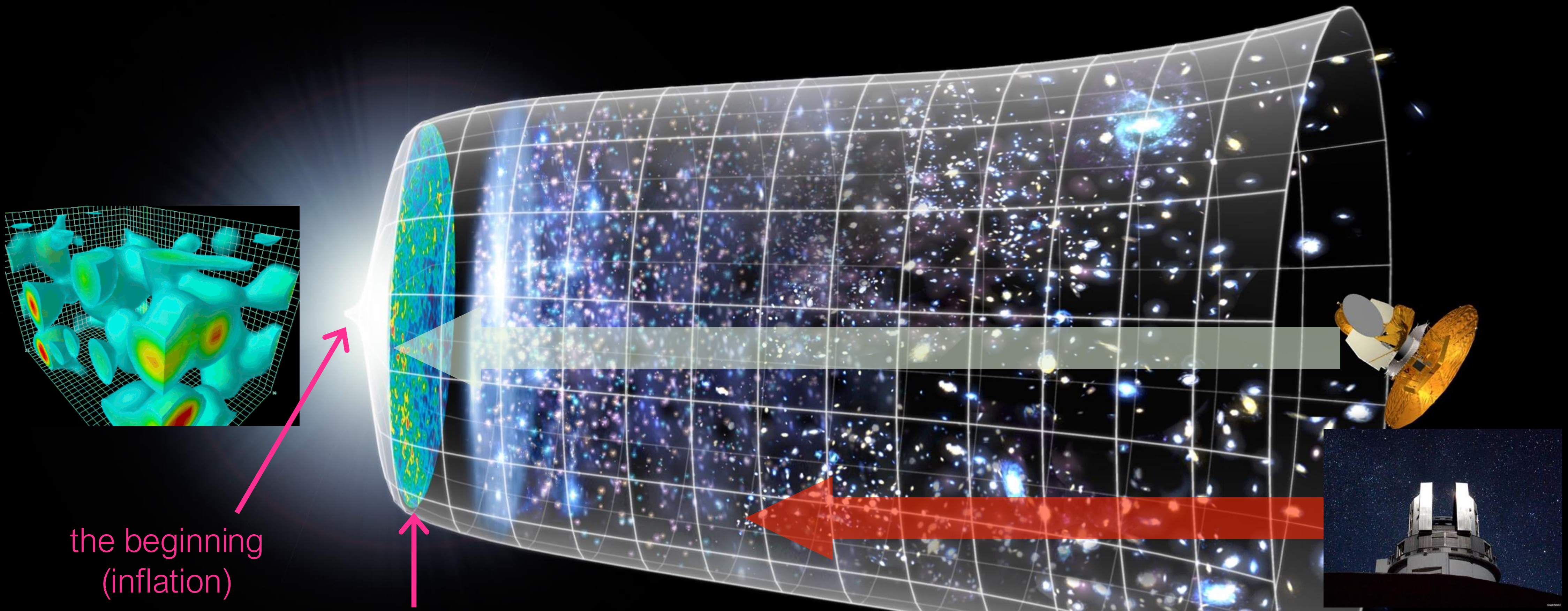
cosmic microwave background (CMB)
=snapshot of the universe at ~380,000 years after the Big Bang



~10^6 pixels

the goal: understanding the history of the universe, from the beginning to the future

see Jinn-Ouk‘ and Elisa‘s talks

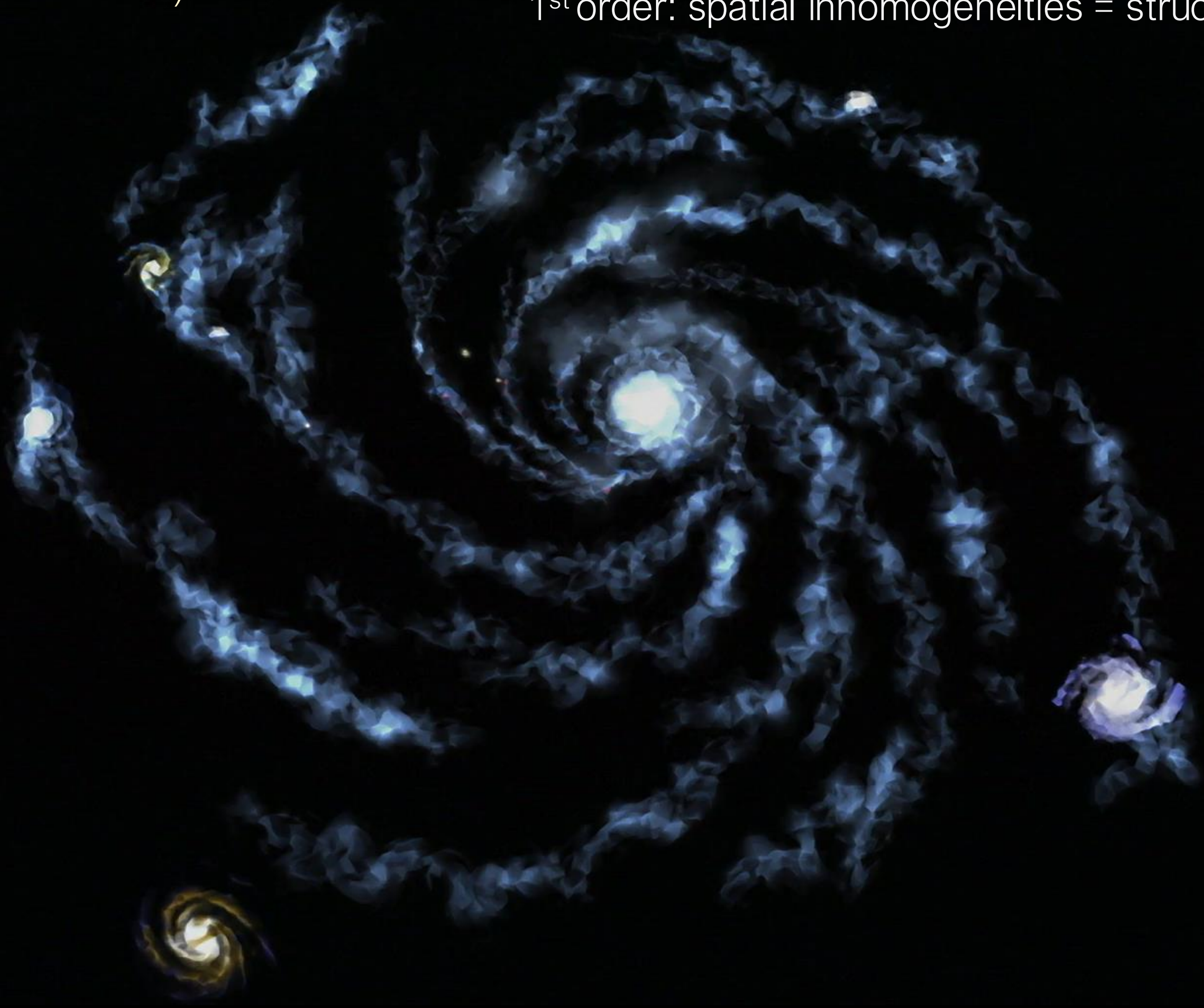


the beginning
(inflation)

CMB
(~380,000yrs)

today
(~14Gyrs)

0^{th} order: isotropic & homogeneous expanding universe
1^{st} order: spatial inhomogeneities = structure formation


NASA WMAP (2001-2010)

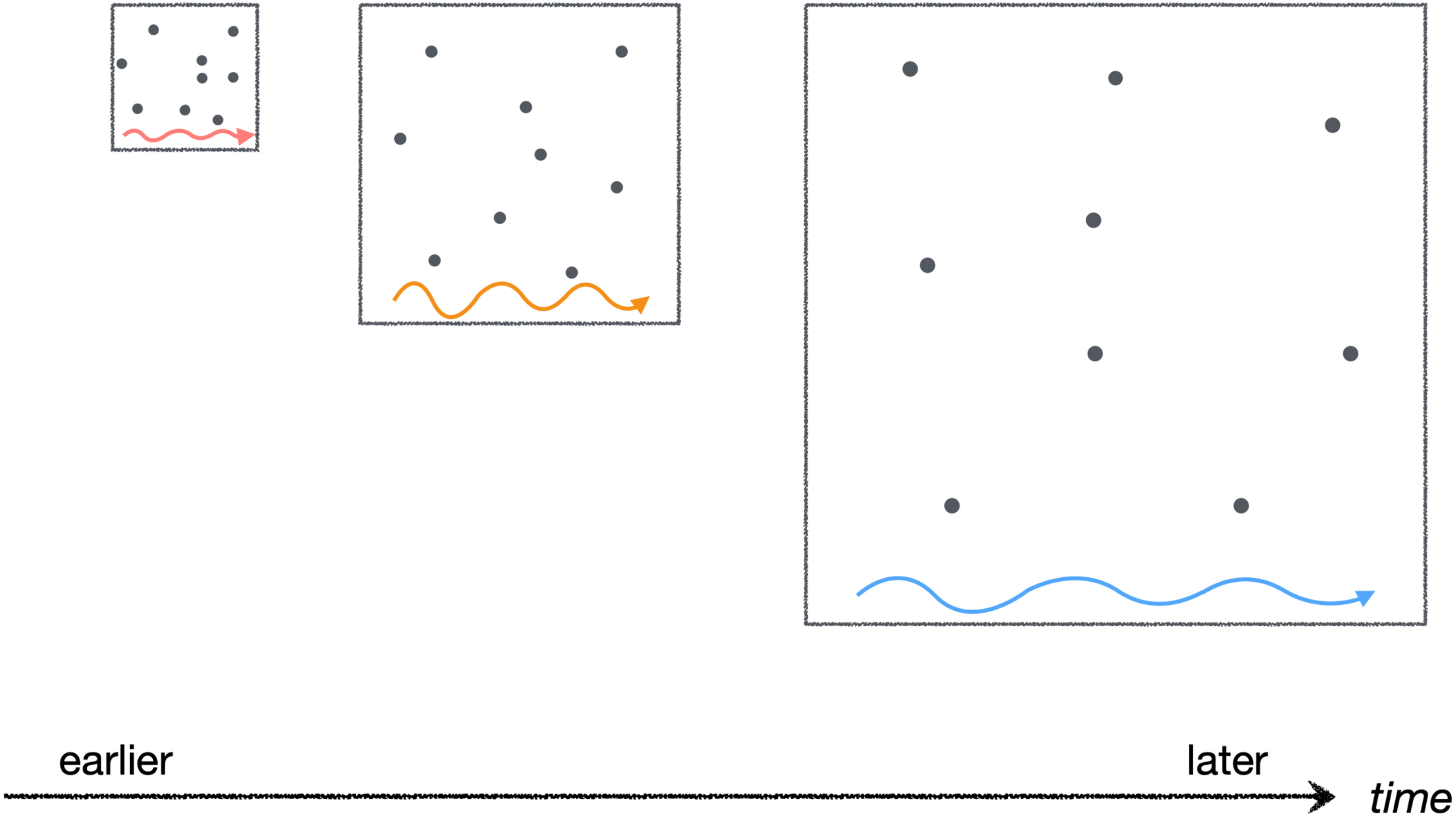
Observable is “temperature” in the blackbody brightness in each angular direction

$$T(\boldsymbol{\theta}; \lambda)$$

Temperature “fluctuations”

$$\frac{\delta T(\boldsymbol{\theta})}{\bar{T}} \equiv \frac{T(\boldsymbol{\theta})}{\bar{T}} - 1$$

*The universe is **expanding**, so it was **denser** and **hotter** in the past.*

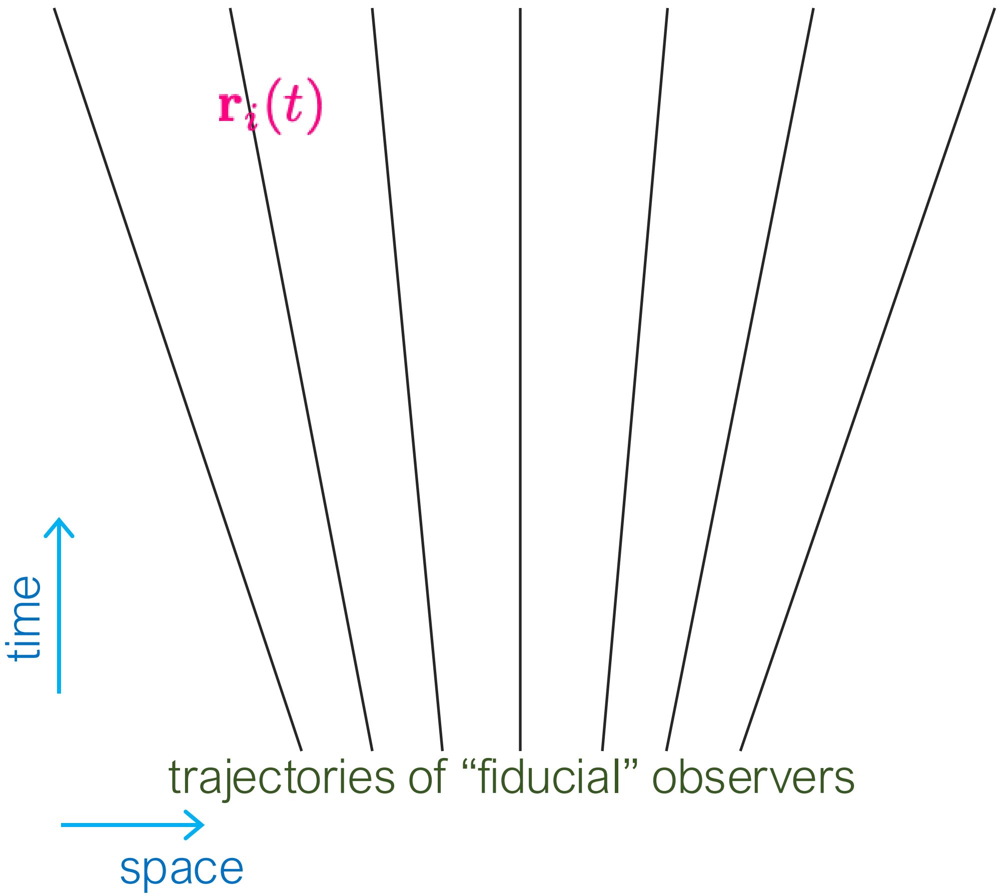


physical and comoving coordinates

Cosmological principles: “homogeneous” & “isotropic”, or no spatial position in the universe

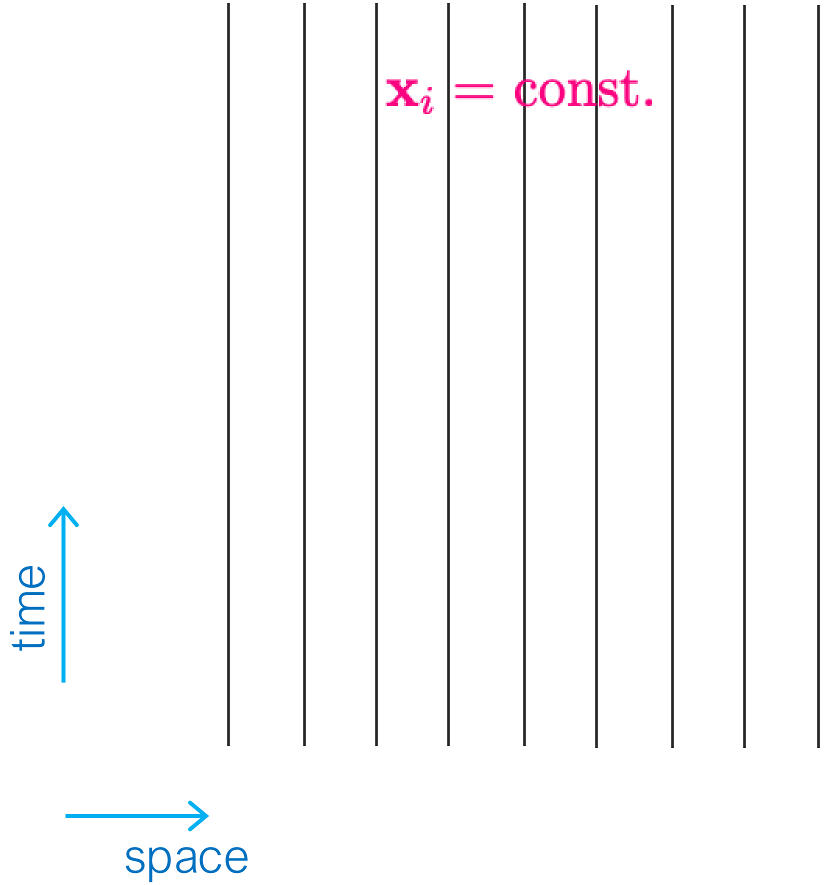
$$\mathbf{r}_{\text{fiducial observer}} = a(t)\mathbf{x}$$

physical coordinate

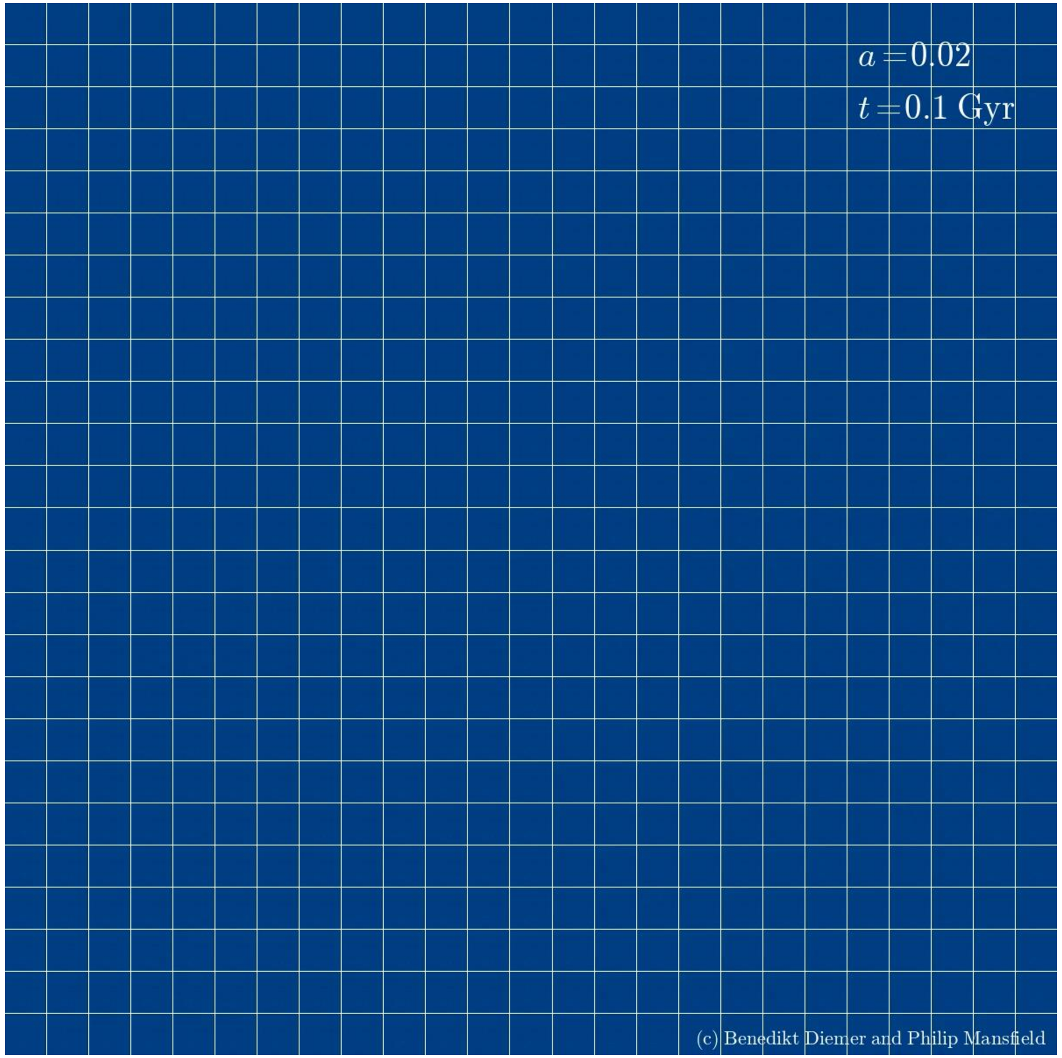


trajectories of “fiducial” observers

comoving coordinate



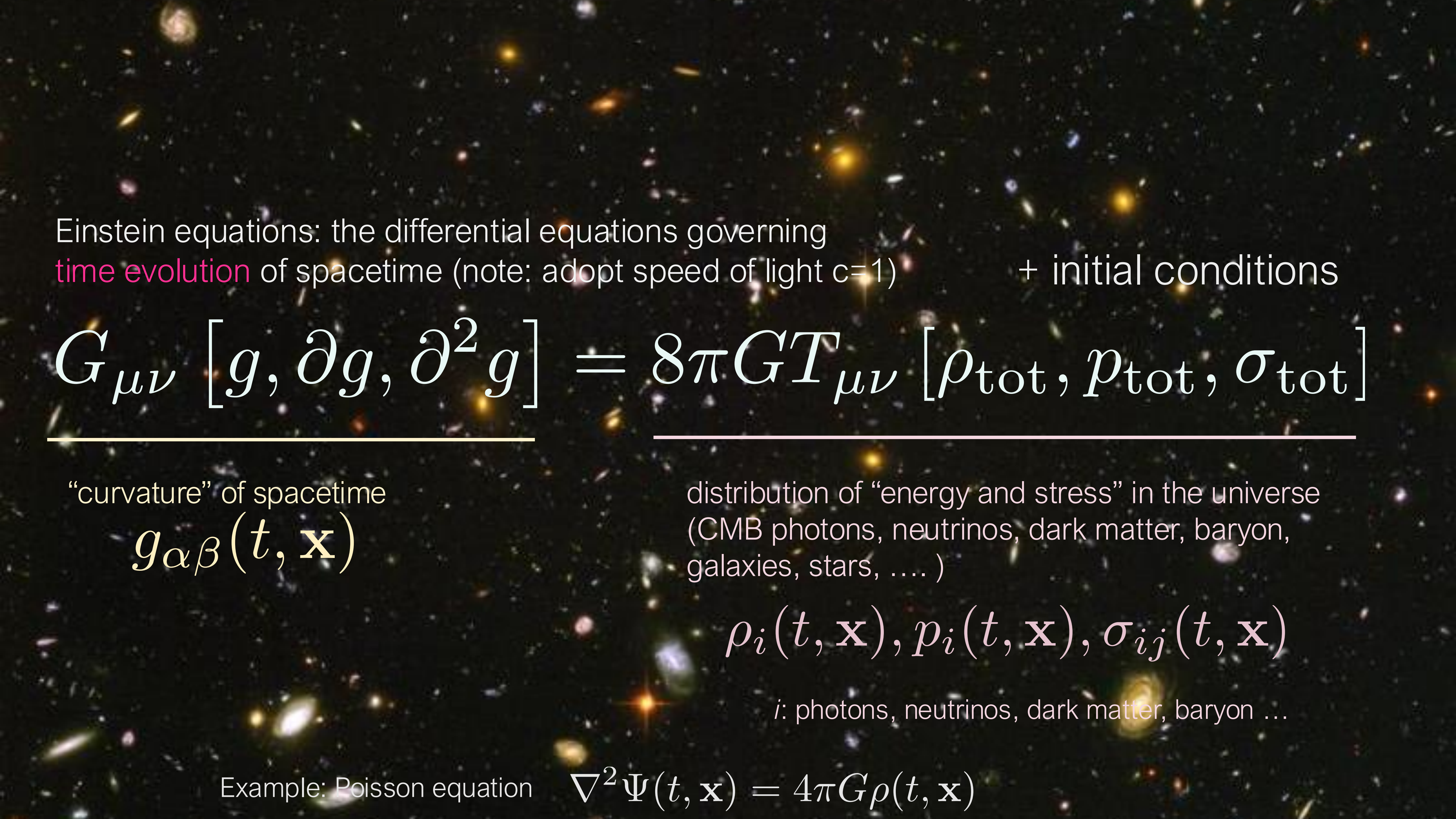
Note “fiducial” observers are defined as observers who see an “isotropic” cosmic microwave background physical coordinate



comoving coordinate

$$L_{\text{phys}} = a(t)\ell_{\text{comoving}}$$

movie: Diemer Benedikt



Einstein equations: the differential equations governing time evolution of spacetime (note: adopt speed of light c=1)

+ initial conditions

$$G_{\mu\nu}\left[g, \partial g, \partial^2 g\right] = 8\pi G T_{\mu\nu}\left[\rho_{\rm tot}, p_{\rm tot}, \sigma_{\rm tot}\right]$$

“curvature” of spacetime

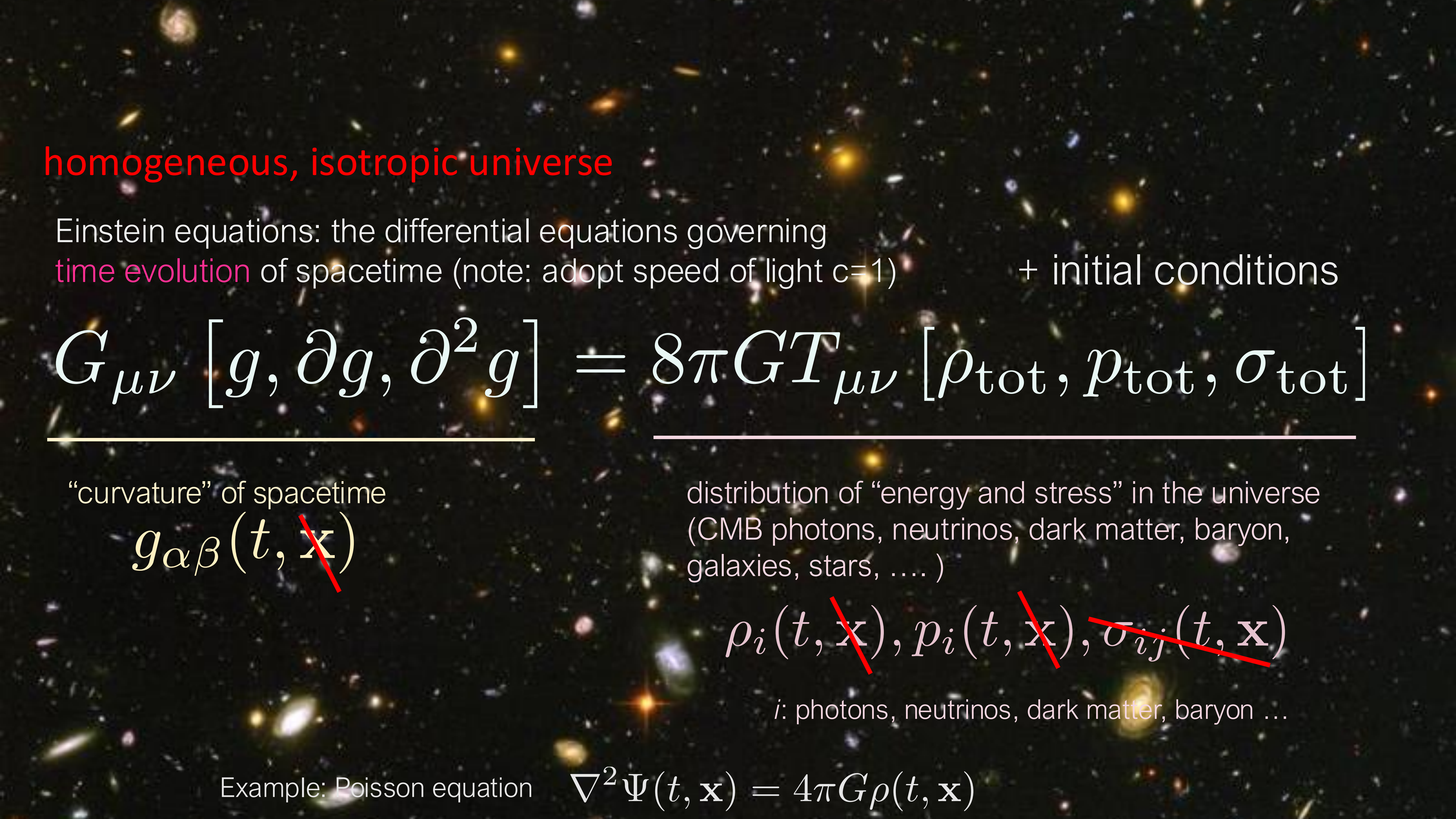
$$g_{\alpha\beta}(t, \mathbf{x})$$

distribution of “energy and stress” in the universe (CMB photons, neutrinos, dark matter, baryon, galaxies, stars, ….)

$$\rho_i(t, \mathbf{x}), p_i(t, \mathbf{x}), \sigma_{ij}(t, \mathbf{x})$$

i: photons, neutrinos, dark matter, baryon …

Example: Poisson equation $\nabla^2\Psi(t, \mathbf{x}) = 4\pi G\rho(t, \mathbf{x})$



homogeneous, isotropic universe

Einstein equations: the differential equations governing time evolution of spacetime (note: adopt speed of light c=1)

+ initial conditions

$$\underline{G_{\mu\nu}\left[g,\partial g,\partial^2 g\right]} = \underline{8\pi G T_{\mu\nu}\left[\rho_{\mathrm{tot}},p_{\mathrm{tot}},\sigma_{\mathrm{tot}}\right]}$$

"curvature" of spacetime

$$g_{\alpha\beta}(t,\cancel{\mathbf{x}})$$

distribution of "energy and stress" in the universe (CMB photons, neutrinos, dark matter, baryon, galaxies, stars, ….)

$$\rho_i(t,\cancel{\mathbf{x}}),\, p_i(t,\cancel{\mathbf{x}}),\, \cancel{\sigma_{ij}(t,\mathbf{x})}$$

i: photons, neutrinos, dark matter, baryon …

Example: Poisson equation $\nabla^2\Psi(t,\mathbf{x}) = 4\pi G\rho(t,\mathbf{x})$

- Einstein equations for the FLRW universe can connect the metric to the energy-matter contents of the universe
  - Cosmological principle: the distribution of energy-matter distribution should be homogeneous and isotropic

$$G_{\nu\mu}[\bar{g}(t)] = 8\pi G T_{\nu\mu}[\bar{\rho}(t), \bar{P}(t)]$$

  - FRLW equations governing time evolution of the metric variable (the scale factor, a(t), in FLRW universe), which can be applied to “classical” universe

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}_{\rm tot}(t) - \frac{K}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\bar{\rho}_{\rm tot}(t) + 3\bar{P}_{\rm tot}(t)\right]$$

See Elisa’s talk

    - Total energy and pressure density; what we know so far, photons, neutrinos, matter (CDM+baryon), dark energy (see later)

$$\bar{\rho}_{\rm tot} = \underbrace{\bar{\rho}_\gamma}_{\text{photons}} + \underbrace{\bar{\rho}_\nu}_{\text{neutrinos}} + \underbrace{\bar{\rho}_{\rm m}}_{\text{matter}} + \underbrace{\bar{\rho}_{\rm de}}_{\text{dark energy}}$$

$$\bar{\rho}_{\rm de} \sim a^0$$

$$\bar{\rho}_\gamma(t) \propto a^{-4}$$

$$\bar{\rho}_{\rm m} \propto a^{-3}$$

$$\bar{\rho}_\nu \propto \begin{cases} a^{-4} & (T_\nu \gg m_\nu) \\ a^{-3} & (T_\nu \ll m_\nu) \end{cases}$$

$$\nabla_\mu T^\mu{}_\nu = 0$$

$$\partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu} T^\mu{}_\alpha = 0$$

$$\boxed{\dot{\bar{\rho}}_a = -3H(\bar{\rho}_a + \bar{P}_a)}$$

a=photons, neutrinos, DM, baryon, dark energy

How small was the universe?

- The age of the universe: ~13.8Gyr
- The size of the observable universe: $L_0 \sim 14\text{G light yrs} \sim 10^{28}\text{cm}$
- Assume that the end of the inflation ~ the epoch of creation of the Standard Model particles (the beginning of the "hot" Big Bang): $T_{\rm rh} \sim 10^{14}\ \text{GeV}$
- Assume that, during the inflation, the universe expanded by ~62 e-folds
- The current "observable" universe should have begun with the size of …

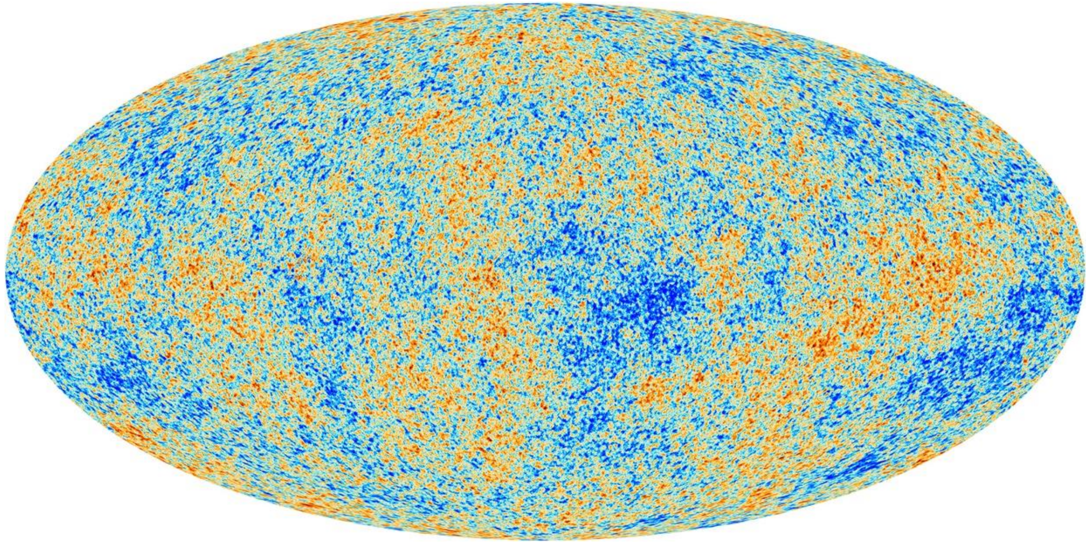
$$L_{\rm ini} \sim L_0 \frac{a_{\rm rh}}{a_0} \frac{a_{\rm ini}}{a_{\rm end}} \sim 10^{28}\text{cm} \left(\frac{10^{-4}\text{eV}}{10^{14}\text{GeV}} \right) \left(\frac{1}{\exp(62)} \right)$$

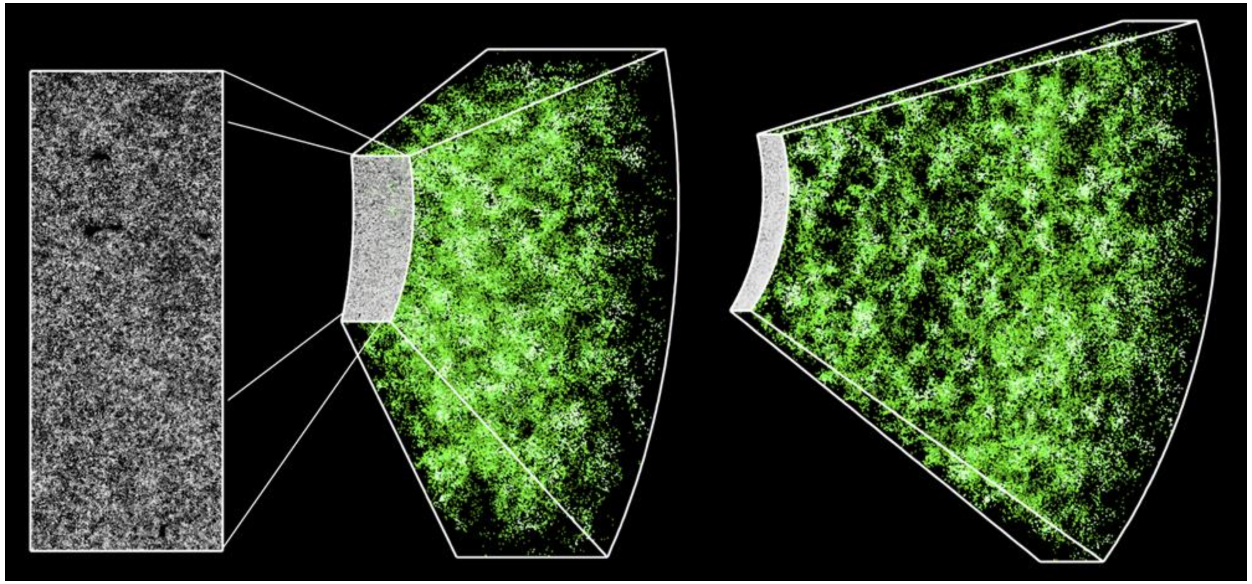
$$\sim 10^{-26}\ \text{cm}$$ ⇔ atom size ~ 10^{-8} cm, LHC ~ 10^{-16} cm

The whole universe should have been "quantum" (not yet fully in the quantum gravity regime)

$$l_{\rm Pl} \sim 10^{-33}\text{cm}$$

structure formation = time evolution of spatial "inhomogeneities" (spacetime and matter-energy)

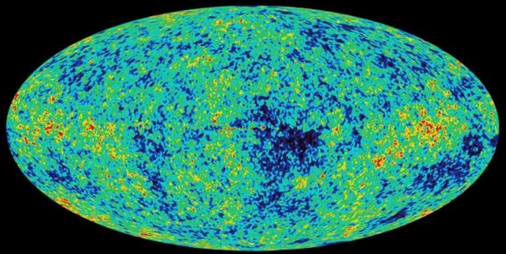




The Universe has inhomogeneities!

- Where do these inhomogeneities come from?
- What is the origin of these inhomogeneities/structures?
- Structure formation = time evolution of the inhomogeneities. Here we need to consider both fluctuations of "spacetime" (metric) and inhomogeneities of "matter-energy distribution"
- Note that, on very large scales, the universe looks homogeneous and isotropic: FLRW model gives a good description of the universe in the average sense

ΛCDM structure formation model

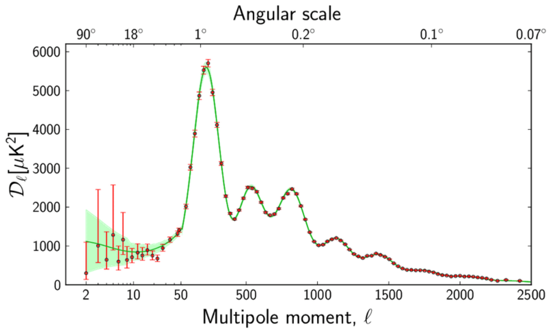


CMB at z~1000

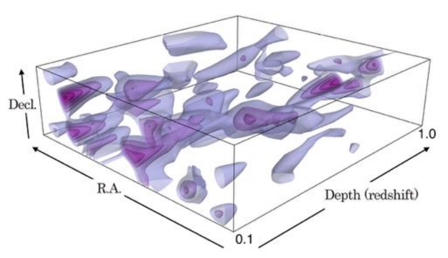
time

gravity (dark matter)

cosmic expansion (dark energy)



ΛCDM=~6 parameters



Galaxy surveys directly measure “**lumpiness**” of the late universe

large-scale structure = “spatial” inhomogeneities = fluctuations

- background

$$G_{\mu\nu}[a(t), \dot{a}(t), \ddot{a}(t)] = 8\pi G T_{\mu\nu}[\bar{\rho}_i(t), \bar{P}_i(t)]$$

- Einstein equations for the spatial inhomogeneities or fluctuations

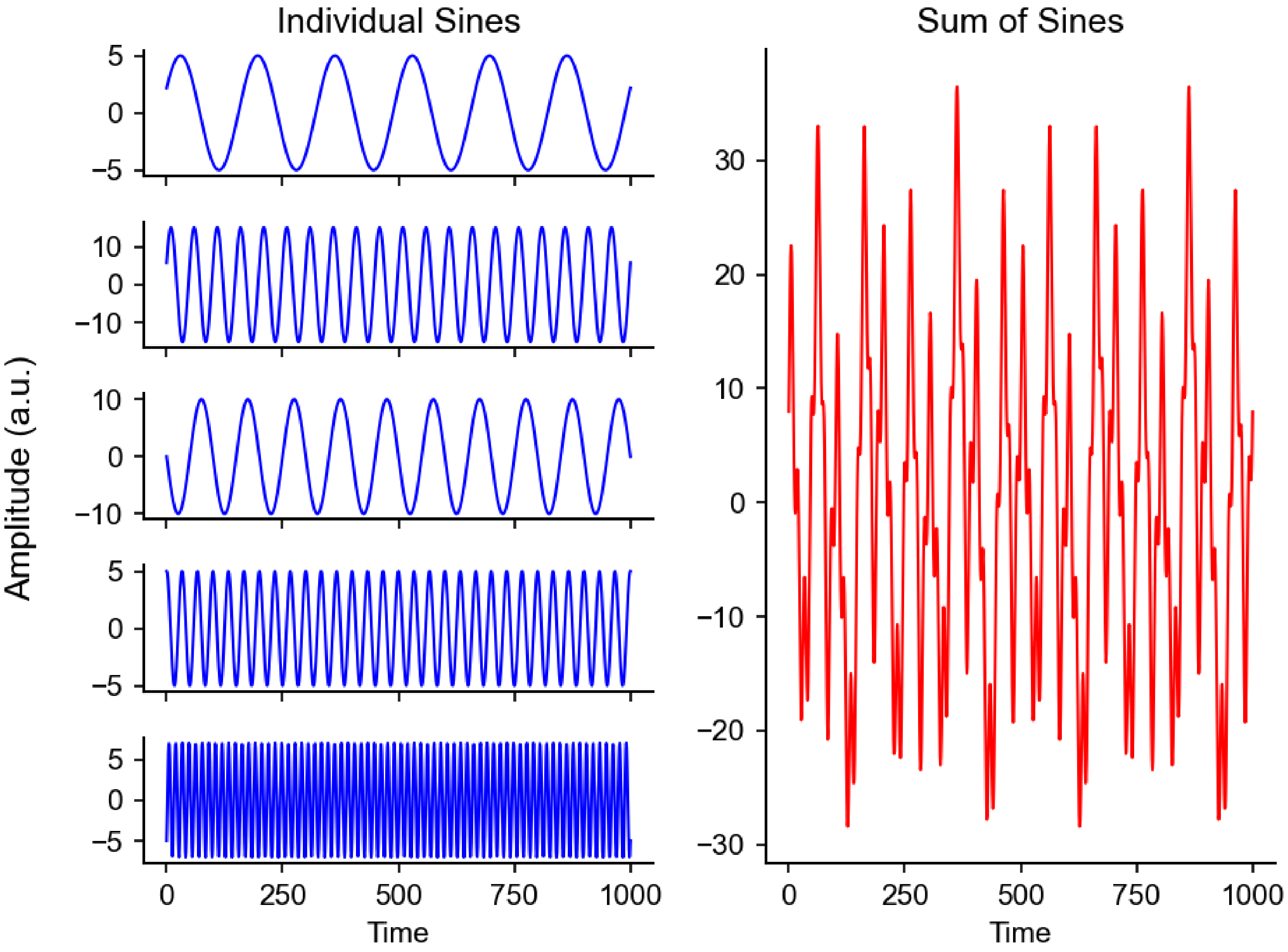
$$\delta G_{\mu\nu}[\underline{\bar{g}_{\alpha,\beta}(t) + \delta g_{\alpha\beta}(\mathbf{x}, t), \partial g_{\alpha\beta}, \partial\partial g_{\alpha\beta}}] = 8\pi G\ \delta T_{\mu\nu}[\underline{\bar{\rho}_i + \delta\rho_i(\mathbf{x}, t), \bar{P}_i + \delta P(\mathbf{x}, t), \Pi(\mathbf{x}, t)}]$$

isotropic & anisotropic expansion
curvature perturbations
gravitational potential
gravitational wave

photons
DM
baryon (galaxies, stars, gas, …)
neutrinos
density fluctuations
pressure perturbations
anisotropic stress

+ Boltzmann equations for each species

Fourier decomposition



- Fourier decomposition of the fluctuation field

$$f(\mathbf{x}) = \int \mathrm{d}^3\mathbf{k}\ f_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

- “wavenumber” or “wavelength”

$$\mathbf{k} = (k_x, k_y, k_z)$$

$$\lambda = \frac{2\pi}{k}$$

- In the linear regime, different Fourier modes evolve independently

Angular power spectrum of CMB temperature fluctuations

Fourier decomposition in 3D Euclidean space ⇔ Harmonic decomposition in 2D sphere

Data analysis (cosmological observable)

$$a_{\ell m} \equiv \int \mathrm{d}\Omega \; \frac{\delta T(\varphi,\theta)}{\bar{T}} Y_{\ell m}(\varphi,\theta)$$

or

$$\frac{\delta T(\varphi,\theta)}{\bar{T}} \equiv \sum_{\ell m} a_{\ell m} Y_{\ell m}(\varphi,\theta)$$

$$C_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$$

large-scale structure = “spatial” inhomogeneities = fluctuations

- Fourier decomposition is useful for describing spatial inhomogeneities

$$\rho(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\;\widetilde{\rho}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$$

homogeneous field $\widetilde{\rho}_{\mathbf{k}} \neq 0 \;\; \text{for } k=0, \quad \widetilde{\rho}_{\mathbf{k}} = 0 \;\; \text{for } \mathbf{k} \neq \mathbf{0}$

$$\longrightarrow \rho(\mathbf{x},t) \to \rho(t)$$

- linearized perturbation theory (all perturbations ≪1)

$$|\Phi|, |\Psi|, \left|\frac{\delta\rho}{\bar{\rho}}\right| \ll 1$$

e.g.,

$$\nabla^2\Phi(\mathbf{x},\eta) - 3\mathcal{H}\left[\Phi'(\mathbf{x},\eta) + \mathcal{H}\Phi(\mathbf{x},\eta)\right] = 4\pi G a^2 \delta\rho(\mathbf{x},\eta)$$

$$\longrightarrow \underline{-k^2}\,\underline{\Phi_{\mathbf{k}}(\eta)} - \underline{3\mathcal{H}}\left[\underline{\Phi'_{\mathbf{k}}(\eta)} + \underline{\mathcal{H}}\,\underline{\Phi_{\mathbf{k}}(\eta)}\right] = \underline{4\pi G a^2\bar{\rho}}\,\underline{\delta_{\mathbf{k}}(\eta)}$$

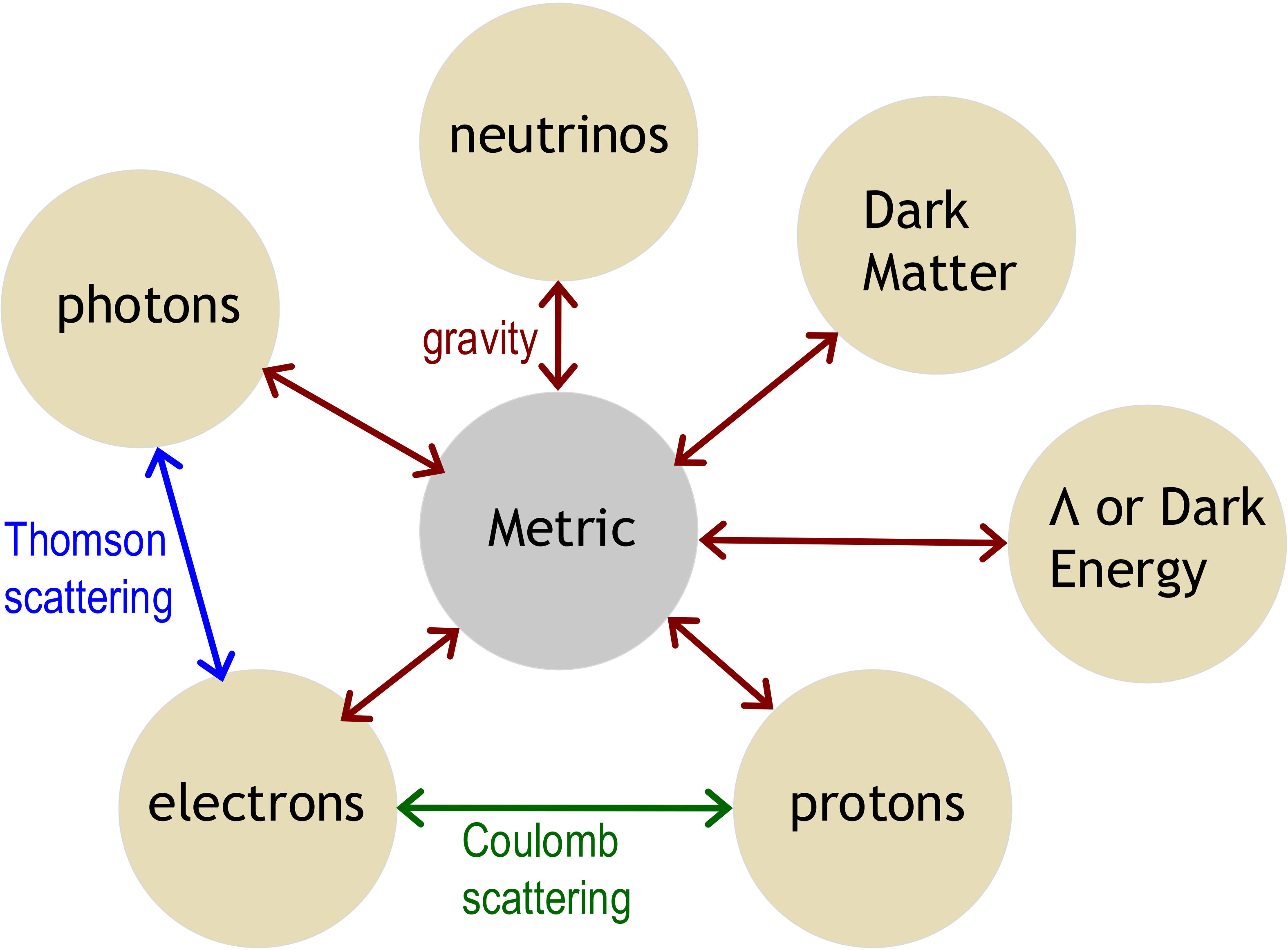
background

perturbations

we can solve the differential equations separately, for each “k”

Fourier decomposition is very useful

Cosmological linearized perturbation theory



- gravity affects all particle species or energy
- need to solve time-evolution of this multi-component system

cosmological linearized PT

$$\mathrm{d}s^2 = a(\eta)^2\left[-(1+2\Psi)\mathrm{d}\eta^2 + (1-2\Phi)\mathrm{d}\mathbf{x}^2\right]$$

- the linearized Einstein equations

Bardeen; Kodama & Sasaki; Mukhanov+, Bond; Ma & Bertschinger, …

$$\boxed{\begin{aligned} -k^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Psi\right) &= 4\pi G a^2 \delta\rho \\ -k\left(\Phi' + \mathcal{H}\Psi\right) &= 4\pi G a^2 q \\ -k^2(\Phi - \Psi) &= 8\pi G a^3 \Pi \\ \Phi'' + \mathcal{H}\Psi' + 2\mathcal{H}\Phi' - \frac{k^2}{3}(\Psi - \Phi) + (2\mathcal{H}' + \mathcal{H}^2)\Pi &= 4\pi G a^2 \delta P \end{aligned}}$$

$$\boxed{\begin{aligned} \delta' &= \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(kv + 3\Phi') - 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta \\ v' &= -\left(\mathcal{H} + \frac{\bar{P}'}{\bar{\rho} + \bar{P}}\right)v - \frac{1}{\bar{\rho} + \bar{P}}k\delta P + \frac{2}{3}k\Pi - k\Psi \end{aligned}}$$

$$\bar{\rho} = \sum_a \bar{\rho}_a, \delta\rho = \sum_a \delta\rho_a, \delta \equiv \frac{\sum_a \delta\rho}{\bar{\rho}}$$

$$\bar{P} = \sum_a P_a, \delta P = \sum_a \delta P_a, \cdots$$

matter $\delta_{\mathrm{m}}'' + \mathcal{H}\delta_{\mathrm{m}}' = -k^2\Psi + 3\left(\Phi'' + \mathcal{H}\Phi'\right)$

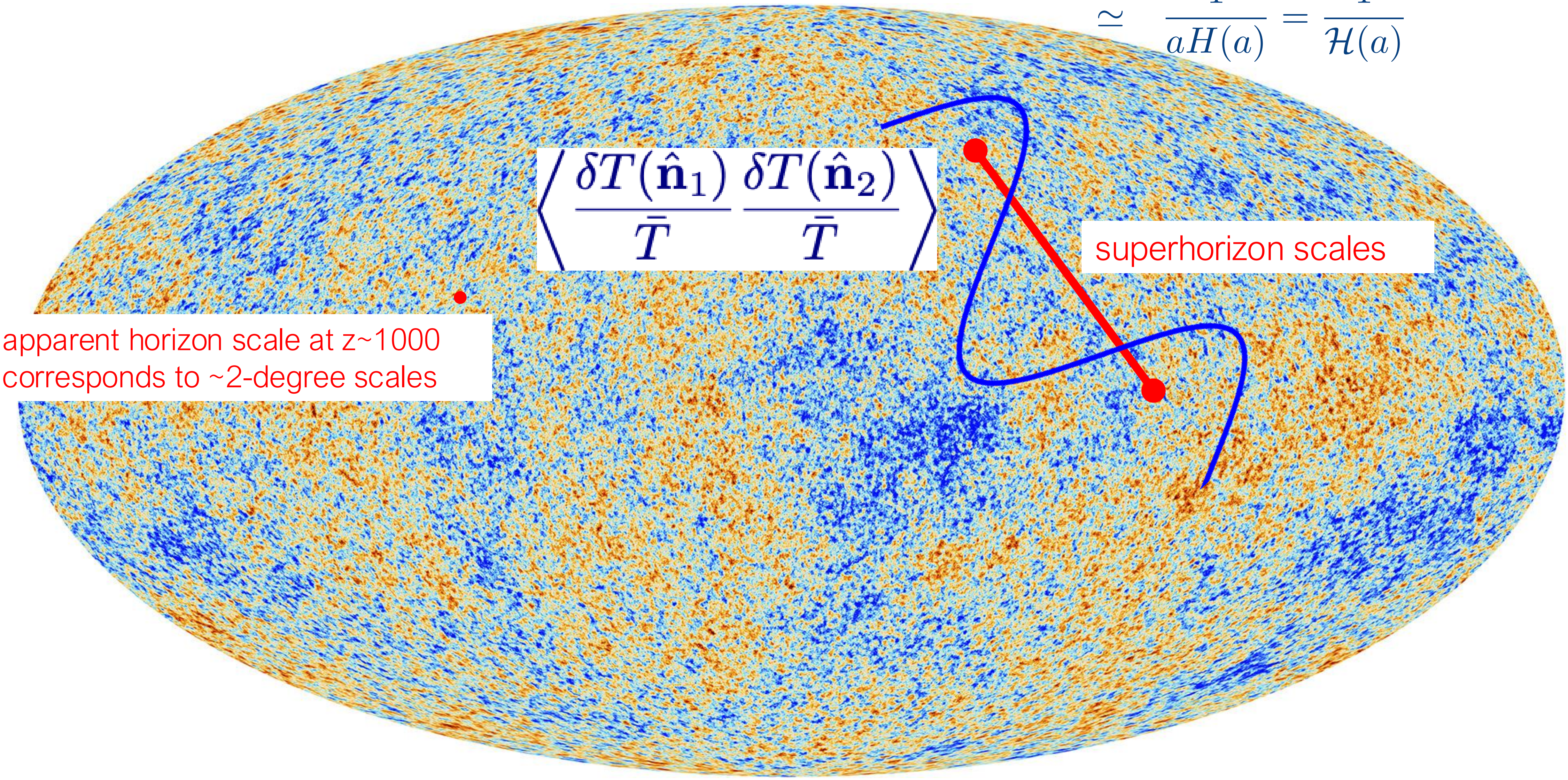
$$\left\{\begin{aligned} \delta_{\mathrm{m}}' &= -kv_{\mathrm{m}} + 3\Phi' \\ v_{\mathrm{m}}' &= -\mathcal{H}v_{\mathrm{m}} - k\Psi \end{aligned}\right.$$

radiation $\delta_{\mathrm{r}}'' + \dfrac{k^2}{3}\delta_{\mathrm{r}} = -\dfrac{4}{3}\Psi + 4\Phi''$

$$\left\{\begin{aligned} \delta_{\mathrm{r}}' &= -\tfrac{4}{3}kv_{\mathrm{r}} + 4\Phi' \\ v_{\mathrm{r}}' &= -\tfrac{1}{4}\delta_{\mathrm{r}} - k\Psi \end{aligned}\right.$$

superhorizon-scale fluctuations exist!

$$\begin{aligned}\lambda_H(a) &\equiv \int_{t_i}^{t} \frac{\mathrm{d}t'}{a(t')} = \int_{a_i}^{a} \frac{\mathrm{d}a'}{H(a')(a')^2} \\ &\simeq \frac{1}{aH(a)} = \frac{1}{\mathcal{H}(a)}\end{aligned}$$



$$\left\langle \frac{\delta T(\hat{\mathbf{n}}_1)}{\bar{T}} \frac{\delta T(\hat{\mathbf{n}}_2)}{\bar{T}} \right\rangle$$

superhorizon scales

apparent horizon scale at z~1000
corresponds to ~2-degree scales

Inflationary scenario (yet to be proven)

see Jinn-Ouk' and Elisa's talks

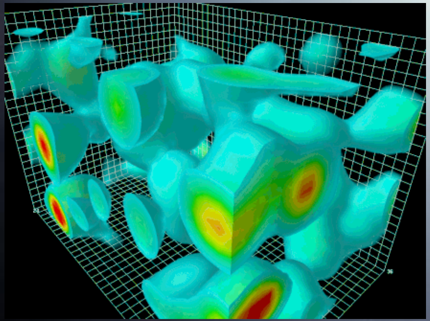
- The size of our "observable" universe

$$\sim 10^{-26}\mathrm{cm}$$

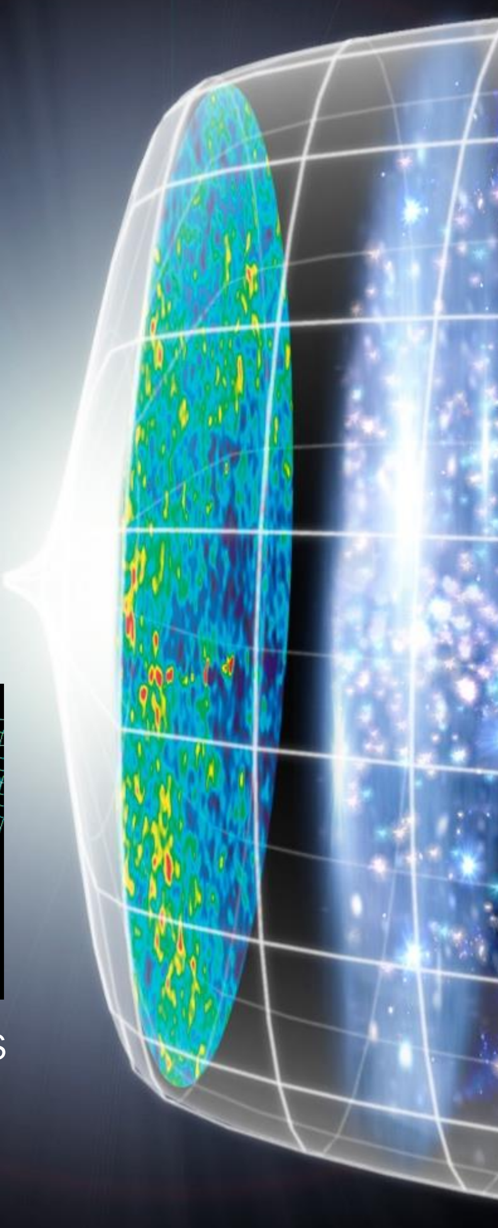
- Accelerated expansion

$$\frac{\mathrm{d}}{\mathrm{d}\eta}(aH)^{-1} < 0 \rightarrow \ddot{a} > 0$$

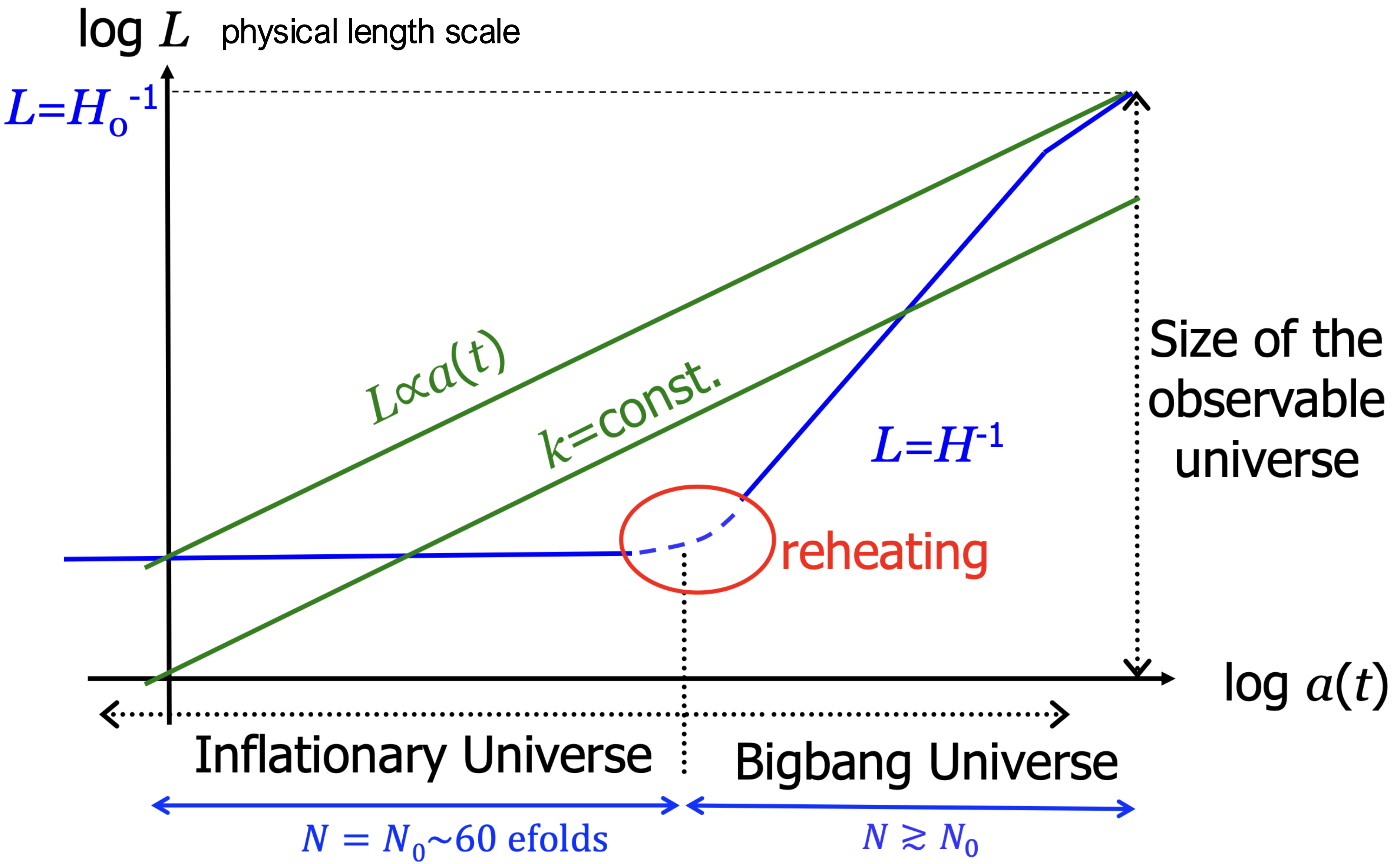
  - Solve Horizon problem and flatness problem
- quantum fluctuations ⇒ primordial fluctuations (see later): the origin of all structures

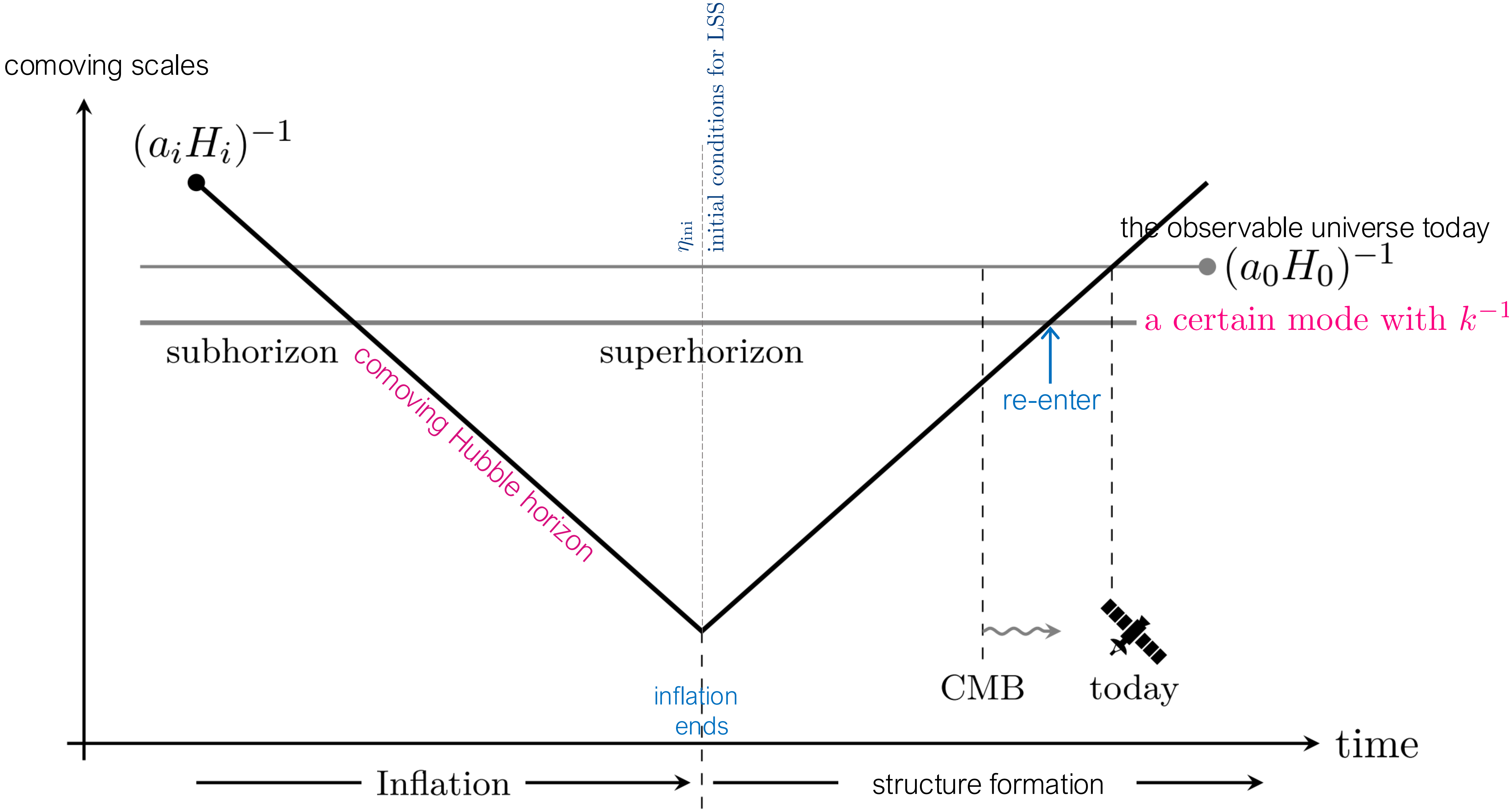


quantum fluctuations



taken from Misao's slides




comoving scales
$(a_i H_i)^{-1}$
initial conditions for LSS
$\eta_{\rm ini}$
the observable universe today
$(a_0 H_0)^{-1}$
a certain mode with k^{-1}
subhorizon
superhorizon
comoving Hubble horizon
re-enter
inflation ends
CMB
today
time
Inflation
structure formation

Adiabatic initial conditions – single degree-of-freedom conditions

- The adiabatic initial conditions: all fluid components on superhorizon scales are perfectly synchronized – very, very intriguing initial conditions

the coordinate transformation $\boxed{\tilde{\eta} = \eta + \pi(\mathbf{x}, \eta)}$

$$\bar{\rho}_\gamma(\tilde{\eta}) \quad \rightarrow \quad \bar{\rho}_\gamma(\eta + \pi) \simeq \bar{\rho}_\gamma(\eta) + \frac{1}{4}\mathcal{H}\bar{\rho}_\gamma \pi(\mathbf{x}, \eta)$$

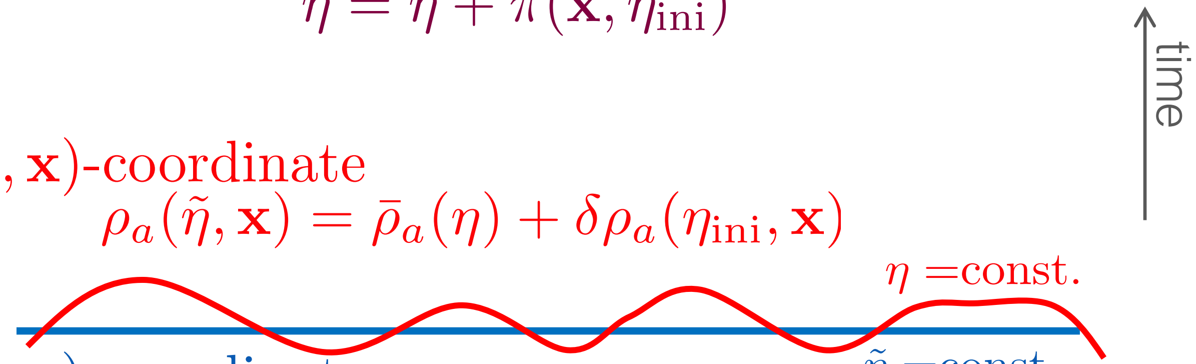
$$\rightarrow \delta_\gamma(\eta, \mathbf{x}) = \frac{1}{4}\mathcal{H}\pi(\mathbf{x}, \eta)$$

Similarly

$$\boxed{\begin{aligned} \bar{\rho}_\gamma(\tilde{\eta}) &\mapsto \delta_\gamma(\eta, \mathbf{x}) = \frac{1}{4}\mathcal{H}\pi(\mathbf{x}, \eta) \\ \bar{\rho}_\nu(\tilde{\eta}) &\mapsto \delta_\nu(\eta, \mathbf{x}) = \frac{1}{4}\mathcal{H}\pi(\mathbf{x}, \eta) \\ \bar{\rho}_\mathrm{c}(\tilde{\eta}) &\mapsto \delta_\mathrm{c}(\eta, \mathbf{x}) = \frac{1}{3}\mathcal{H}\pi(\mathbf{x}, \eta) \\ \bar{\rho}_\mathrm{b}(\tilde{\eta}) &\mapsto \delta_\mathrm{b}(\eta, \mathbf{x}) = \frac{1}{3}\mathcal{H}\pi(\mathbf{x}, \eta) \end{aligned}}$$

furthermore, $\pi(\mathbf{x}, \eta) = \pi(\mathbf{x}, \eta_\mathrm{ini})$

set by the initial conditions!



$$\tilde{\eta} = \eta + \pi(\mathbf{x}, \eta_\mathrm{ini})$$

time

$(\eta, \mathbf{x})$-coordinate

$$\rho_a(\tilde{\eta}, \mathbf{x}) = \bar{\rho}_a(\eta) + \delta\rho_a(\eta_\mathrm{ini}, \mathbf{x})$$

η =const.

$\tilde{\eta}$ =const.

$(\tilde{\eta}, \mathbf{x})$-coordinate

$$\tilde{\rho}_a(\tilde{\eta}, \mathbf{x}) = \bar{\rho}_a(\tilde{\eta})$$

for all components (photons, neutrinos, CDM, baryon)

Question:
How can we naturally generate the adiabatic initial conditions?

The leading model is inflation

slow-roll inflation

- Assumption: an unknown scalar field is dominated

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right)$$

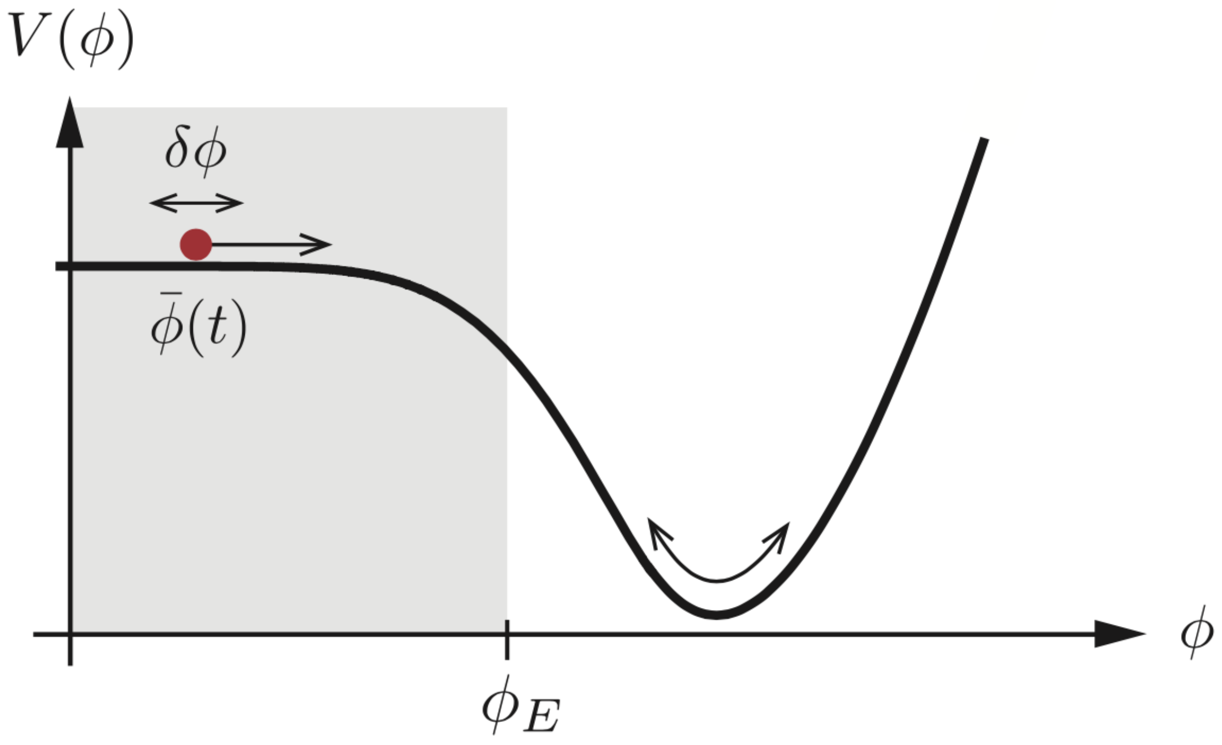
- Einstein equations

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V\right]$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[2\dot{\phi}^2 - 2V\right]$$

- Slow-roll assumption $\dot{\phi}^2 \ll V$

⇒ $a \propto \exp[Ht]$



quantum fluctuations (inflaton fluctuations)

- Inflaton would have quantum fluctuations

$$\phi(\eta, \mathbf{x}) = \bar{\phi}(\eta) + \delta\phi(\eta, \mathbf{x})$$

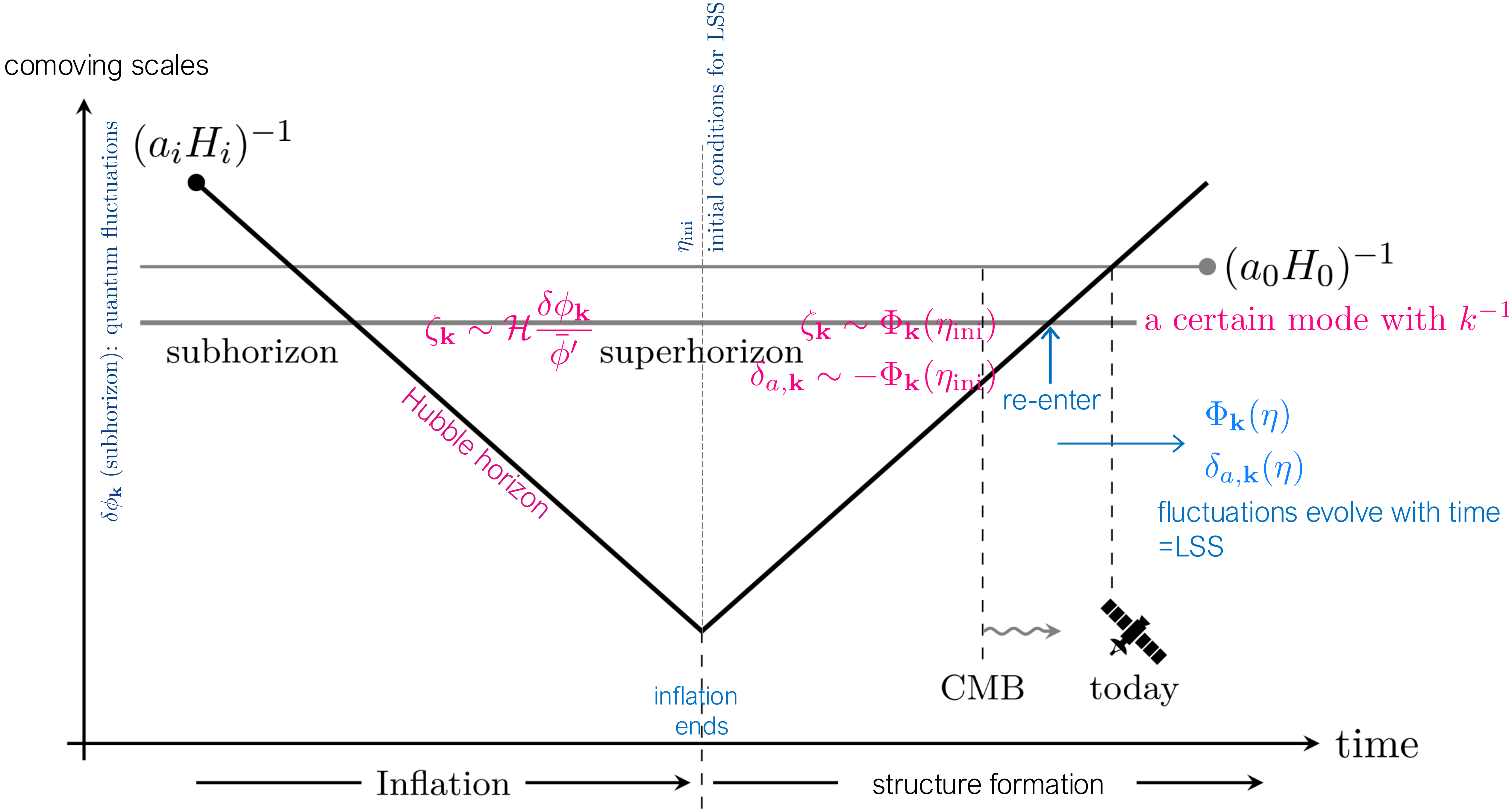
- Equation of motion for the perturbed inflaton field (under slow-roll inflation)

$$\delta\phi(\eta, \mathbf{x}) = a f(\eta, \mathbf{x})$$

$$f''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) f_{\mathbf{k}} = 0, \quad f_{\mathbf{k}}(\eta) \equiv \int \mathrm{d}^3\mathbf{x}\ f(\eta, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

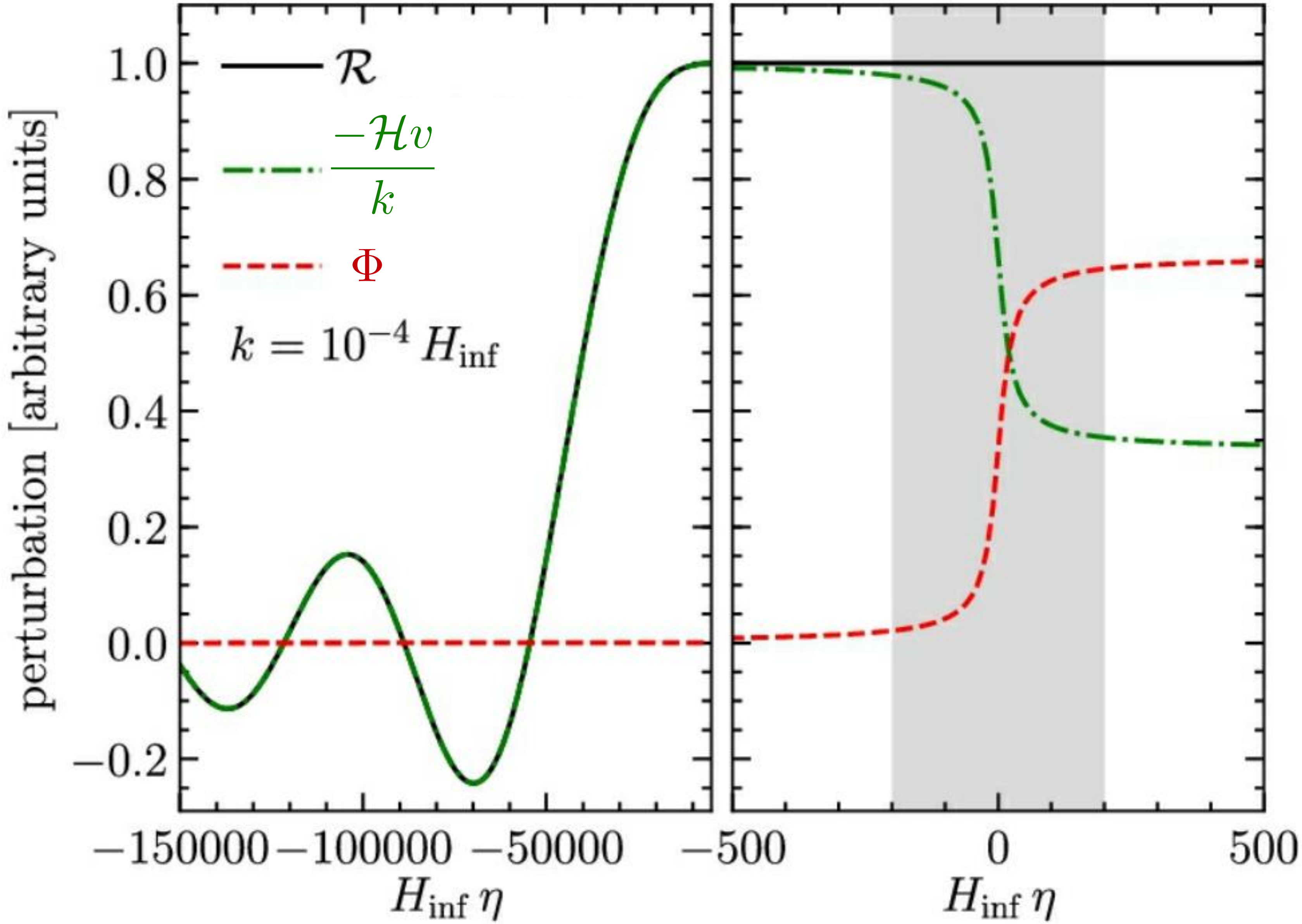
Sasaki-Mukhanov equation!

- Quantization of the perturbed inflaton field on subhorizon scales ⇒ quantum fluctuations
- Quantum fluctuations at each wavenumber are stretched out by inflation, and turn into "classical" fluctuations


comoving scales
$\delta\phi_{\mathbf{k}}$ (subhorizon): quantum fluctuations
$(a_i H_i)^{-1}$
$\eta_{\rm ini}$
initial conditions for LSS
$(a_0 H_0)^{-1}$
a certain mode with k^{-1}
subhorizon
$\zeta_{\mathbf{k}} \sim \mathcal{H}\frac{\delta\phi_{\mathbf{k}}}{\bar{\phi}'}$
superhorizon
$\zeta_{\mathbf{k}} \sim \Phi_{\mathbf{k}}(\eta_{\rm ini})$
$\delta_{a,\mathbf{k}} \sim -\Phi_{\mathbf{k}}(\eta_{\rm ini})$
Hubble horizon
re-enter
$\Phi_{\mathbf{k}}(\eta)$
$\delta_{a,\mathbf{k}}(\eta)$
fluctuations evolve with time
=LSS
inflation ends
CMB
today
time
Inflation
structure formation

$$\mathrm{d}s^2 = a(\eta)^2\left[-(1+2\Psi)\mathrm{d}\eta^2 + (1-2\Phi)\mathrm{d}\mathbf{x}^2\right]$$

$$\mathcal{R} = \Phi - \frac{\mathcal{H}}{k}v = \Phi - \frac{\mathcal{H}}{k}\left(-\frac{k\eta}{2}\Phi\right) = \frac{3}{2}\Phi \text{ (post inflation for superhorizon curvature perturbation)}$$



$$\zeta = \mathcal{R}$$

$$ds^2 = a(\eta)^2\left[-(1+2\Psi)d\eta^2 + (1-2\Phi)d\mathbf{x}^2\right]$$

initial conditions of structure formation

inflaton fluctuations (subhorizon)
$\delta\phi(\mathbf{k},\eta)$

$\xrightarrow[\text{by inflation}]{\text{stretching}}$

$\zeta_{\mathbf{k}} \sim \mathcal{H}\dfrac{\delta\phi_{\mathbf{k}}}{\bar{\phi}'}$
$\Phi_{\mathbf{k}} \sim 0$

$\xrightarrow[\text{inflation ends}]{}$

$\zeta_{\mathbf{k}} \sim \Phi_{\mathbf{k}}(\eta_{\rm ini}), \;\; \delta_{a,\mathbf{k}} \sim -\Phi_{\mathbf{k}}(\eta_{\rm ini})$
initial conditions of LSS

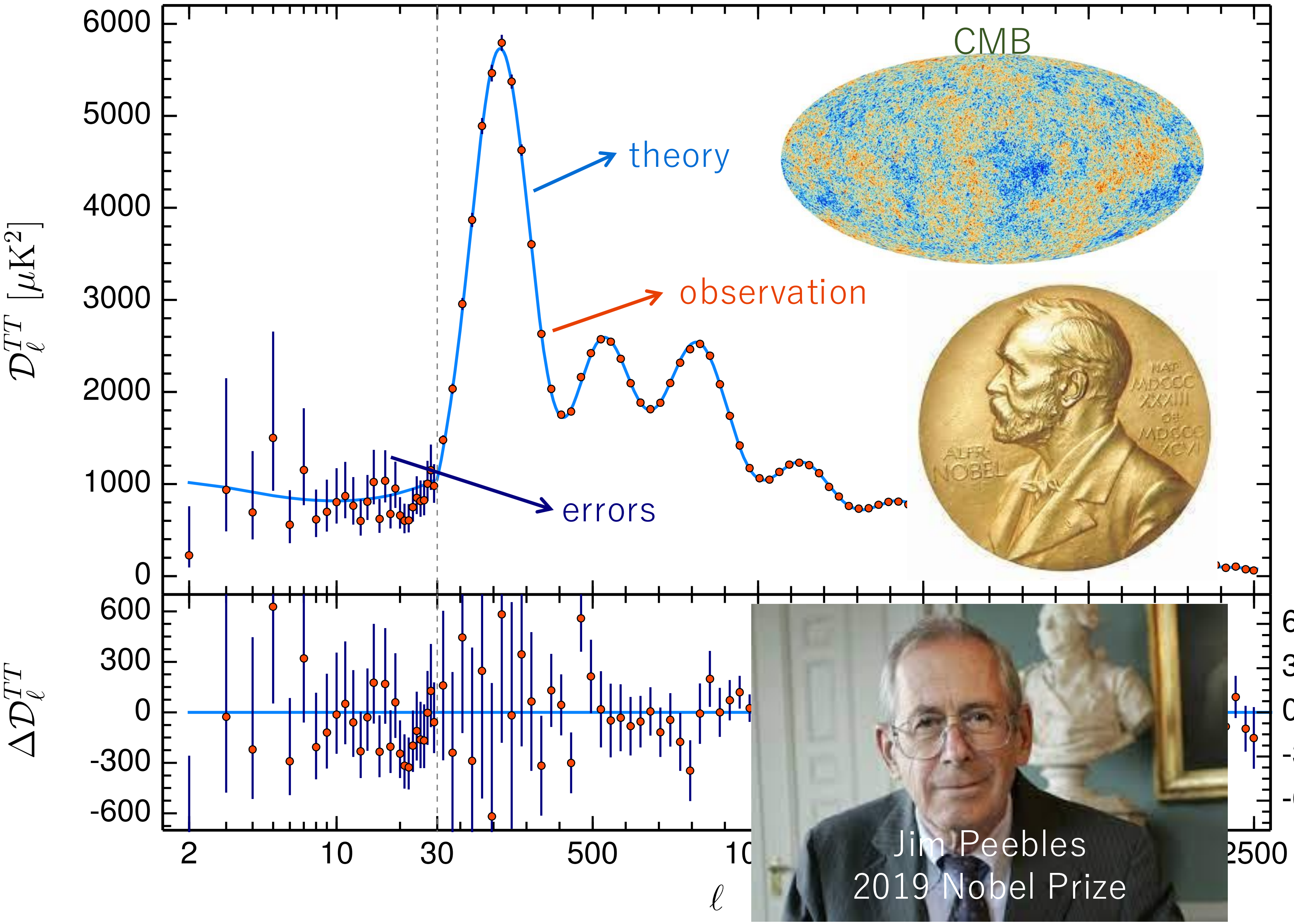
- the inflationary scenario provides a physical mechanism for generating primordial fluctuations (i.e. the initial conditions of cosmic structures)
- A single-field inflation field (at least a dominant one inflaton field even in multi-field inflation) gives the adiabatic initial conditions

$$\delta_\gamma(\mathbf{k},\eta) = \delta_\nu(\mathbf{k},\eta) = \frac{4}{3}\delta_{\rm c}(\mathbf{k},\eta) = \frac{4}{3}\delta_{\rm b}(\mathbf{k},\eta) = -2\Phi_{\rm ini}(\mathbf{k})$$

- A slow-roll inflation naturally predicts a nearly scale-invariant primordial power spectrum (also slightly red tilted spectrum)

$$\Delta^2_{\delta\phi}(k) = \left(\frac{H(t)}{2\pi}\right)^2\Bigg|_{k=aH(t)}$$

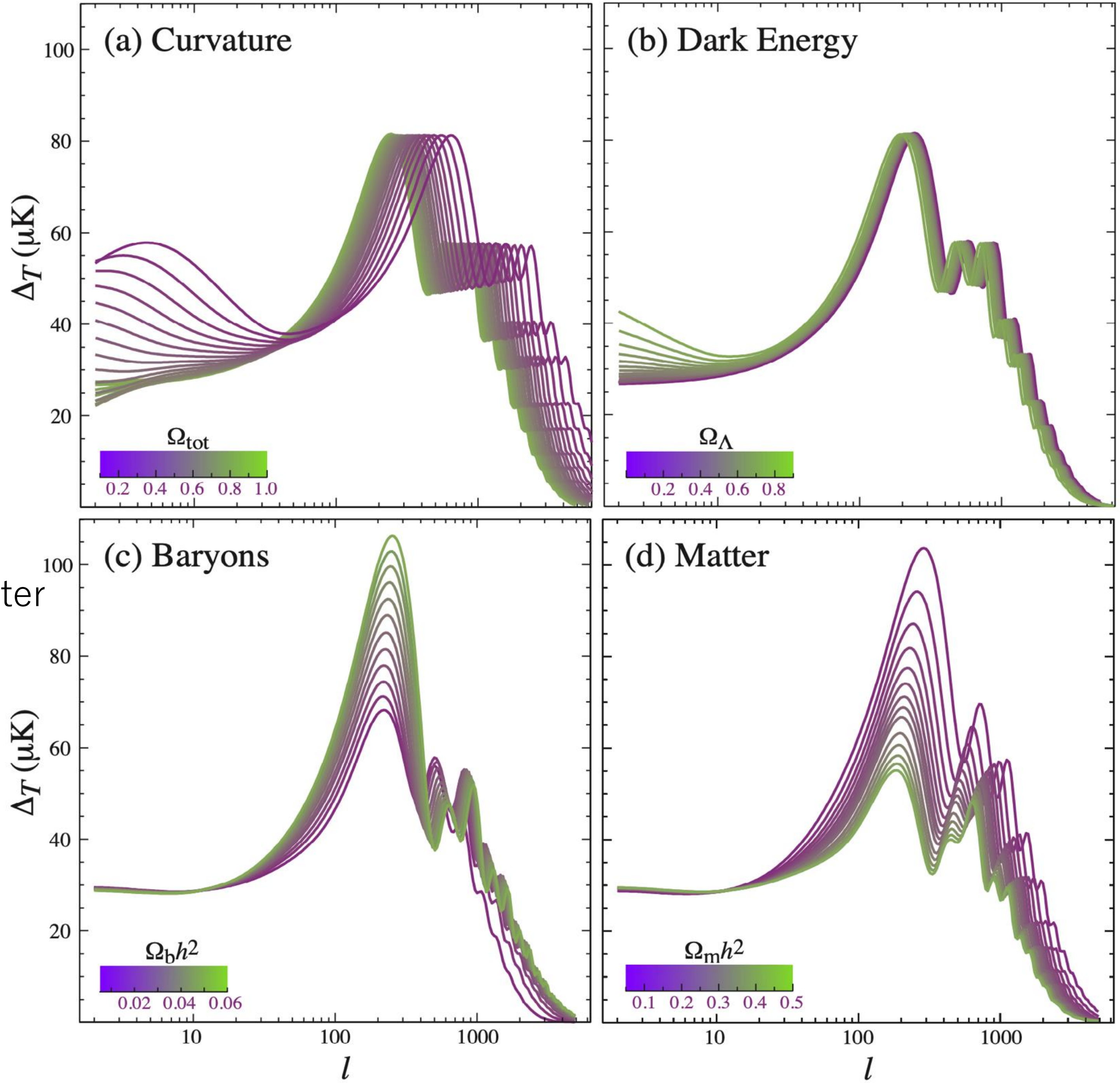
Understanding the universe with physics: ΛCDM model



Jim Peebles
2019 Nobel Prize

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi + \sigma_T\bar{n}_e\left[\Theta_0 - \Theta + \mu v_b\right]$$
$$\dot{\delta}_{\rm dm} + ikv_{\rm dm} = -3\dot{\Phi}$$
$$\dot{v}_{\rm dm} + \frac{\dot{a}}{a}v_{\rm dm} = -ik\Psi$$
$$\dot{\delta}_{\rm b} + ikv_{\rm b} = -3\dot{\Phi}$$
$$\dot{v}_{\rm b} + \frac{\dot{a}}{a}v_{\rm b} = -ik\Psi - \frac{\sigma_T\bar{n}_e}{R}\left[v_{\rm b} + 3i\Theta_1\right]$$
$$\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot{\Phi} - ik\mu\Psi$$

- Dark matter
- Dark energy
- Ordinary matter
- Primordial perturbations (2 parameters)
- (the amount of free electron in the late universe)



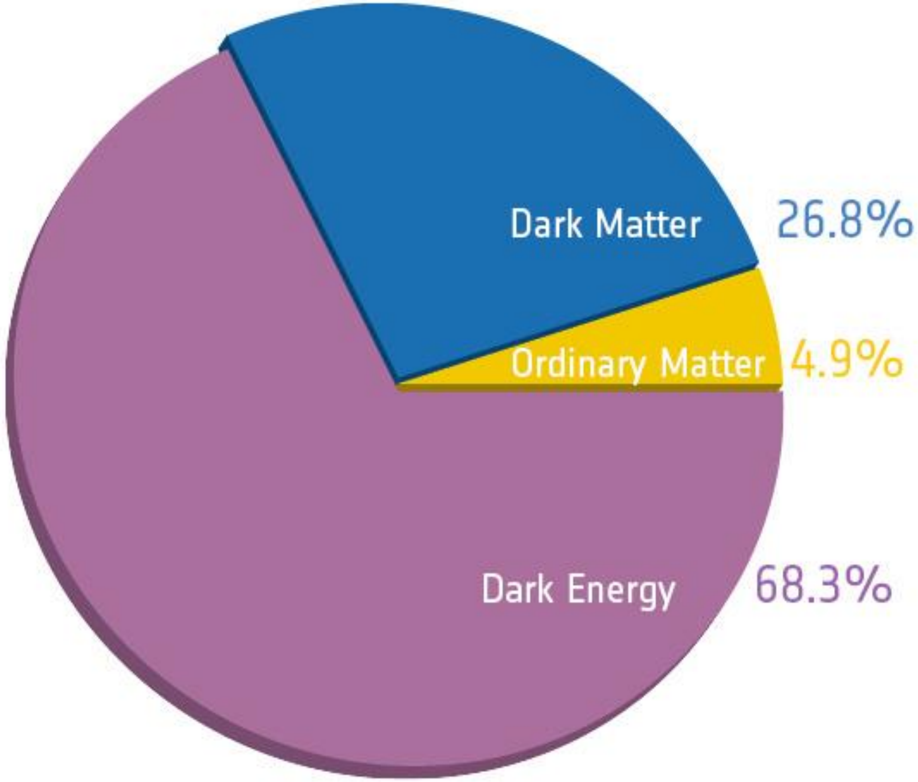
Curvature

Dark energy

Ordinary matter

Dark matter

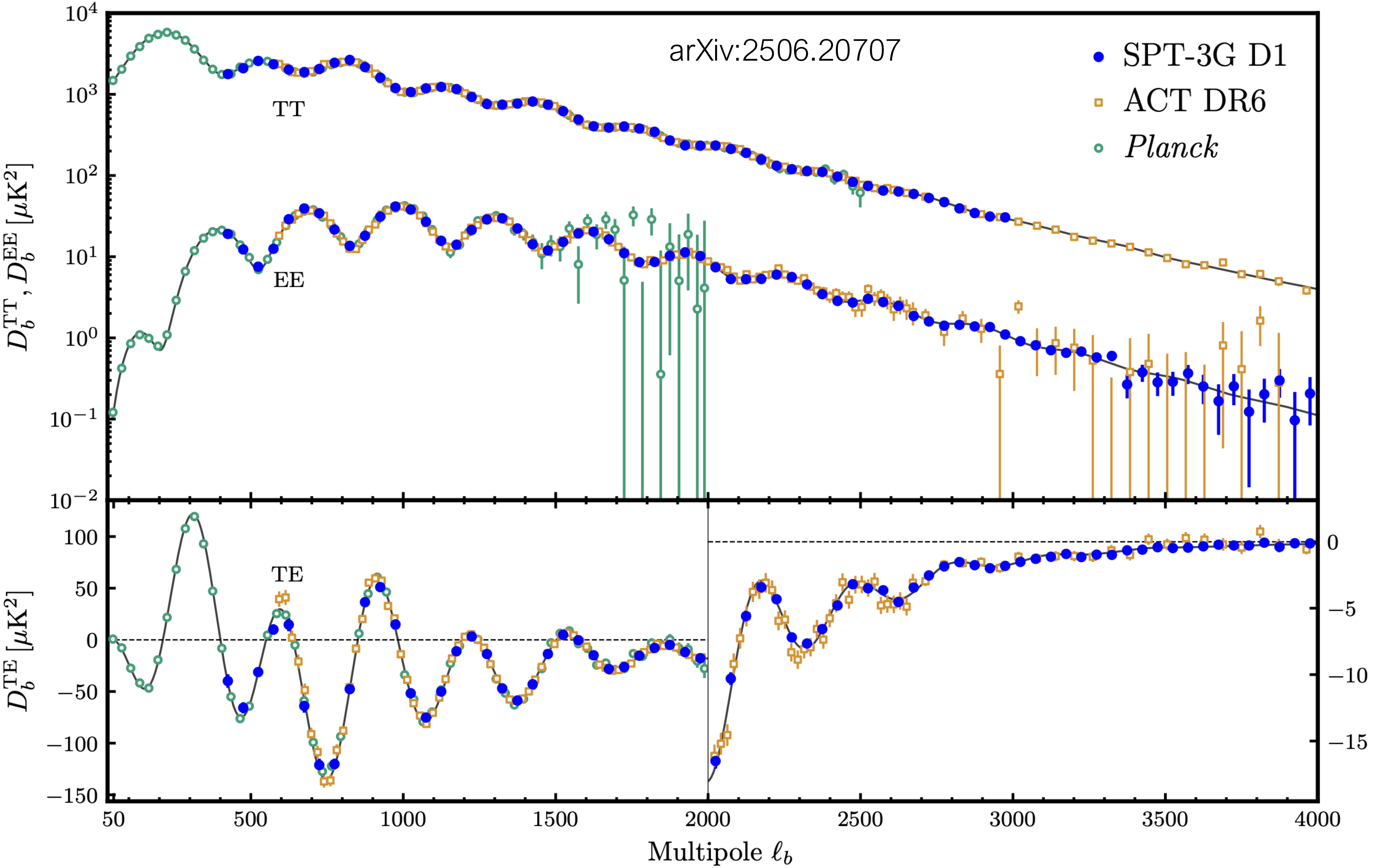
Hu & Dodelson

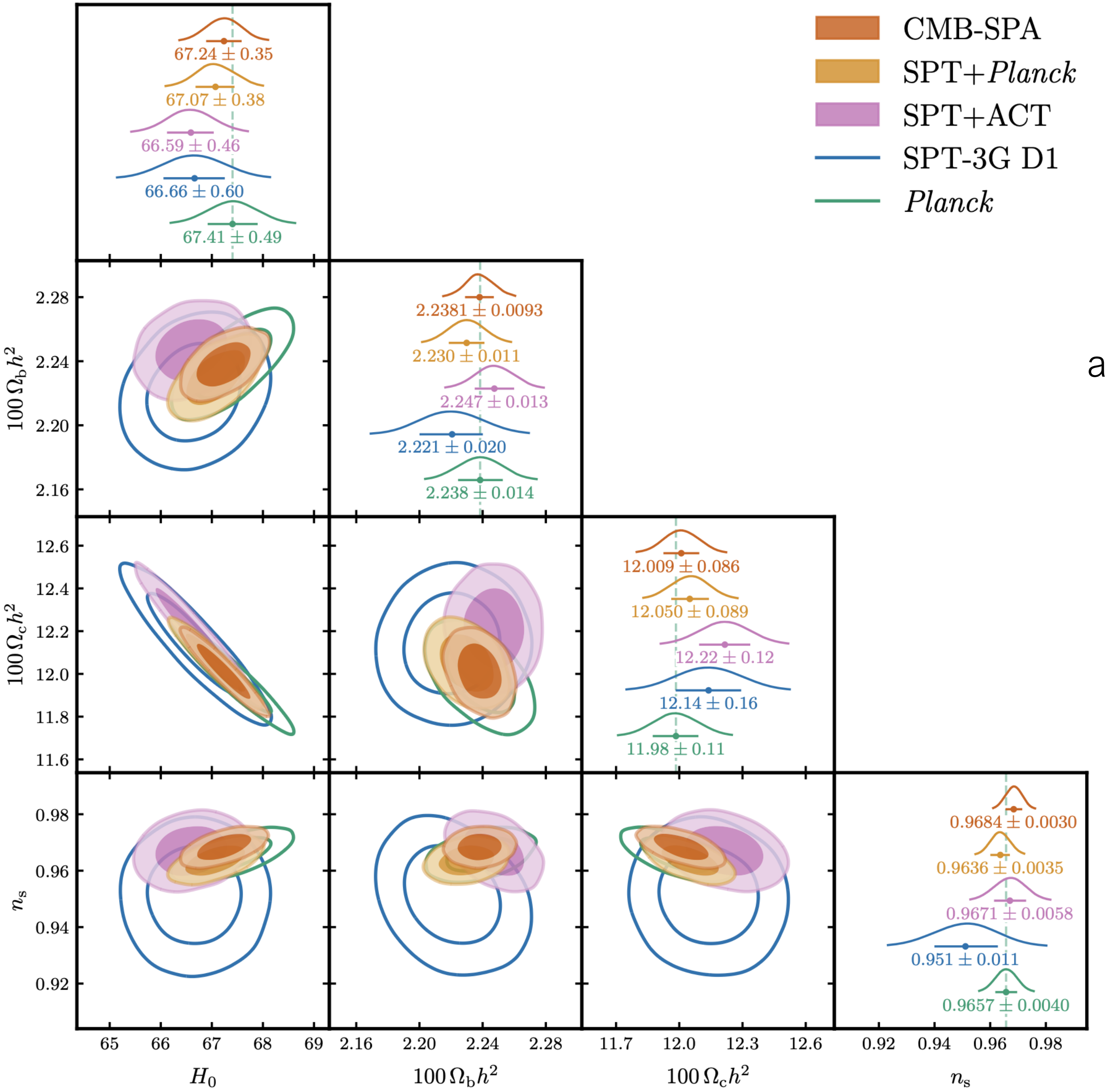


Dark Energy: 68.3%
Dark matter: 26.8%
Baryon (ordinary mater): 4.9%

95% of the Universe is “dark” unknown components

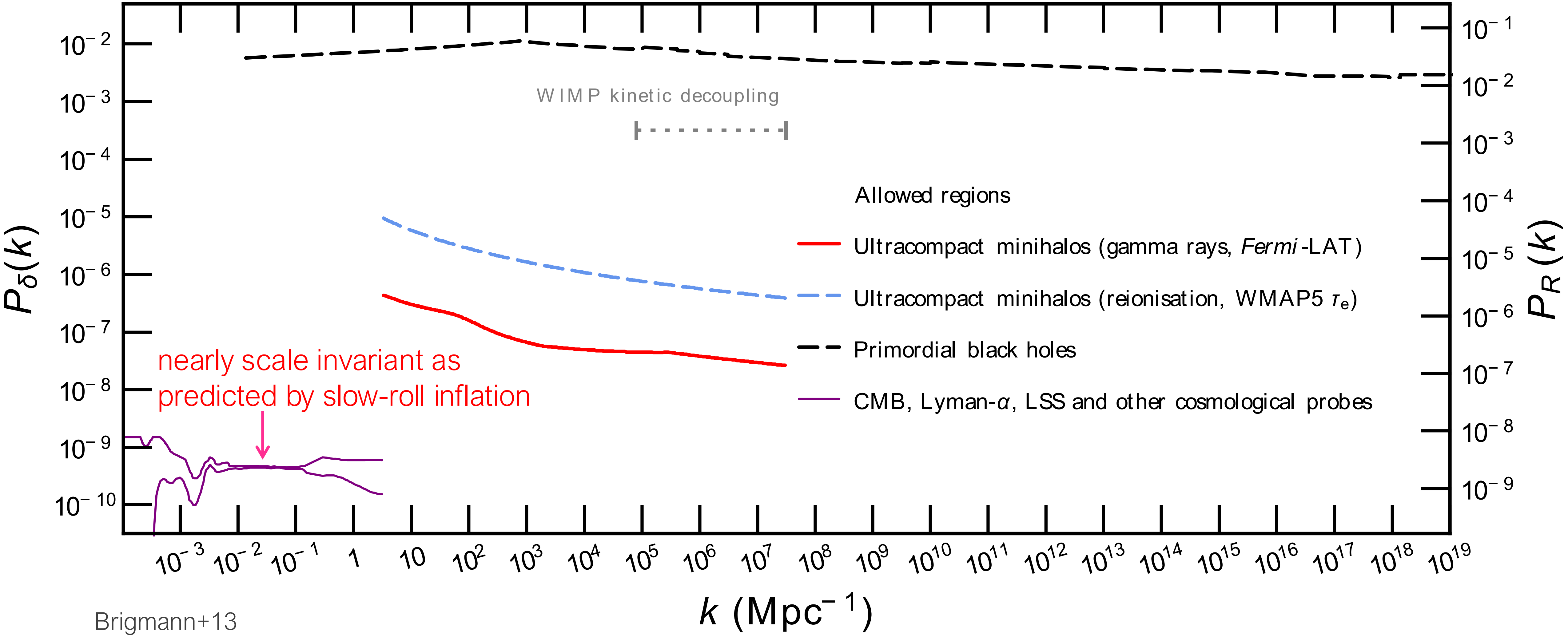
Parameter	TT+lowE 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022
$\Omega_c h^2$	0.1206 ± 0.0021
$100\theta_{MC}$	1.04077 ± 0.00047
τ	0.0522 ± 0.0080
$\ln(10^{10} A_s)$	3.040 ± 0.016
n_s	0.9626 ± 0.0057
H_0 [km s^{-1} Mpc^{-1}] . .	66.88 ± 0.92
Ω_Λ	0.679 ± 0.013
Ω_m	0.321 ± 0.013

arXiv:2506.20707
SPT-3G D1
ACT DR6
Planck
TT
EE
TE
$D_b^{\mathrm{TT}}, D_b^{\mathrm{EE}}\,[\mu\mathrm{K}^2]$
$D_b^{\mathrm{TE}}\,[\mu\mathrm{K}^2]$
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
100
50
0
−50
−100
−150
0
−5
−10
−15
50
500
1000
1500
2000
2500
3000
3500
4000
Multipole ℓ_b



arXiv:2506.20707

Primordial fluctuations are scale-invariant on all the scales?



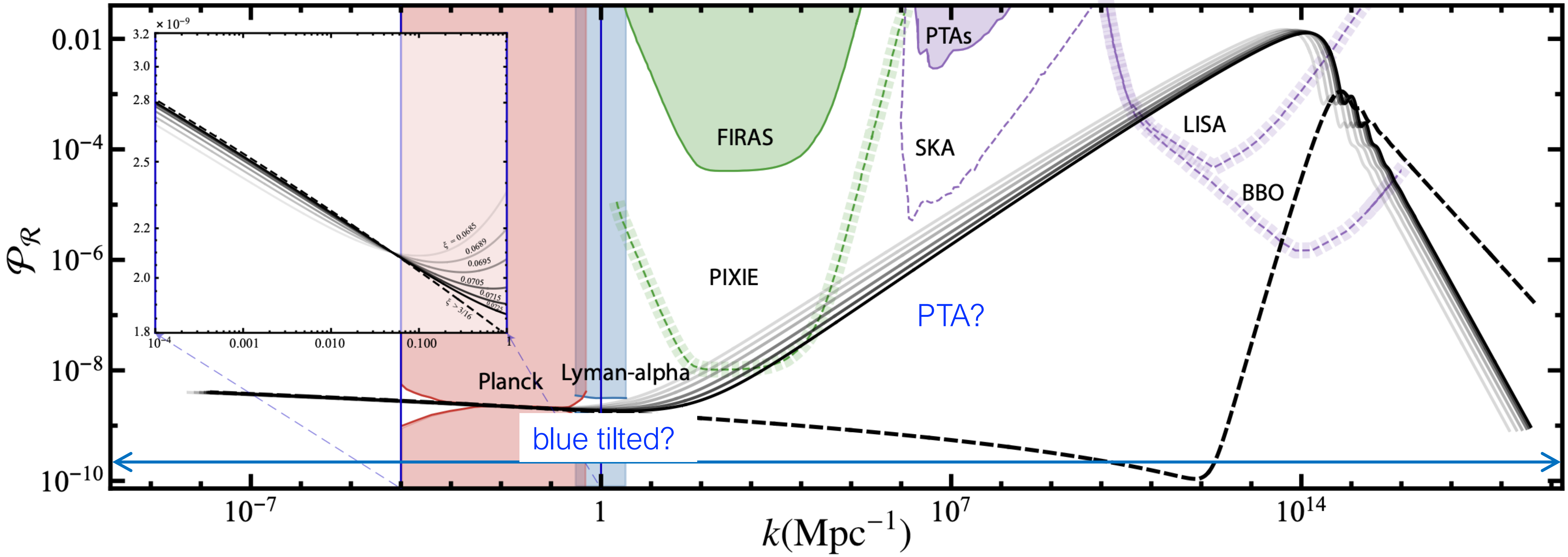
nearly scale invariant as
predicted by slow-roll inflation

Brigmann+13

Kohri & Wang, arXiv:2506.06797

we should be open-minded!

PBH? secondary GW

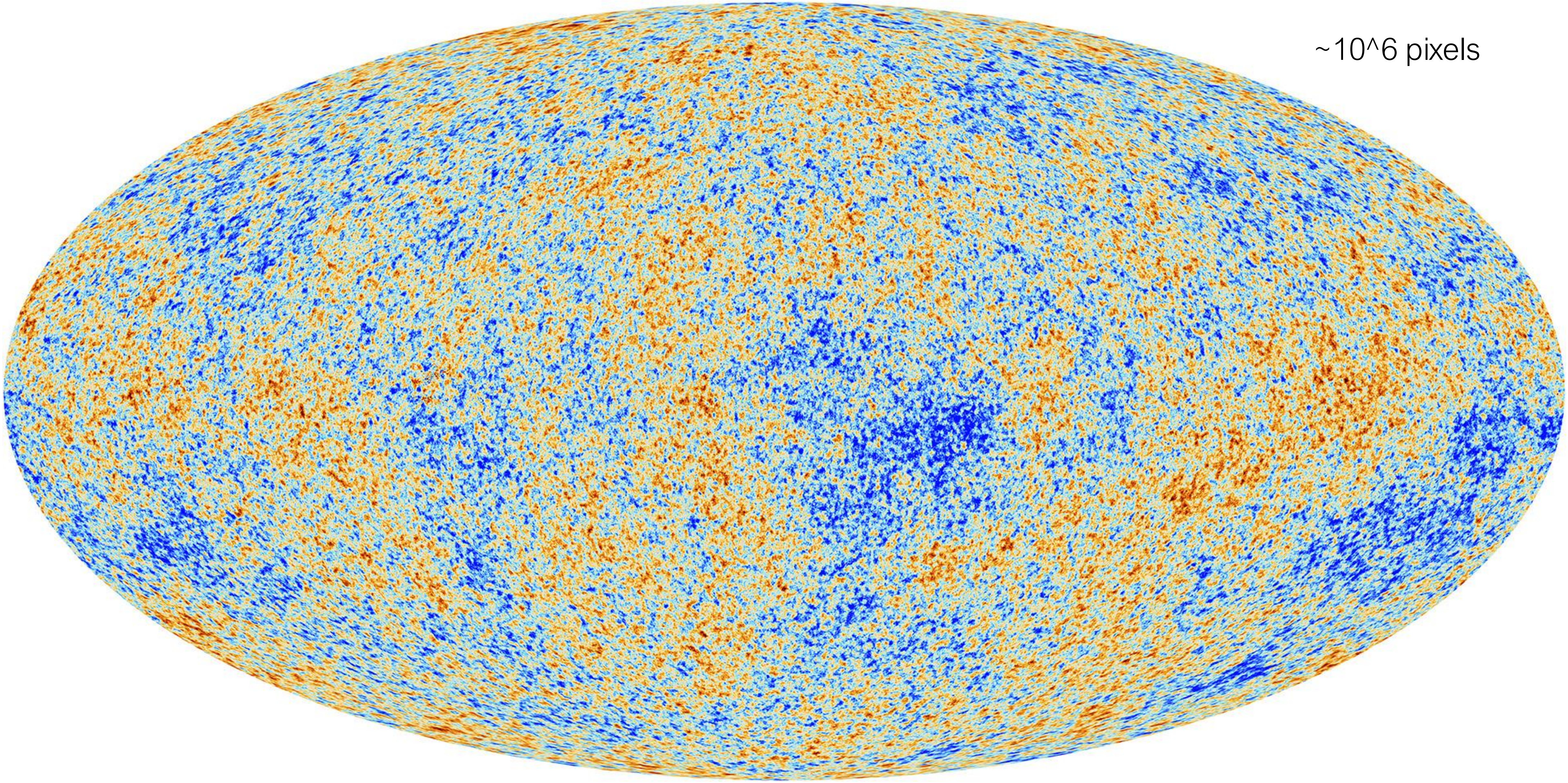


PTA?

blue tilted?

in the range of 10^26

how to analyze “data”?



~10^6 pixels

random Gaussian field

- cosmological fluctuation field

$$\mathbf{d} \equiv \{d(\mathbf{x}_1), d(\mathbf{x}_2), \ldots, d(\mathbf{x}_{N_{\rm grid}})\}$$

the field can be …

$$\frac{T(\hat{\mathbf{n}})}{\bar{T}} - 1, \frac{n_g(\mathbf{x})}{\bar{n}_g} - 1, \ldots$$

- multivariate Gaussian field (BBKS 84; Bond & Efstathiou 87)

$$\mathcal{P}(\mathbf{d}) = \frac{1}{\sqrt{(2\pi)^{N_{\rm grid}} \det(C)}} \exp\left[-\frac{1}{2} d_i C^{-1}_{ij} d_j\right]$$

- two-point correlation function and power spectrum

$$\underline{C_{ij}} \equiv \langle d(\mathbf{x}_i) d(\mathbf{x}_j) \rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \underline{P_d(|\mathbf{k}|)} e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}$$

2pt function

power spectrum

$$\tilde{d}_{\mathbf{k}} = |\tilde{d}_{\mathbf{k}}| e^{i\phi_{\mathbf{k}}} \to P_d(|\mathbf{k}|) = \langle |\tilde{d}_{\mathbf{k}}|^2$$

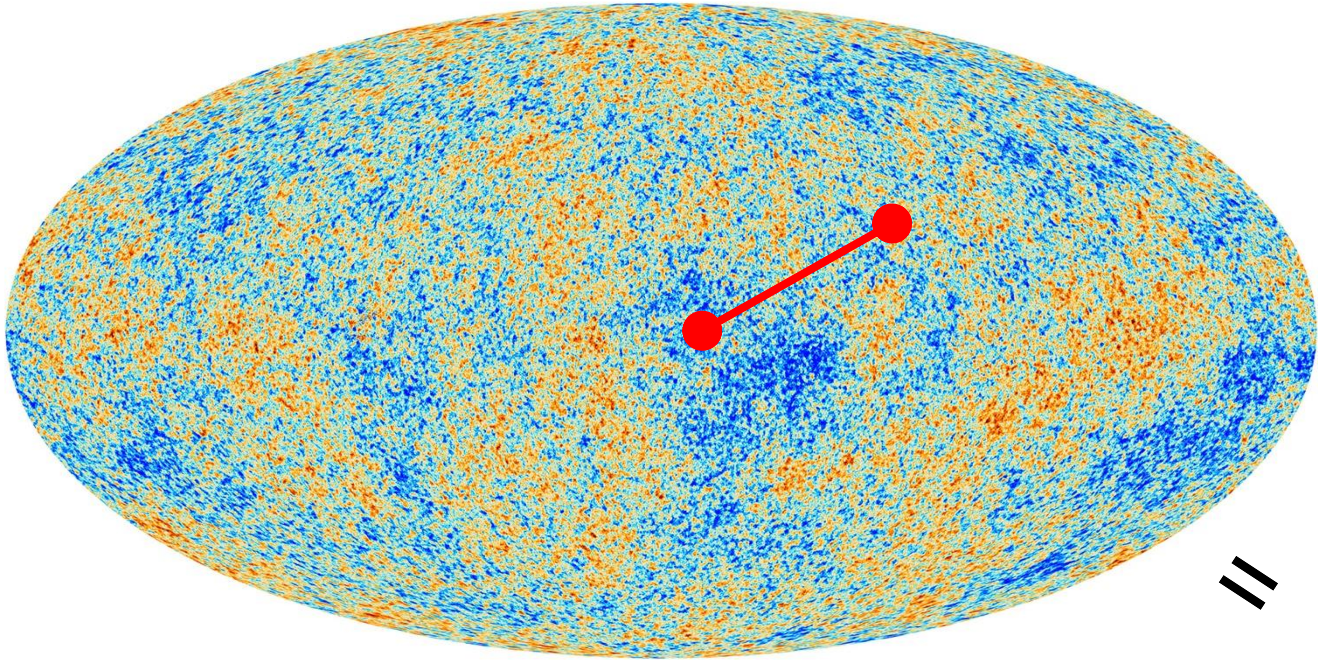
- You can compute any statistics (e.g, n-point correlation function), but 2pt func. (power spectrum) contains the full information

$$\langle d(\mathbf{x}_1) d(\mathbf{x}_2) d(\mathbf{x}_3) \rangle = \int \mathrm{d}d_1 \cdots \mathrm{d}d_{N_{\rm grid}}\ d_1 d_2 d_3 \mathcal{P}(\mathbf{d})$$

$$\langle x^2 \rangle = \sigma^2$$

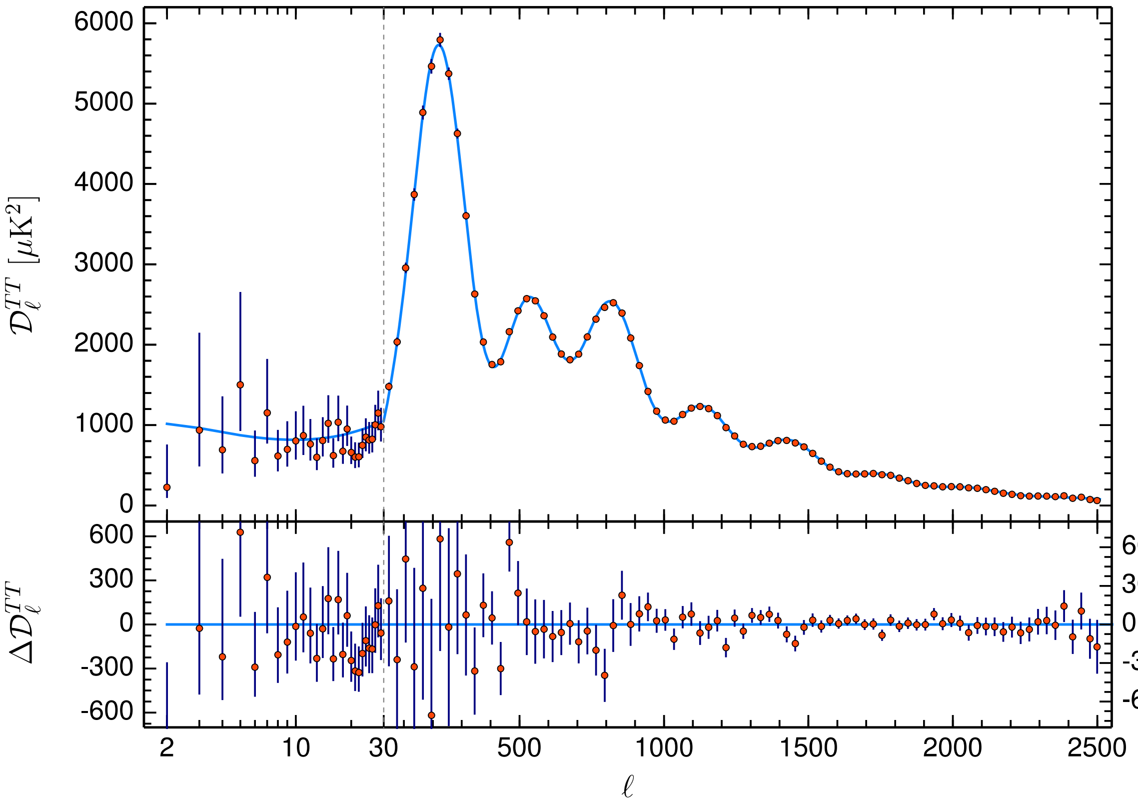
$$\langle x^4 \rangle = 3\sigma^4$$

$$\ldots$$



$$\frac{\delta T(\hat{\mathbf{n}})}{\bar{T}} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

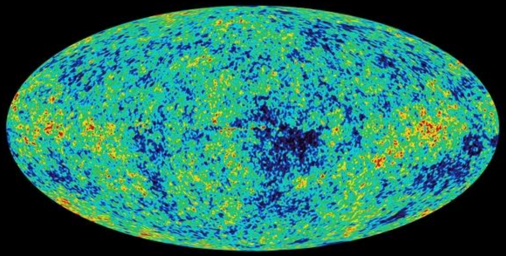
=



huge data reduction: ~10^6 ⇒ ~10^3 or even less

for CMB (which is very close to Gaussian), the power spectrum is sufficient to extract its full information

ΛCDM structure formation model

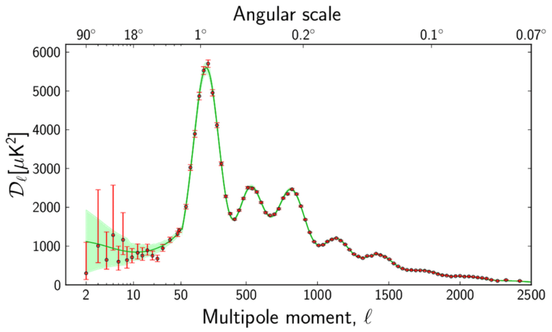


CMB at z~1000

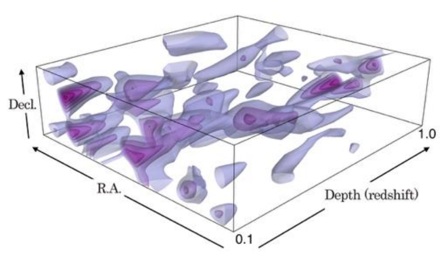
time

gravity (dark matter)

cosmic expansion (dark energy)

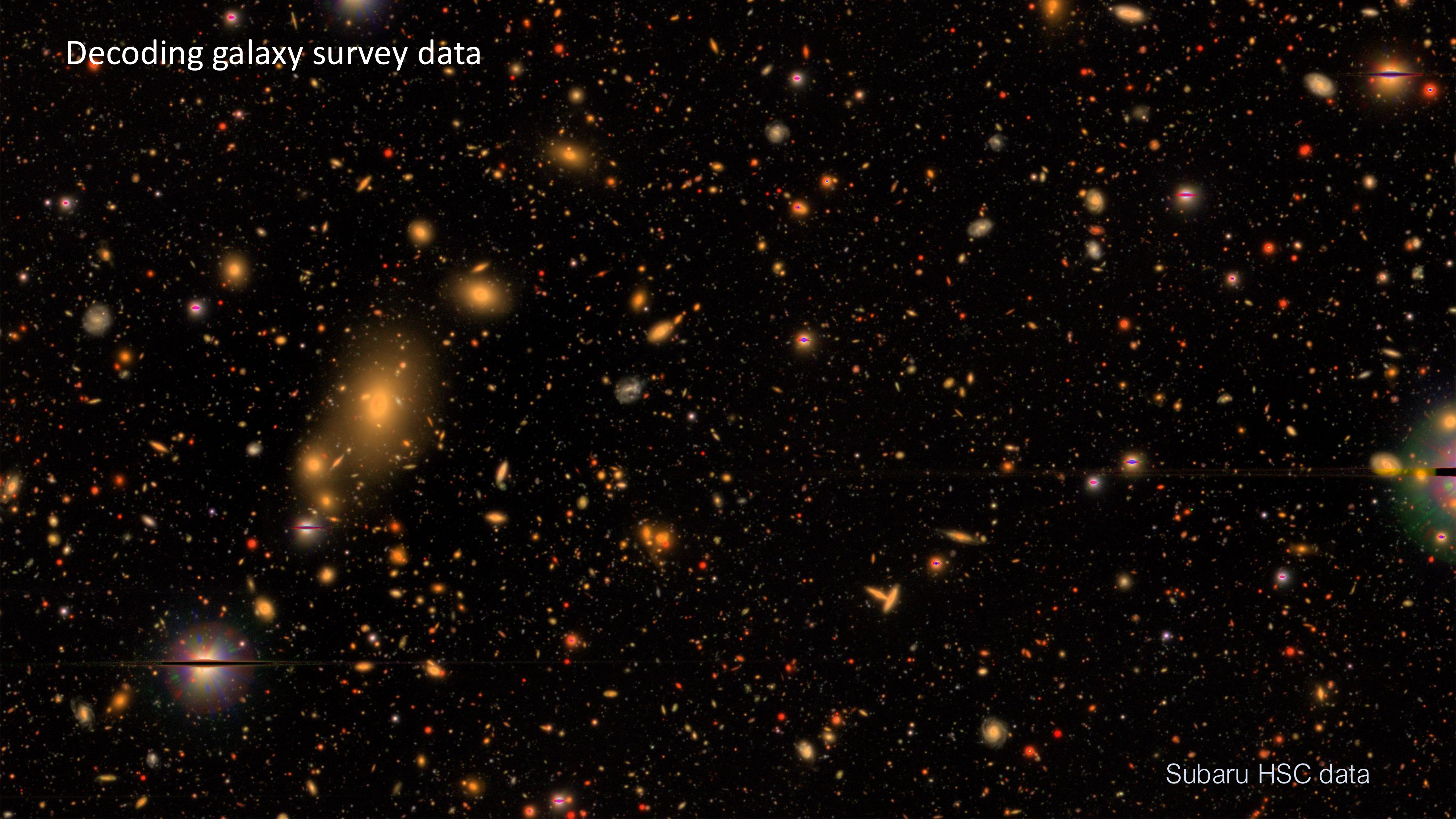


ΛCDM=~6 parameters



Galaxy surveys directly measure “**lumpiness**” of the late universe

Decoding galaxy survey data

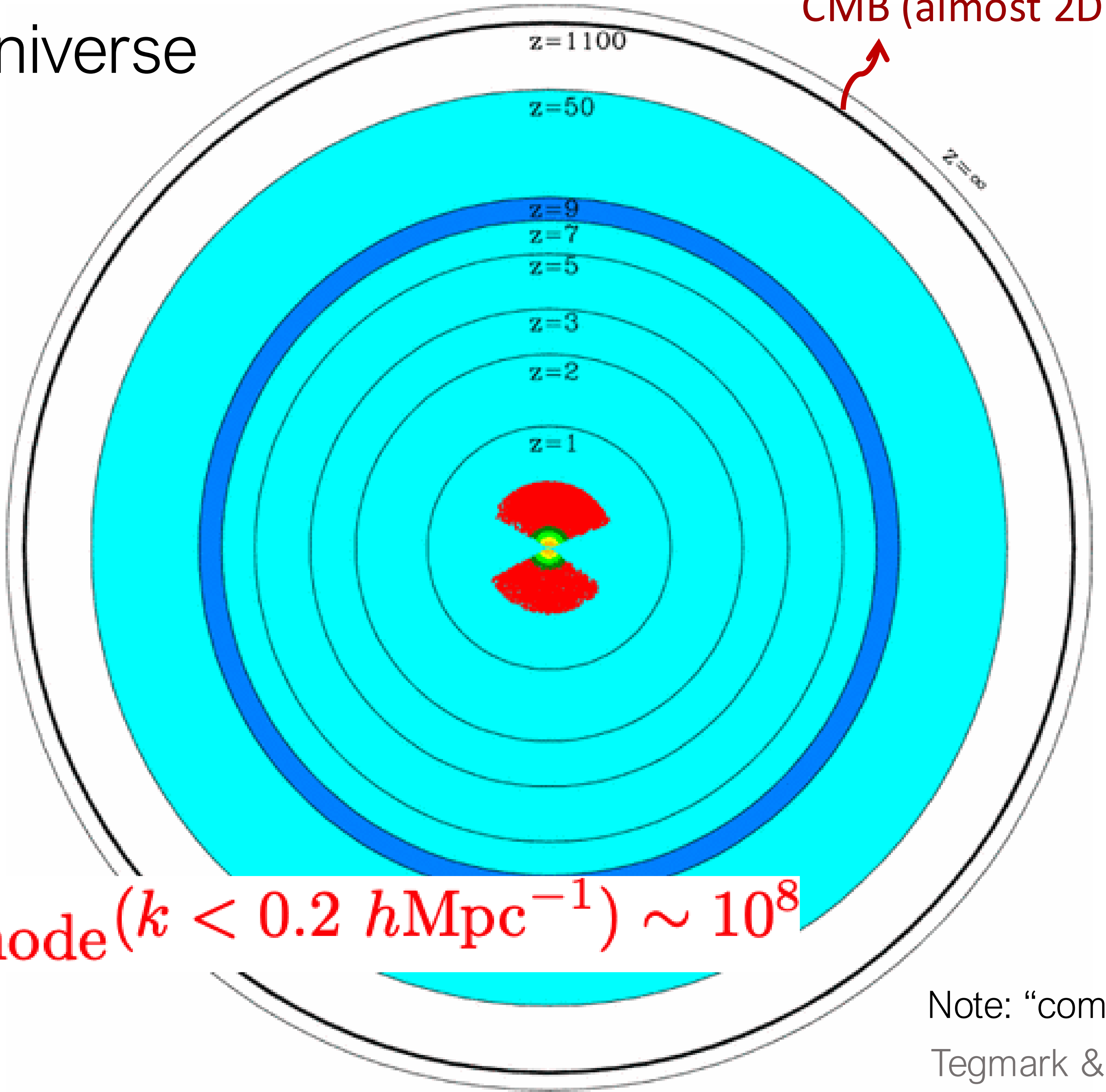


Subaru HSC data

galaxy survey sees (3+1)D universe (time and space)

“Observable” universe

CMB (almost 2D) (width ~ 5Mpc)



$N_{\mathrm{CMB,mode}} \sim 10^6$

$N_{\text{galax-survey, mode}}(k < 0.2\ h\mathrm{Mpc}^{-1}) \sim 10^8$

Note: “comoving” coordinates

Tegmark & Zaldarriaga 09

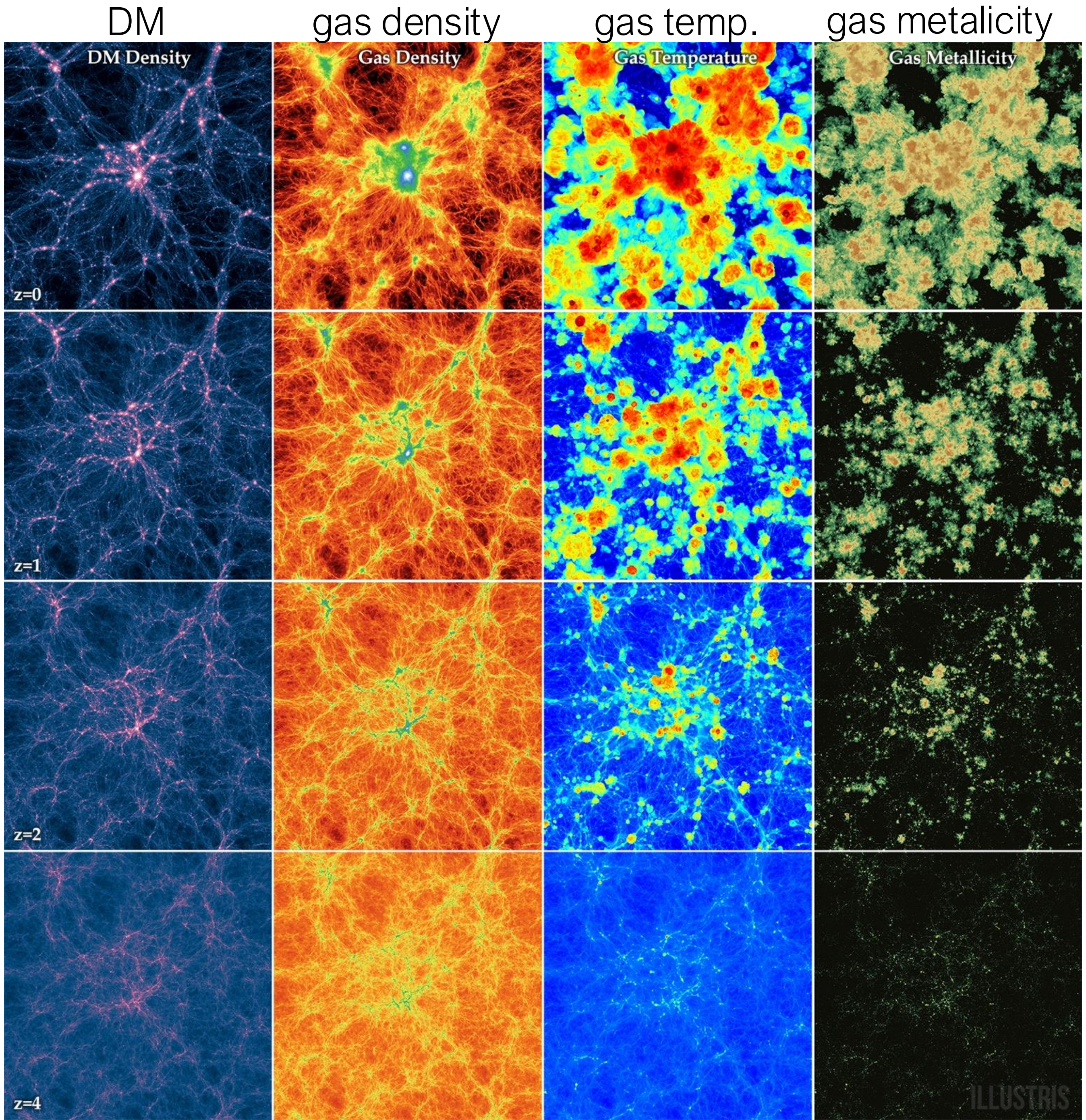
Nonlinear structure formation (late-time universe)

- The fluctuations evolve to form nonlinear structures (stars, galaxies, filaments ...) on small scales $O(\delta_g) > 1$
- The linear theory is no longer valid for describing the time evolution of small-scale structures (note that structures on large scales are still in the linear regime)
- Nonlinear structure formation cause mode coupling of different Fourier modes
  - For instance, a perturbative picture of nonlinear structure formation gives

$$\delta(\mathbf{k}) = \delta^{(1)}(\mathbf{k}) + \delta^{(2)}(\mathbf{k}) + \cdots$$

where $$\delta^{(2)}(\mathbf{k}) = \int \frac{d\mathbf{q}_1}{(2\pi)^3}\frac{d\mathbf{q}_2}{(2\pi)^3}(2\pi)^3\delta_D(\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2)F(\mathbf{q}_1,\mathbf{q}_2)\delta^{(1)}(\mathbf{q}_1)\delta^{(1)}(\mathbf{q}_2)$$

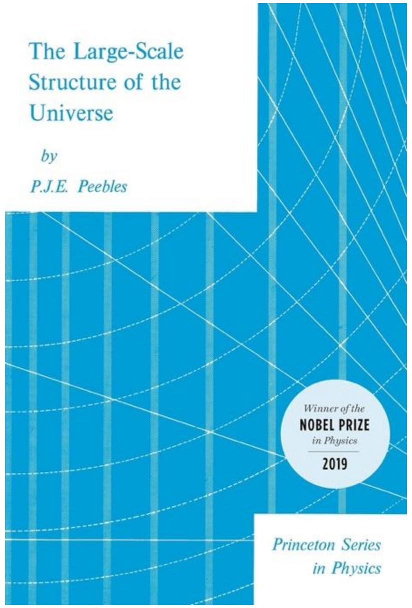
- Approaches to modeling nonlinear structure formation for galaxy surveys
  - Cosmological N-body (gravity only) simulations: now easy to run and very accurate
  - Cosmological hydrodynamical simulations: ideal, but very expensive
  - Analytical cosmological perturbation theory (effective field of theory of LSS): See Zvonimir's talk



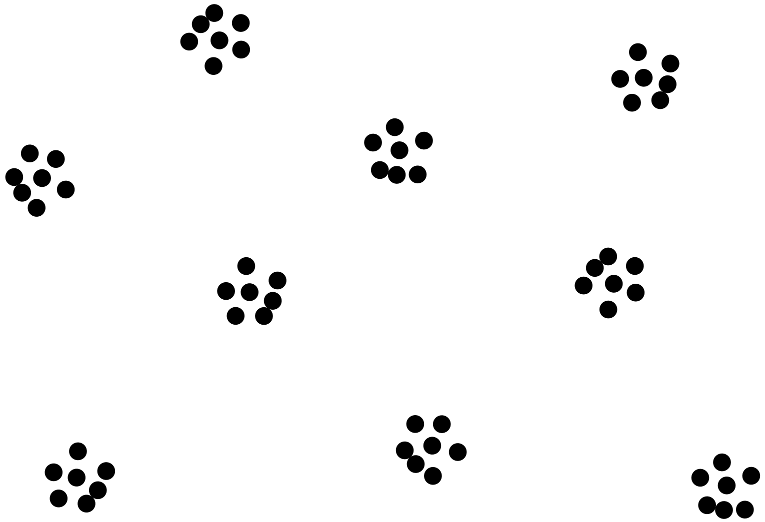
- Nonlinear structure formation on small scales (galaxy formation, …) are very complicated – still an open question
- Still impossible or very difficult to accurately simulate galaxy formation (no simulation using star-by-star particles)
- it involves a wealth of physical processes (also various observational data)

from Illustris simulation project

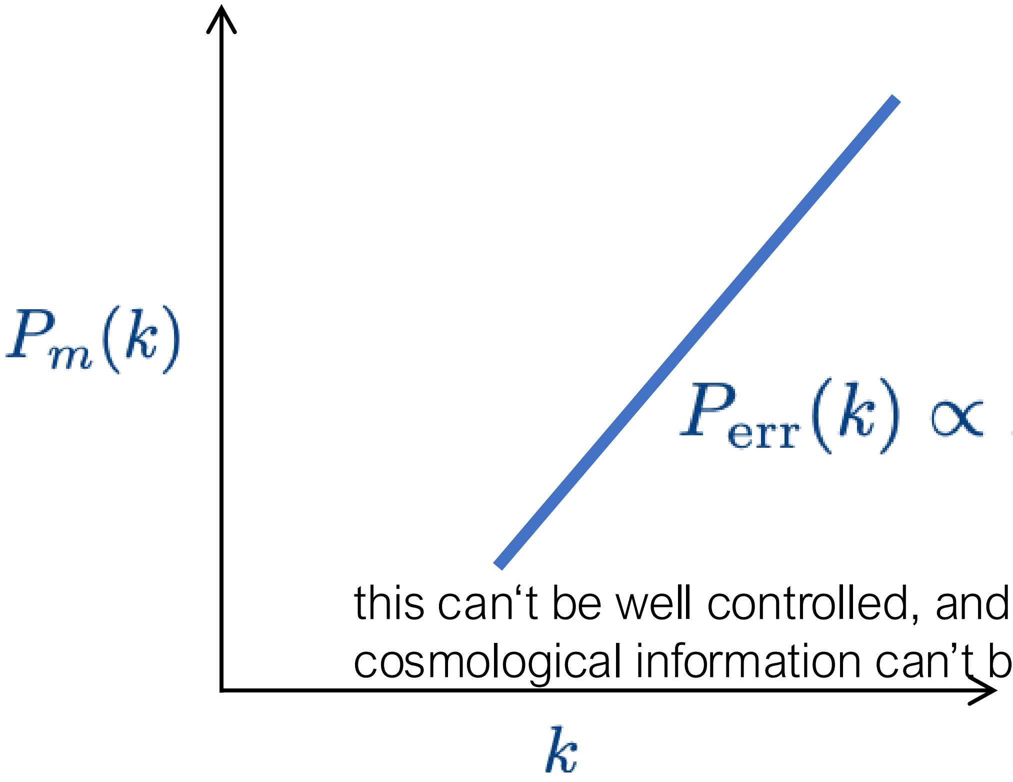
The backreaction from small scales to large scales: the effect of a lumpy matter distribution



- Peebles (1974; 80 textbook) considered an expanding universe containing lumpy (self-gravitating) clumps with a random spatial distribution (also Fry 94)
- Used an argument of mass and momentum conservations

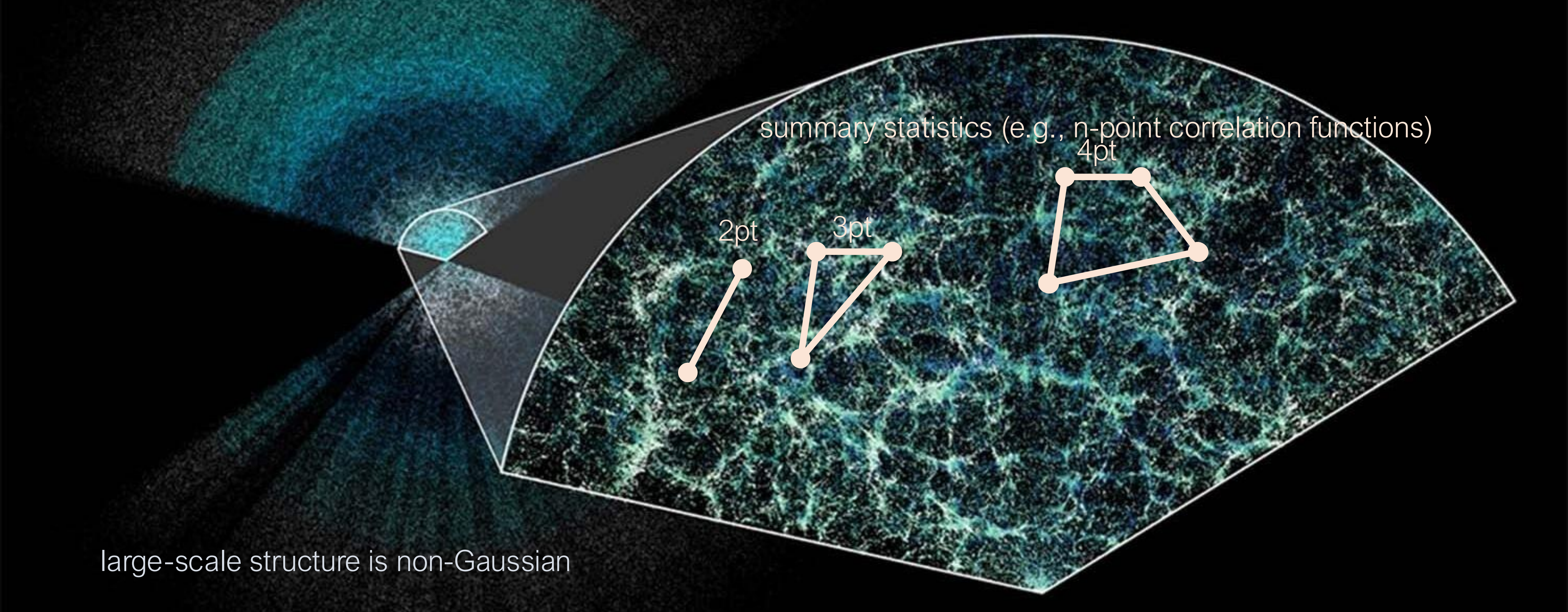


self-gravitating system (e.g., galaxy) that can‘t be well modeled



this can‘t be well controlled, and thus cosmological information can’t be extracted

Problem: How can we extract the “full” cosmological information from galaxy survey data?



summary statistics (e.g., n-point correlation functions)

large-scale structure is non-Gaussian

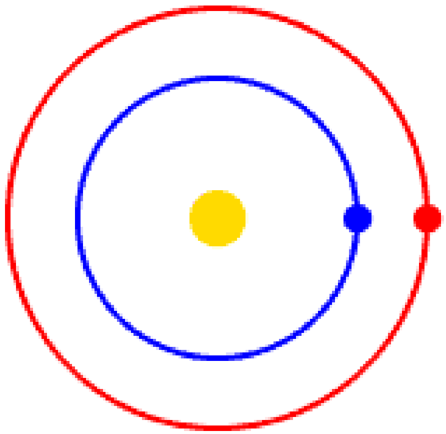
forward modeling

$$\delta_{\rm final}(\mathbf{x}) = F[\delta_{\rm ini}(\mathbf{q}), \delta^2_{\rm ini}, (\partial\partial\Phi_{\rm ini})^2, \cdots]$$

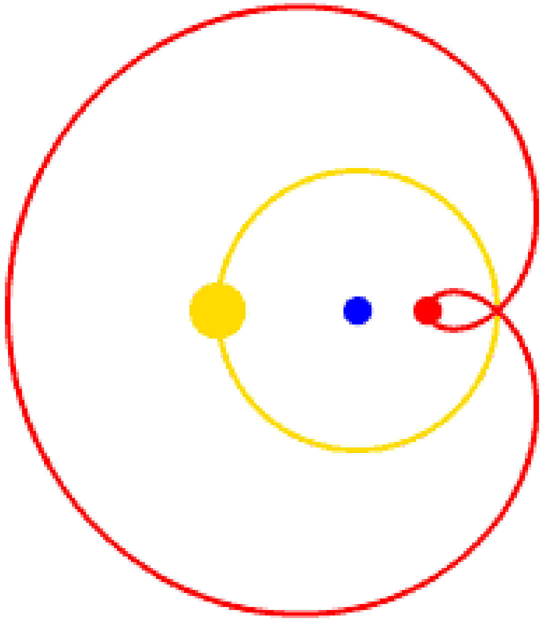
Motion of mars on the sky

Observation = Truth

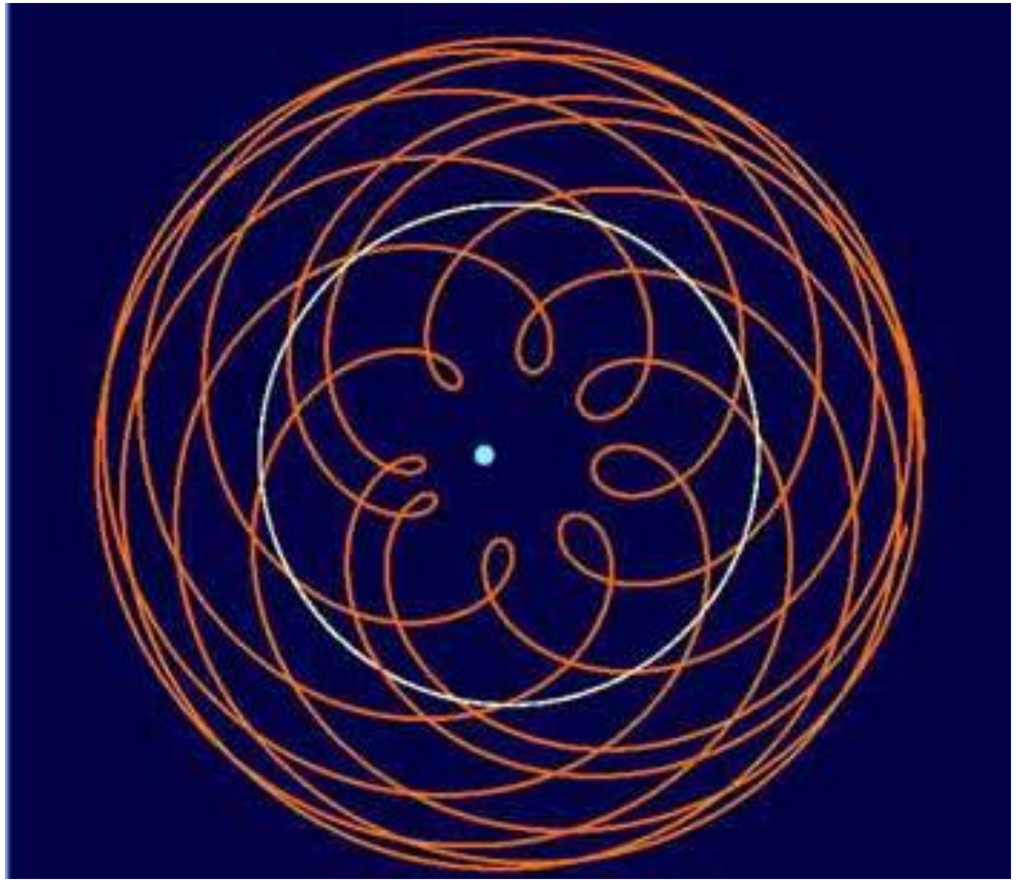
Theory = Understanding/Interpretation



Heliocentric theory – this is much more beautiful (or natural)

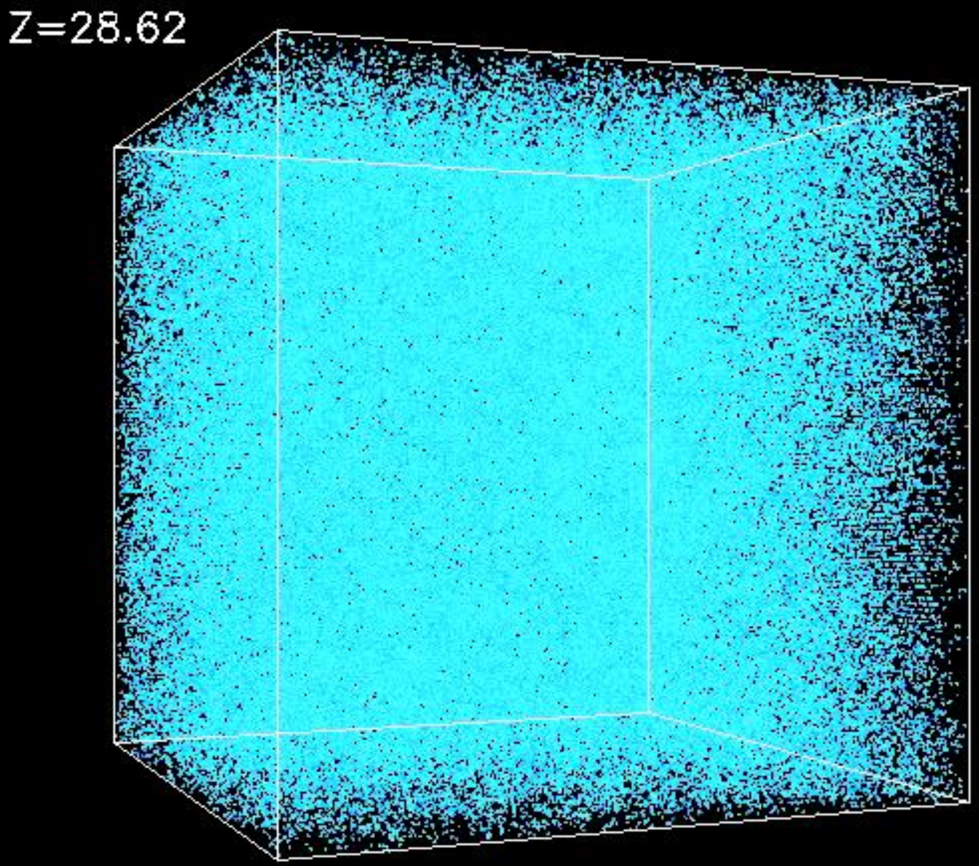


Geocentric theory



The motion of Mars looks like this assuming the geocentric theory

separate universe approach of LSS



- any simulations are done in a finite volume (ignore super-box modes). simulations usually employ periodic boundary conditions
- simulations are expensive: want to use as many particles as possible in a small volume to simulate nonlinear structure formation/galaxy formation
- any galaxy survey covers a finite volume

Separate universe approach (peak-background split)

Baldauf+ 11; Li, Hu & MT 14

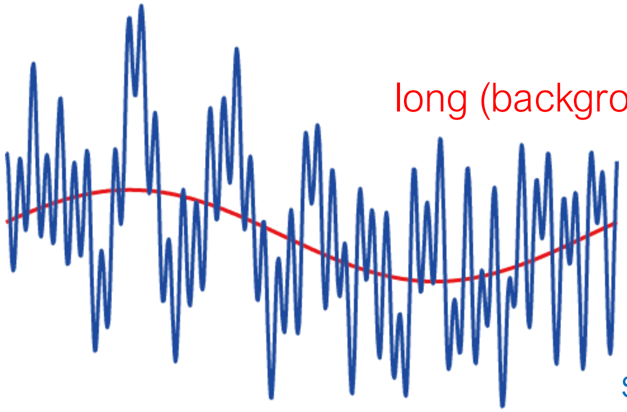
- Cosmological Newtonian metric

$$\mathrm{d}s^2 = -\left[1+2\Phi(\mathbf{x},t)\right]\mathrm{d}t^2 + a(t)^2\left[1-2\Phi(\mathbf{x},t)\right]\mathrm{d}\mathbf{x}^2$$

here $\nabla^2\Phi = 4\pi G a^2\left[\bar{\rho}_\mathrm{c}\delta_\mathrm{c} + \bar{\rho}_\mathrm{b}\delta_\mathrm{b} + \cdots\right]$

- Long- and short-wavelength mode splits

$$\Phi(\mathbf{x},t) \simeq \Phi^{(l)}(\mathbf{x},t) + \Phi^{(s)}(\mathbf{x},t)$$



- Taylor expansion …

$$\begin{aligned}
\Phi^{(l)}(\mathbf{x},t) &\simeq \Phi^{(l)}(\mathbf{x}_0,t) + \partial_i\Phi^{(l)}\Big|_{\mathbf{x}_0}(x-x_0)^i + \frac{1}{2}\,\partial_i\partial_j\Phi^{(l)}\Big|_{\mathbf{x}_0}(x-x_0)^i(x-x_0)^j + O(\partial^3\Phi) \\
&= \Phi^{(l)}(\mathbf{x}_0,t) + \partial_i\Phi^{(l)}\Big|_{\mathbf{x}_0} x^i + \frac{1}{6}\,\nabla^2\Phi^{(l)}\Big|_{\mathbf{x}_0} x^i x_i + \frac{1}{2}\left[\partial_i\partial_j - \frac{\nabla^2}{3}\delta^K_{ij}\right]\Phi^{(l)}\Big|_{\mathbf{x}_0} x^i x^j + O(\partial^3\Phi) \\
&= \Phi^{(l)}(\mathbf{x}_0,t) + \partial_i\Phi^{(l)}\Big|_{\mathbf{x}_0} x^i + \underbrace{\frac{1}{6}\,\nabla^2\Phi^{(l)}}_{\text{isotropic}}\Big|_{\mathbf{x}_0} r^2 + \frac{1}{2}\underbrace{\left[\partial_i\partial_j - \frac{\nabla^2}{3}\delta^K_{ij}\right]\Phi^{(l)}\Big|_{\mathbf{x}_0}}_{\text{trace-less anisotropic (tide)}} x^i x^j + O(\partial^3\Phi)
\end{aligned}$$

separate universe approach (cont'd)

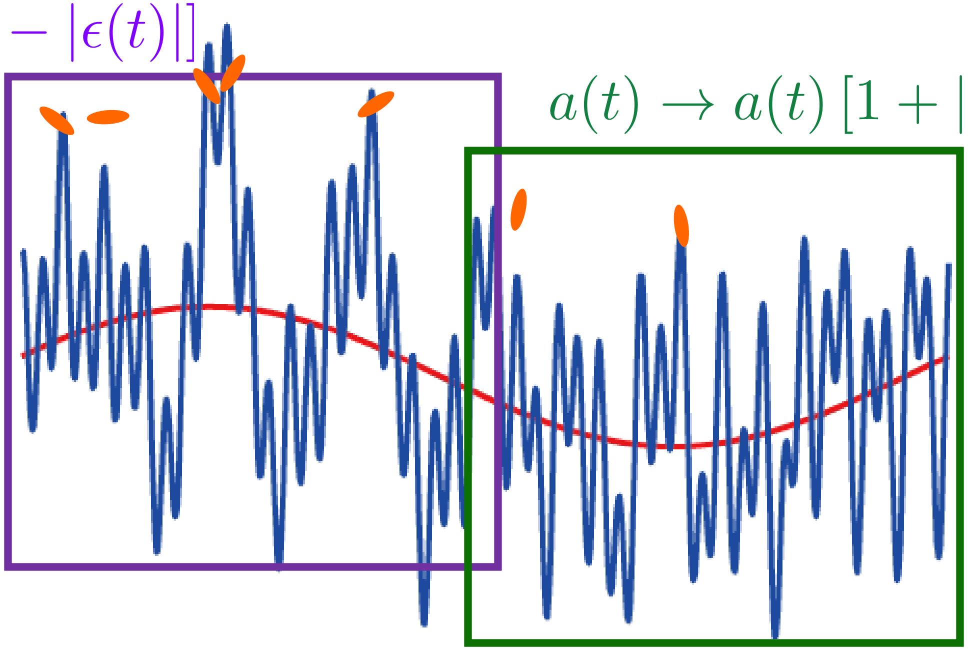
- isotropic part $$\frac{1}{6}\nabla^2\Phi^{(l)}\bigg|_{\mathbf{x}_0} r^2 = \frac{2}{3}\pi G a^2 \bar{\rho}_{\rm m}\delta_l(t) r^2$$

MT & Futamase 99
MT & Hu 13

This is equivalent to $\bar{\rho}_{\rm m} \longmapsto \bar{\rho}_{\rm m}\left[1+\delta_l(t)\right]$

furthermore $\longmapsto \begin{cases} \text{closed universe } K>0, & (\delta_l>0) \\ \text{open universe } K<0, & (\delta_l<0) \end{cases}$ for ΛCDM model

$a(t) \to a(t)\left[1-|\epsilon(t)|\right]$



$a(t) \to a(t)\left[1+|\epsilon(t)|\right]$

$\delta = \delta^{(l)} + \delta^{(s)}$

Galaxy bias for ΛCDM (Kaiser 84)

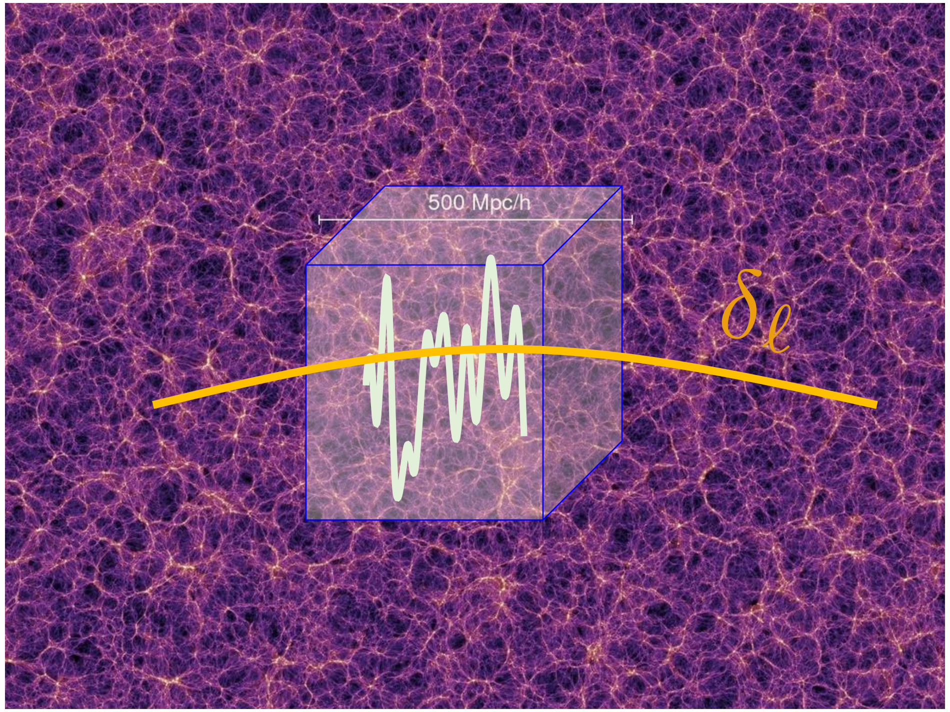
$$\delta_g = b_g \delta_{\rm m}^{(l)}$$

$$b_g \sim \frac{\partial \ln n_g}{\partial \delta^{(l)}} \sim \frac{\partial \ln n_g}{\partial \Omega_K}$$

Separate universe approach

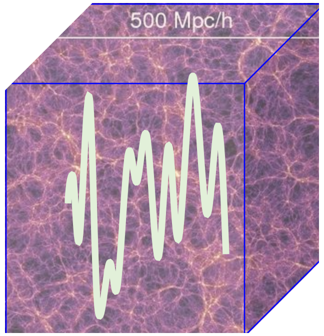
Li, Hu & MT 14a,b; Wagner+15
Baldauf+ 16; Lyzeyras+16
Barreira & Schmidt 17…

$$\Omega_K = 0$$



$$\simeq$$

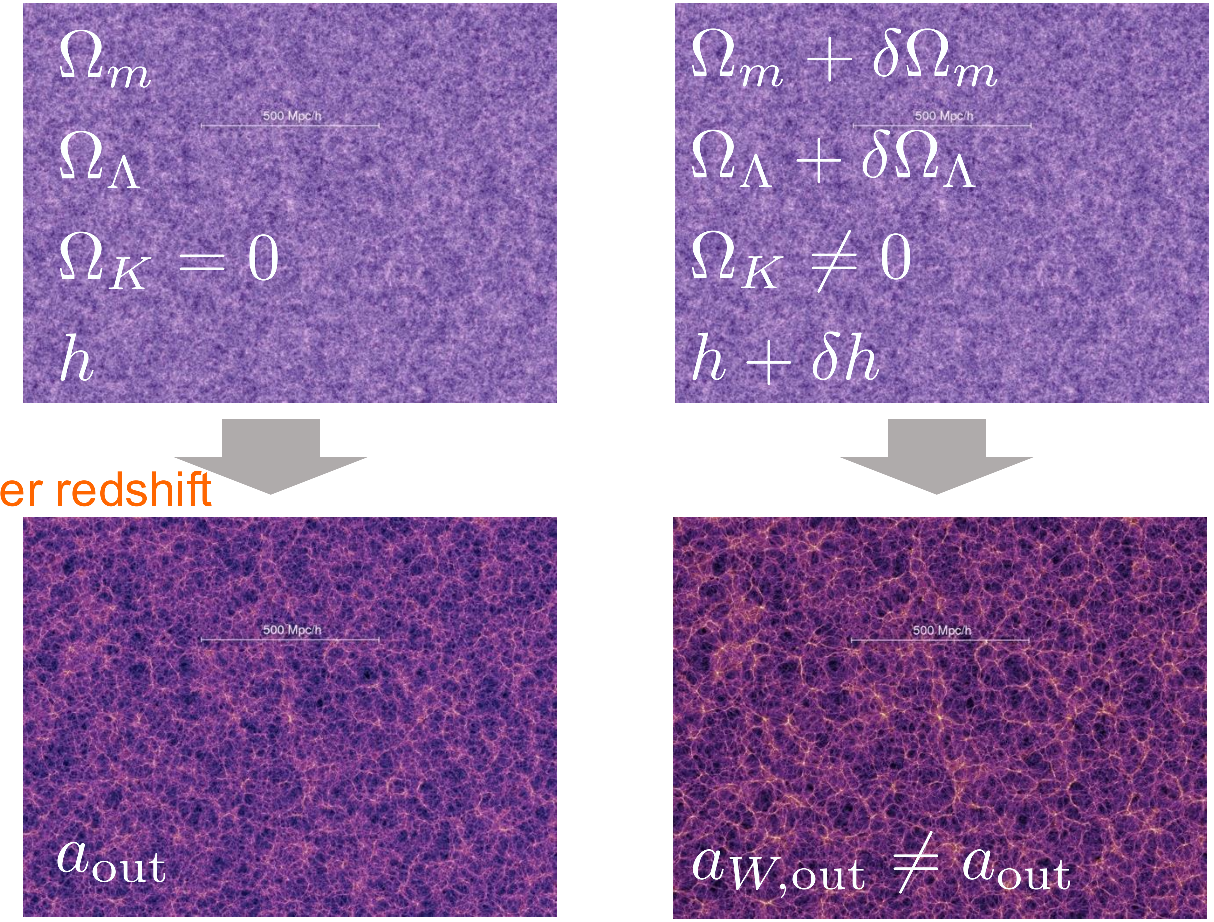
$$\Omega_K \sim \delta_\ell \neq 0$$



- Long-wavelength mode can be absorbed into a change in the background model (for the ΛCDM model, it can be absorbed into the curvature parameter)

Separate universe simulation

initial redshift



later redshift

For flat ΛCDM model case, the SU sim is set by

$$\rho_{\mathrm{m}W} = \bar{\rho}_{\mathrm{m}}\left[1 + \delta_\ell(z)\right]$$

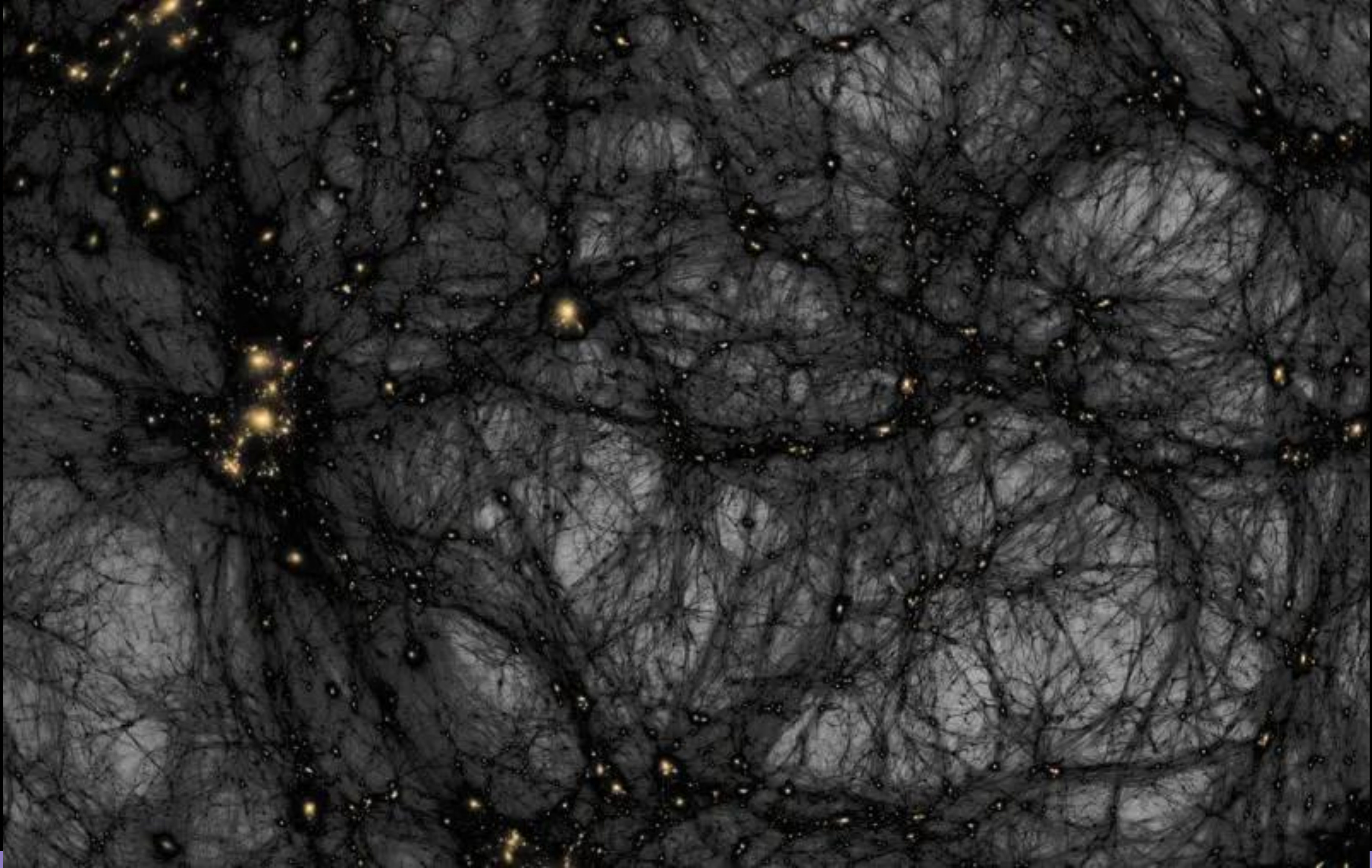
$$a_W \approx a\left(1 - \frac{\delta_\ell}{3}\right)$$

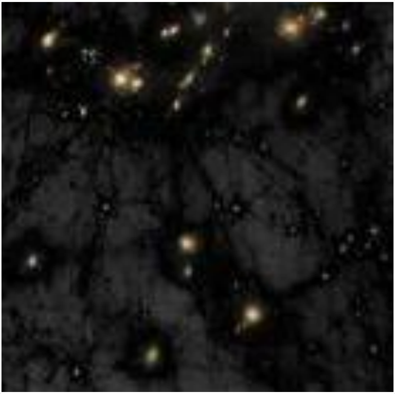
$$\frac{\delta h}{h} \approx -\frac{5\Omega_{\mathrm{m}}}{6}\frac{\delta_\ell}{D}$$

$$\frac{\delta\Omega_{\mathrm{m}}}{\Omega_{\mathrm{m}}} = \frac{\delta\Omega_\Lambda}{\Omega_\Lambda} \approx -2\frac{\delta h}{h}$$

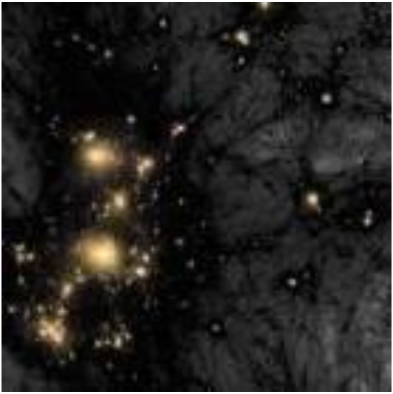
$$\Omega_{KW} = -\frac{5\Omega_{\mathrm{m}}}{3}\frac{\delta_\ell}{D}$$

- We can use the same initial seeds in the paired SU simulations
- We can compute the response of any observable to the long-wavelength mode

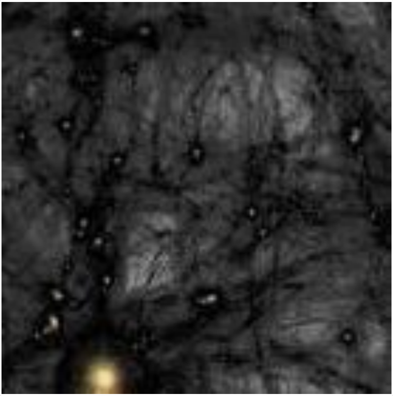




$\Omega_K(\delta_{\ell 1})$

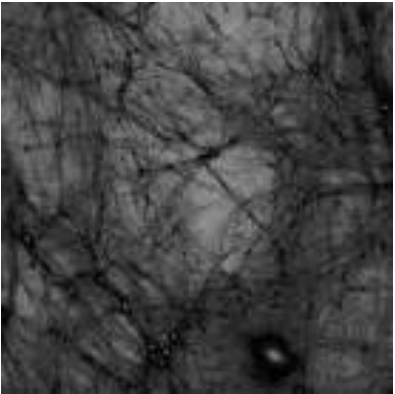


$\Omega_K(\delta_{\ell 2})$

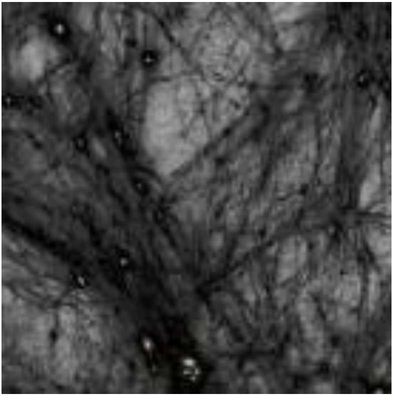


$\Omega_K(\delta_{\ell 3})$

$\Omega_K(\delta_{\ell 4})$



$\Omega_K(\delta_{\ell 5})$



- any observable in each local patch is

$$\mathcal{O}(\delta_\ell) = \mathcal{O}|_{\delta_\ell=0}\left[1+\left.\frac{\mathrm{d}\ln\mathcal{O}}{\mathrm{d}\delta_\ell}\right|_{\delta_\ell=0}\delta_\ell + O(\delta_\ell^2)\right]$$

- E.g., the number density of galaxies in each patch is

$$\begin{aligned} n_g(\delta_\ell) &= n_g|_{\delta_\ell=0}\left[1+\left.\frac{\mathrm{d}\ln n_g}{\mathrm{d}\delta_\ell}\right|_{\delta_\ell=0}\delta_\ell + O(\delta_\ell^2)\right] \\ &= \bar{n}_g\left[1+b_g\delta_\ell+O(\delta_\ell^2)\right] \end{aligned}$$

- galaxy bias (Kaiser 84)

$$\delta_g(\mathbf{x}) = b_g\delta_\ell(\mathbf{x})$$

$$b_g = \left.\frac{\mathrm{d}\ln n_g}{\mathrm{d}\delta_l}\right|_{\delta_l=0} \sim \left.\frac{\mathrm{d}\ln n_g}{\mathrm{d}\Omega_K}\right|_{\Omega_K=0}$$

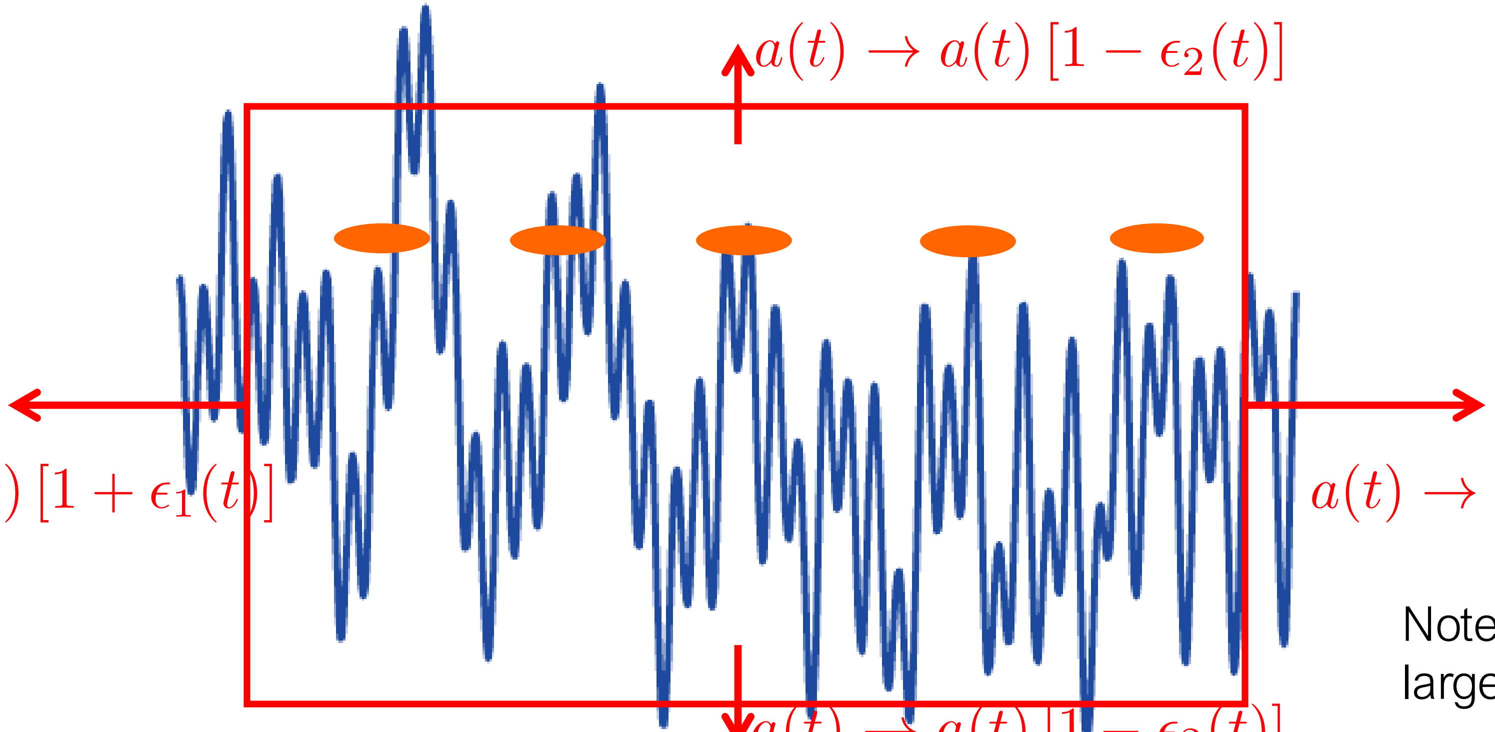
“Anisotropic” separate universe approach

Ip & Schmidt 17; Akitsu & MT 17; Schmidt+18; Masaki+20

- The anisotropic part of long-wavelength mode (tide)

$$\frac{1}{2}\left(\partial_i\partial_j-\frac{\delta_{ij}}{3}\nabla^2\right)\Phi\bigg|_{\mathbf{x}_0} x^i x^j \longrightarrow \tau_{ij}^{(l)}(t)\equiv\nabla^{-2}\left(\partial_i\partial_j-\frac{\delta_{ij}}{3}\nabla^2\right)\Phi\bigg|_{\mathbf{x}_0}$$

- The long-wavelength tide can be realized as an “anisotropic” expansion



$a(t)\rightarrow a(t)\left[1-\epsilon_2(t)\right]$

$a(t)\rightarrow a(t)\left[1+\epsilon_1(t)\right]$

$a(t)\rightarrow a(t)\left[1+\epsilon_1(t)\right]$

$a(t)\rightarrow a(t)\left[1-\epsilon_2(t)\right]$

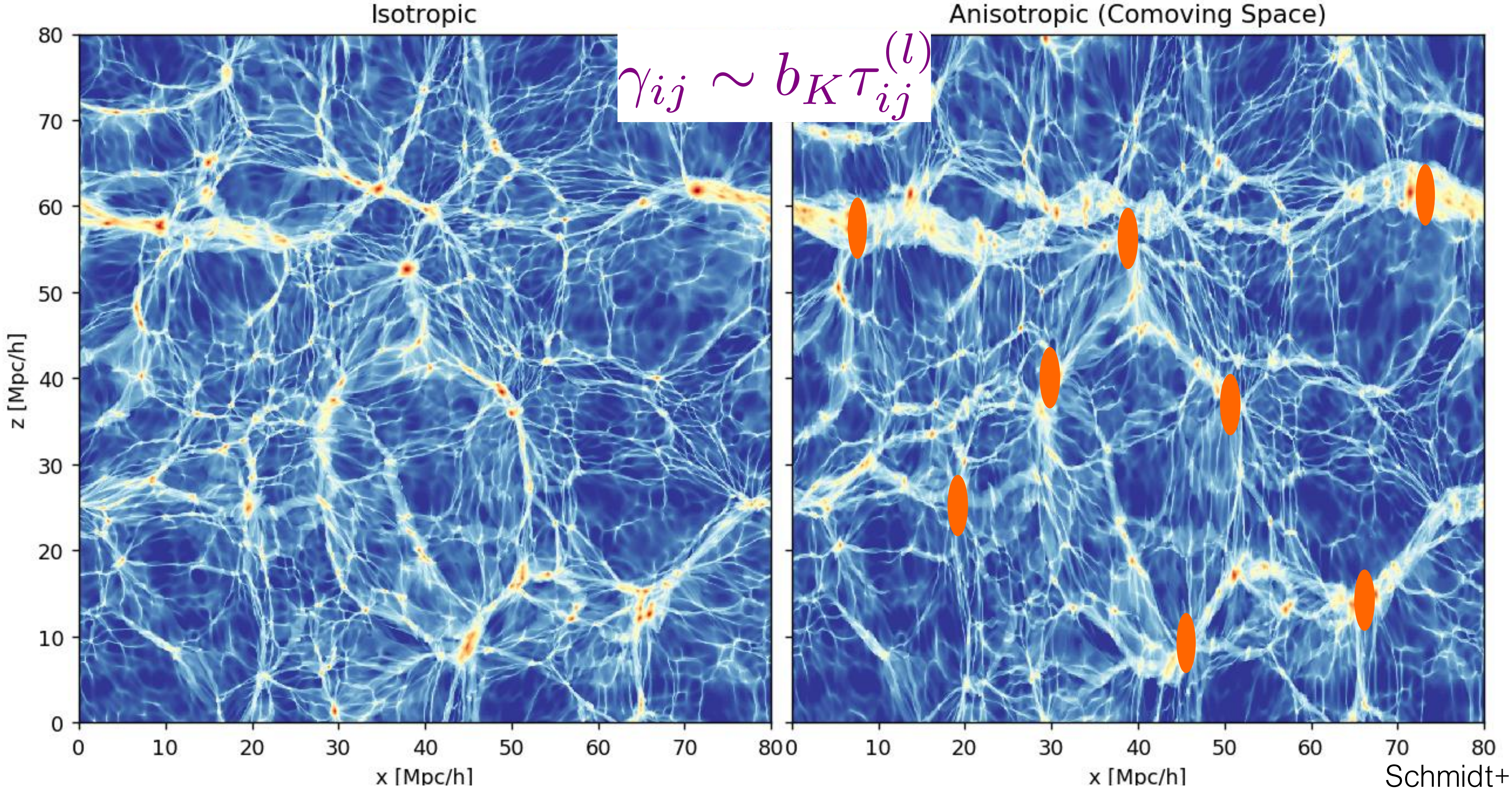
Intrinsic alignments

$$\gamma_{ij}\sim b_K\tau_{ij}^{(l)}$$

Note galaxy “spin” should be vanishing on large scale for LCDM model

Anisotropic separate universe simulation

Also see Akitsu+ 23

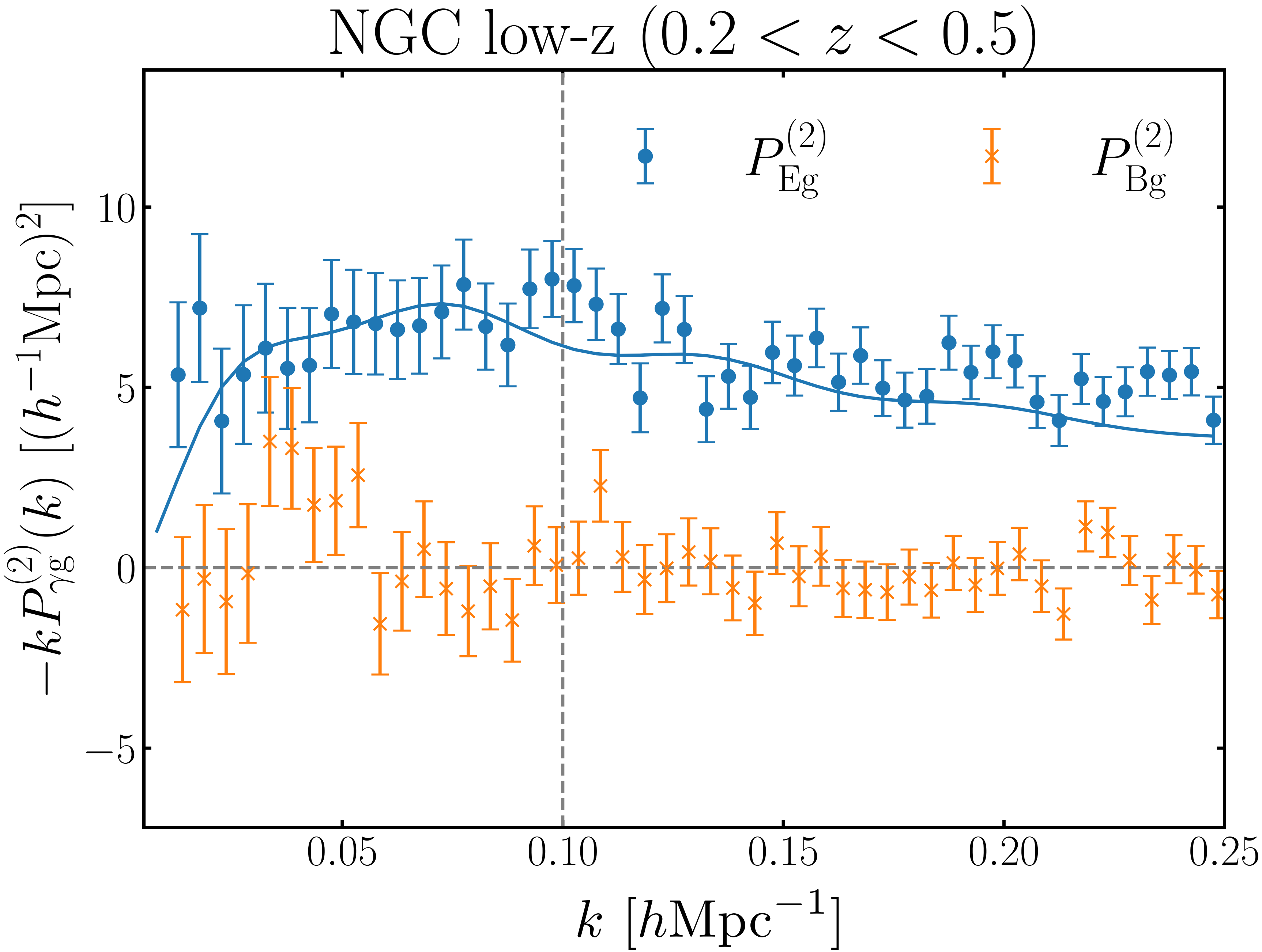


$$\gamma_{ij} \sim b_K \tau_{ij}^{(l)}$$

Schmidt+18

The first measurements of 3D IA power spectrum

Kurita & MT 23

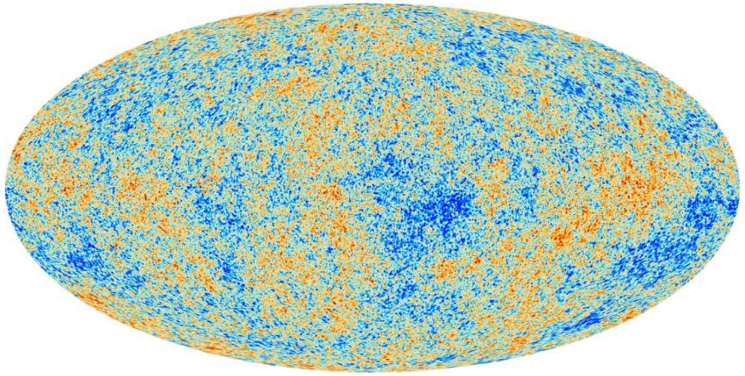


Exercise: Evolution of the primordial density peak

- A useful illustration of BAO physics (Daniel Eisenstein)
- Consider the primordial density peak for the perturbations, assuming the adiabatic initial condition
- Assume that the primordial peak is given by a Gaussian, for illustration

$$\delta_i(\mathbf{r}, \eta_{\rm ini}) \propto \exp\left[-r^2/(2\sigma^2)\right]$$

Primordial density peaks are from the inflation-quantum fluctuations



The "pattern" of temperature fluctuations is one realization of the initial conditions

- Mass profile (3D)

$$\delta m_i(\mathbf{r}, \eta_{\rm ini}) \propto r^2 \exp\left[-r^2/(2\sigma^2)\right]$$

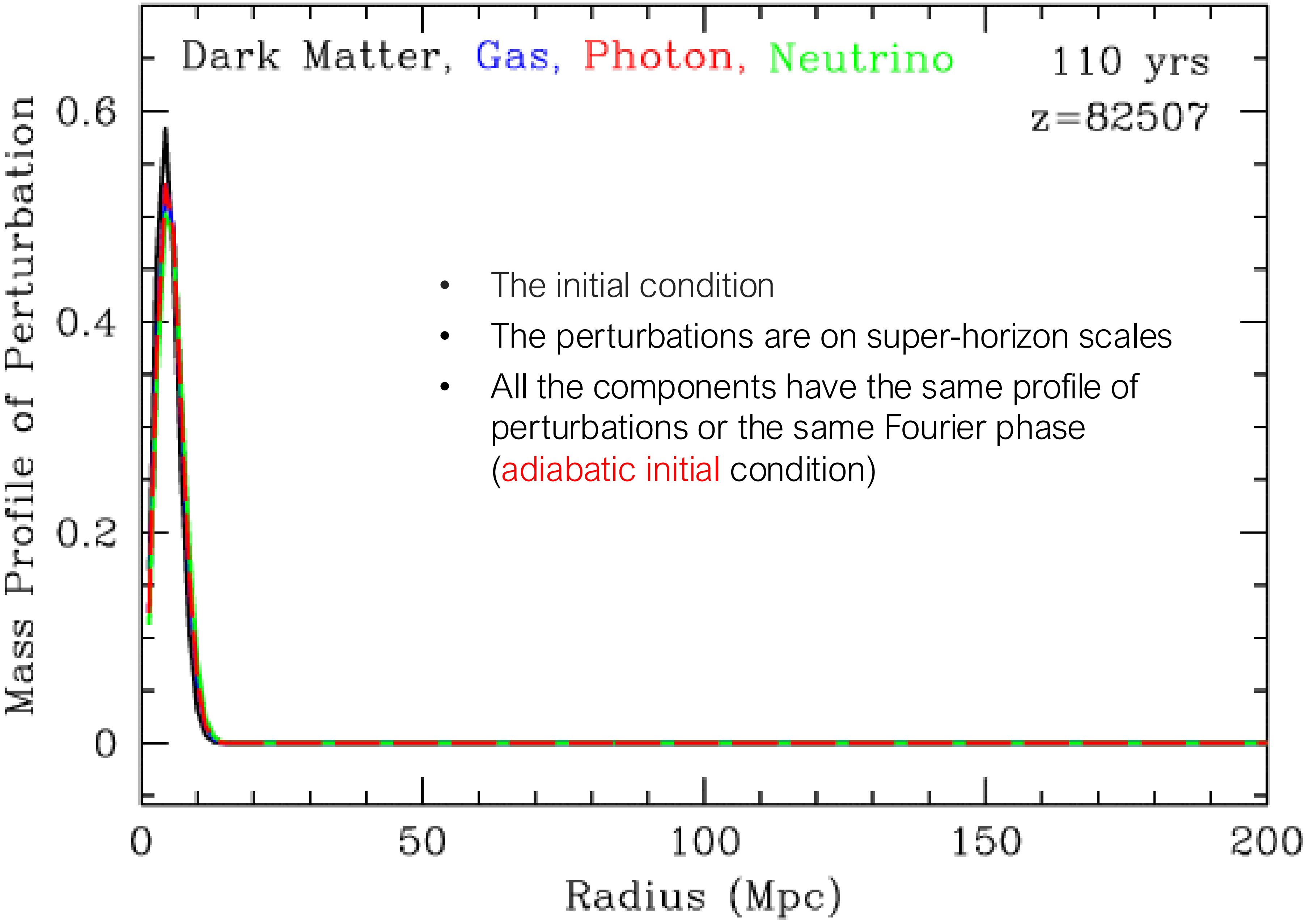
- Use the transfer function of each component to compute the time evolution of its mass profile

$$\delta m_i(\mathbf{r}, \eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \underline{T_i(k, \eta)} \widetilde{\delta m}(\mathbf{k}, \eta_{\rm ini}) e^{i\mathbf{k}\cdot\mathbf{r}}$$

transfer function: use **`CAMB`** to compute it, for LCDM model

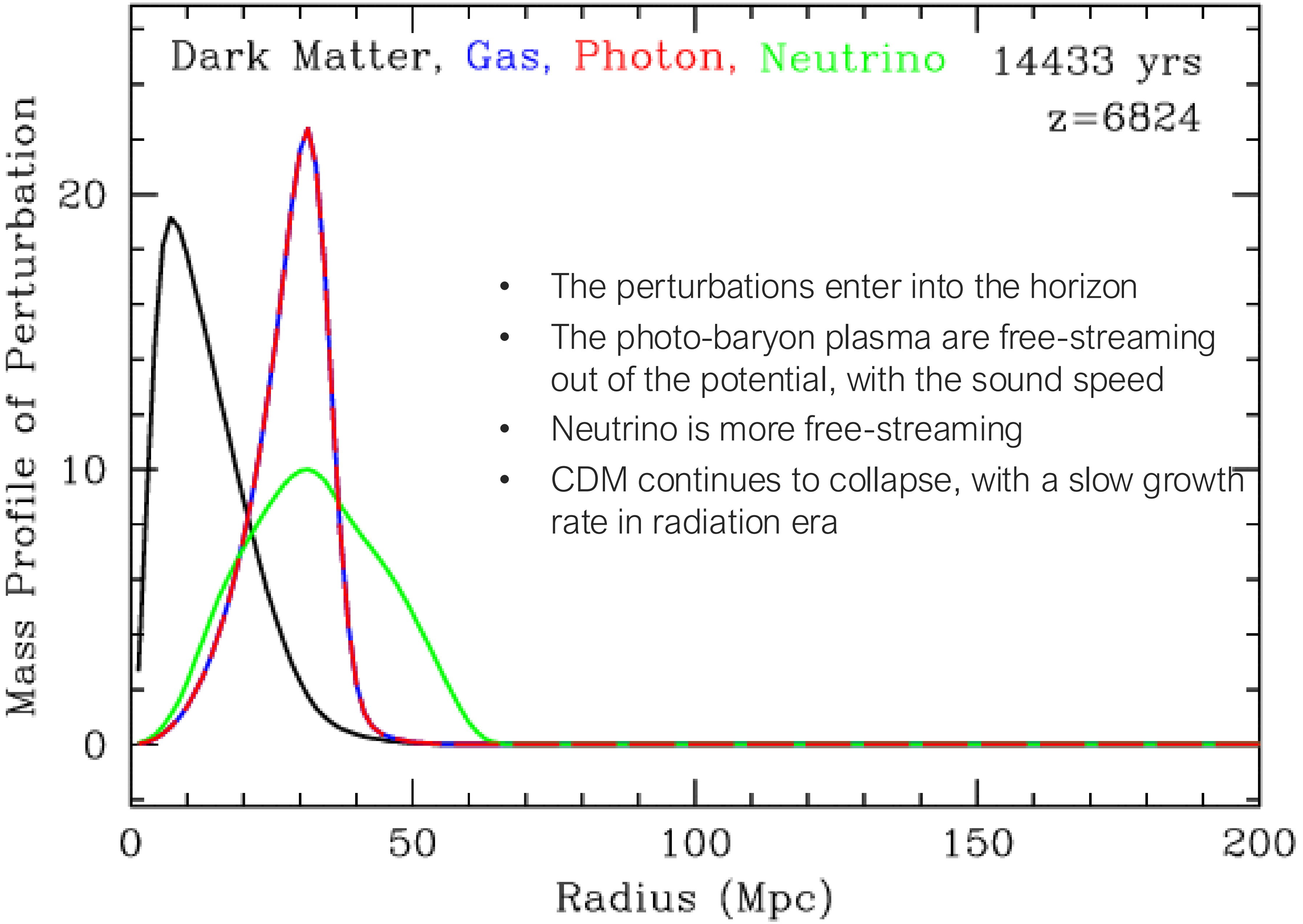
Also Ichiki & MT 09

Credit: D. Eisenstein



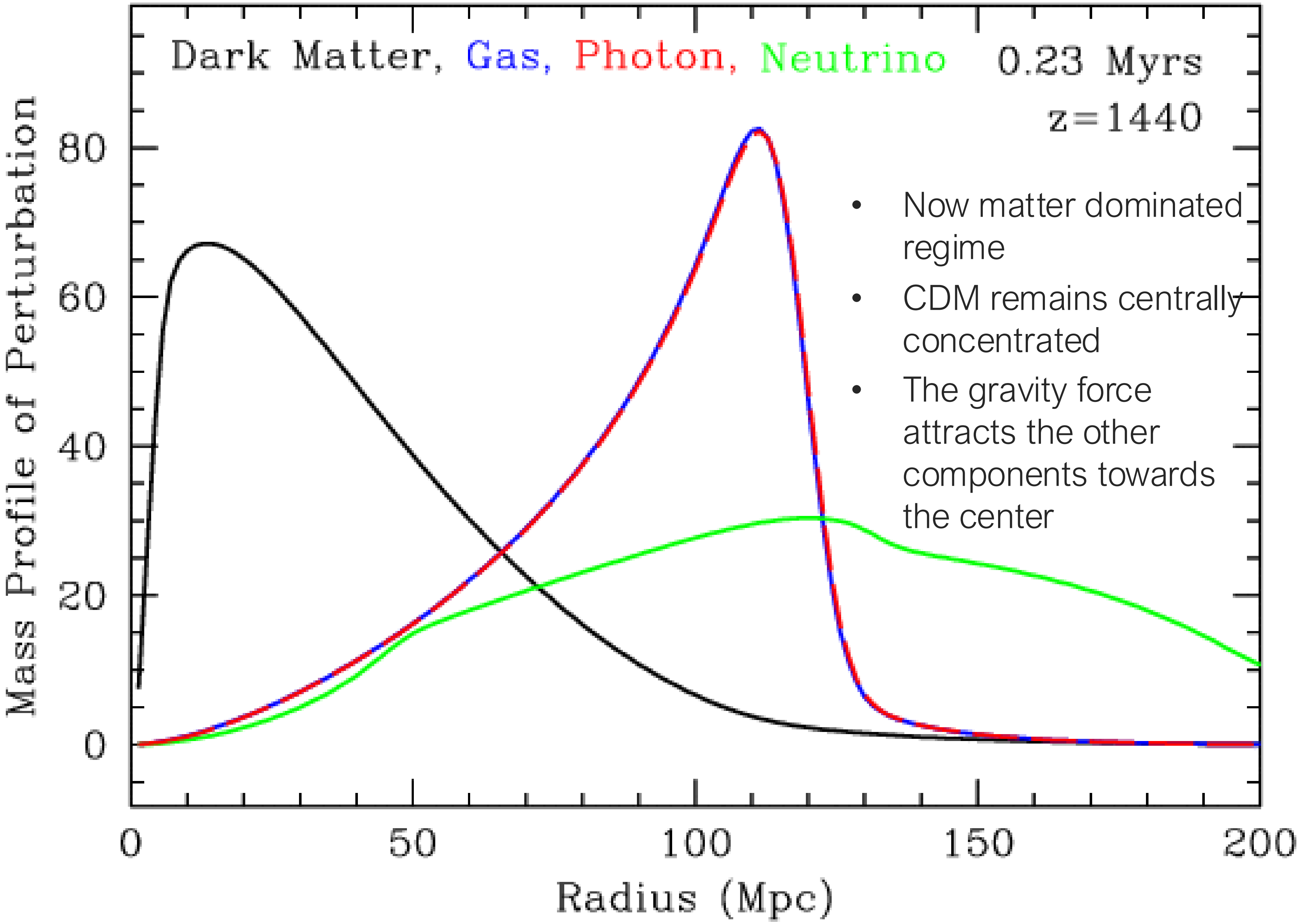
- The initial condition
- The perturbations are on super-horizon scales
- All the components have the same profile of perturbations or the same Fourier phase (adiabatic initial condition)

Credit: D. Eisenstein



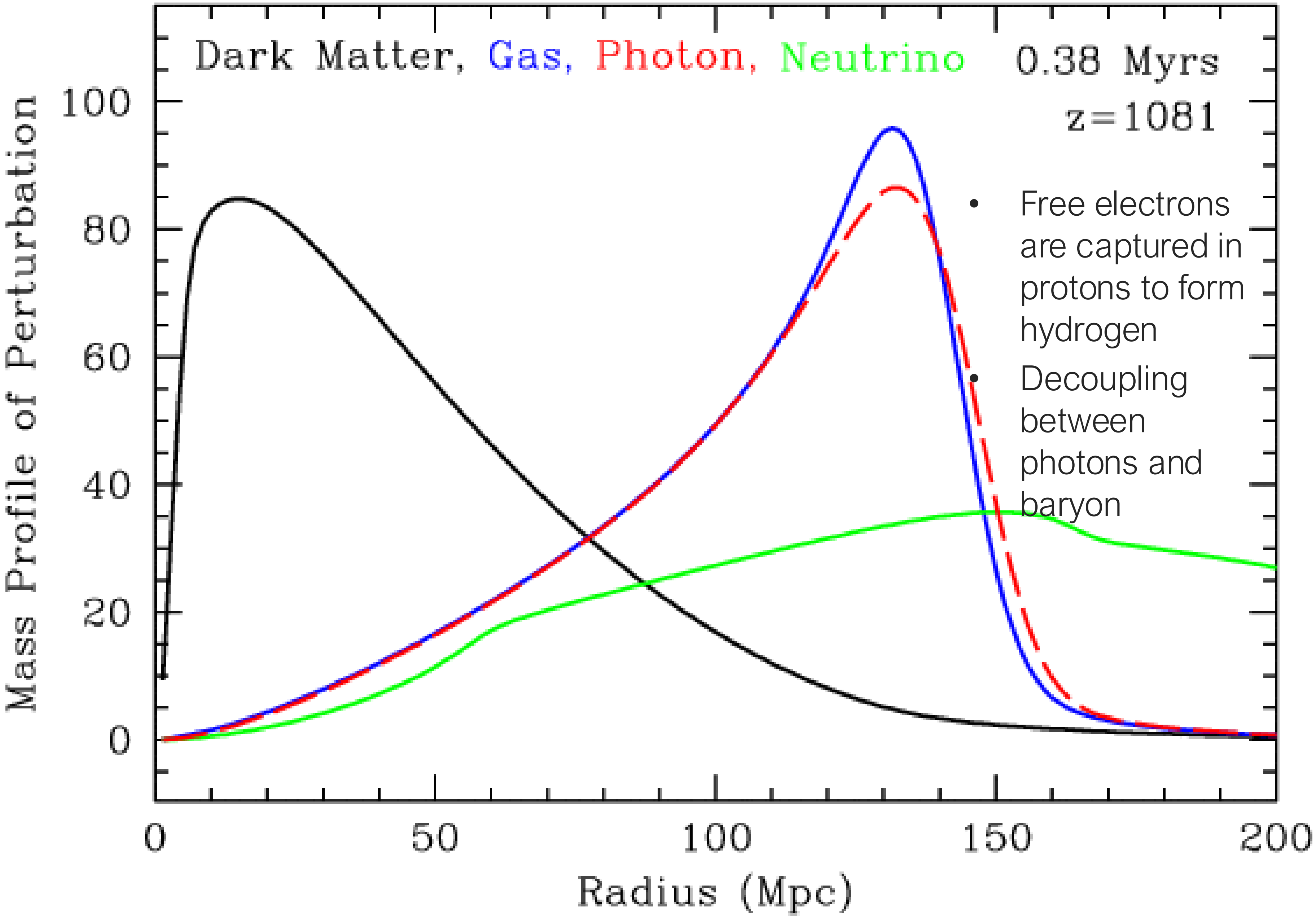
- The perturbations enter into the horizon
- The photo-baryon plasma are free-streaming out of the potential, with the sound speed
- Neutrino is more free-streaming
- CDM continues to collapse, with a slow growth rate in radiation era

Credit: D. Eisenstein

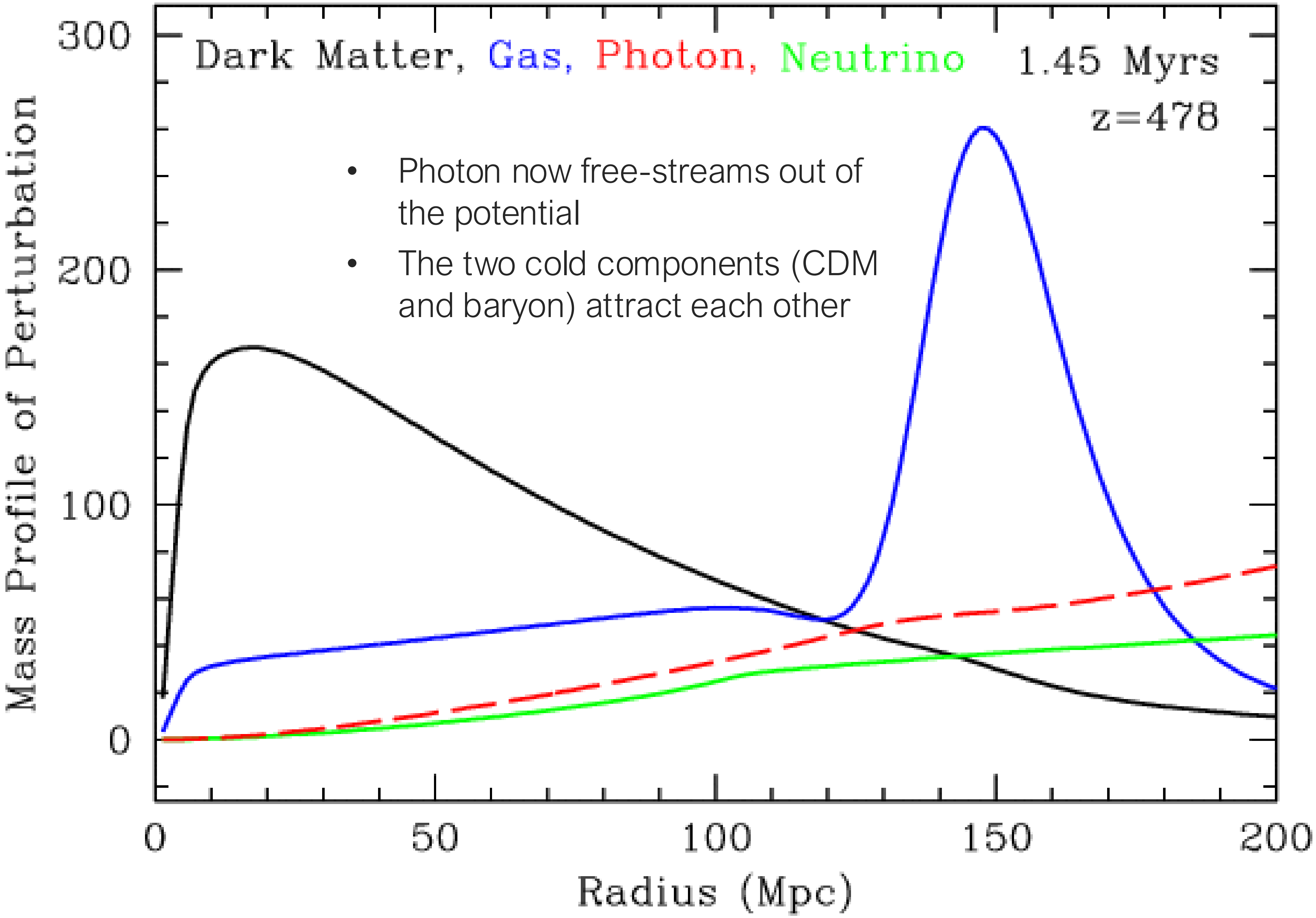


- Now matter dominated regime
- CDM remains centrally concentrated
- The gravity force attracts the other components towards the center

Credit: D. Eisenstein



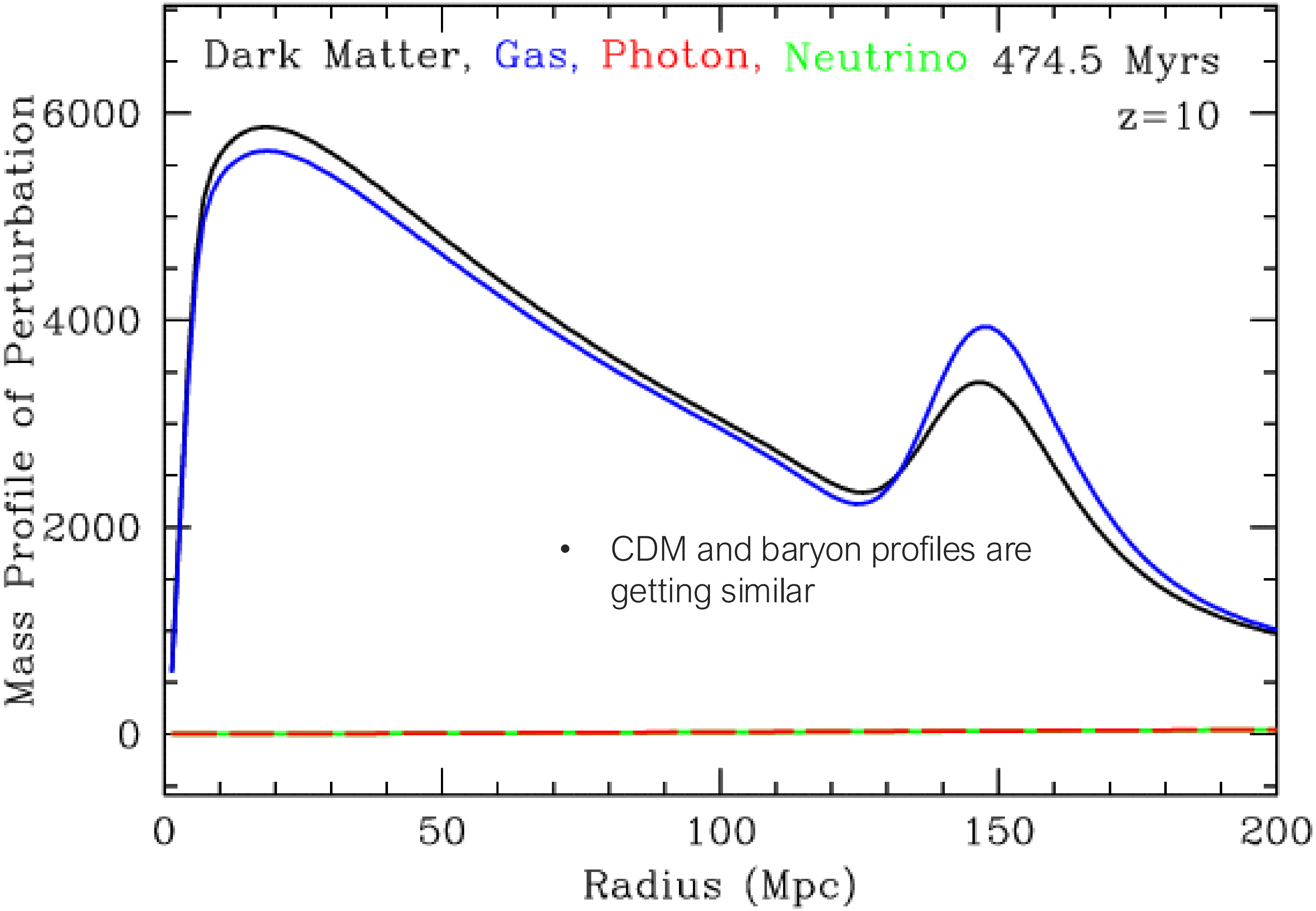
- Free electrons are captured in protons to form hydrogen
- Decoupling between photons and baryon



- Photon now free-streams out of the potential
- The two cold components (CDM and baryon) attract each other

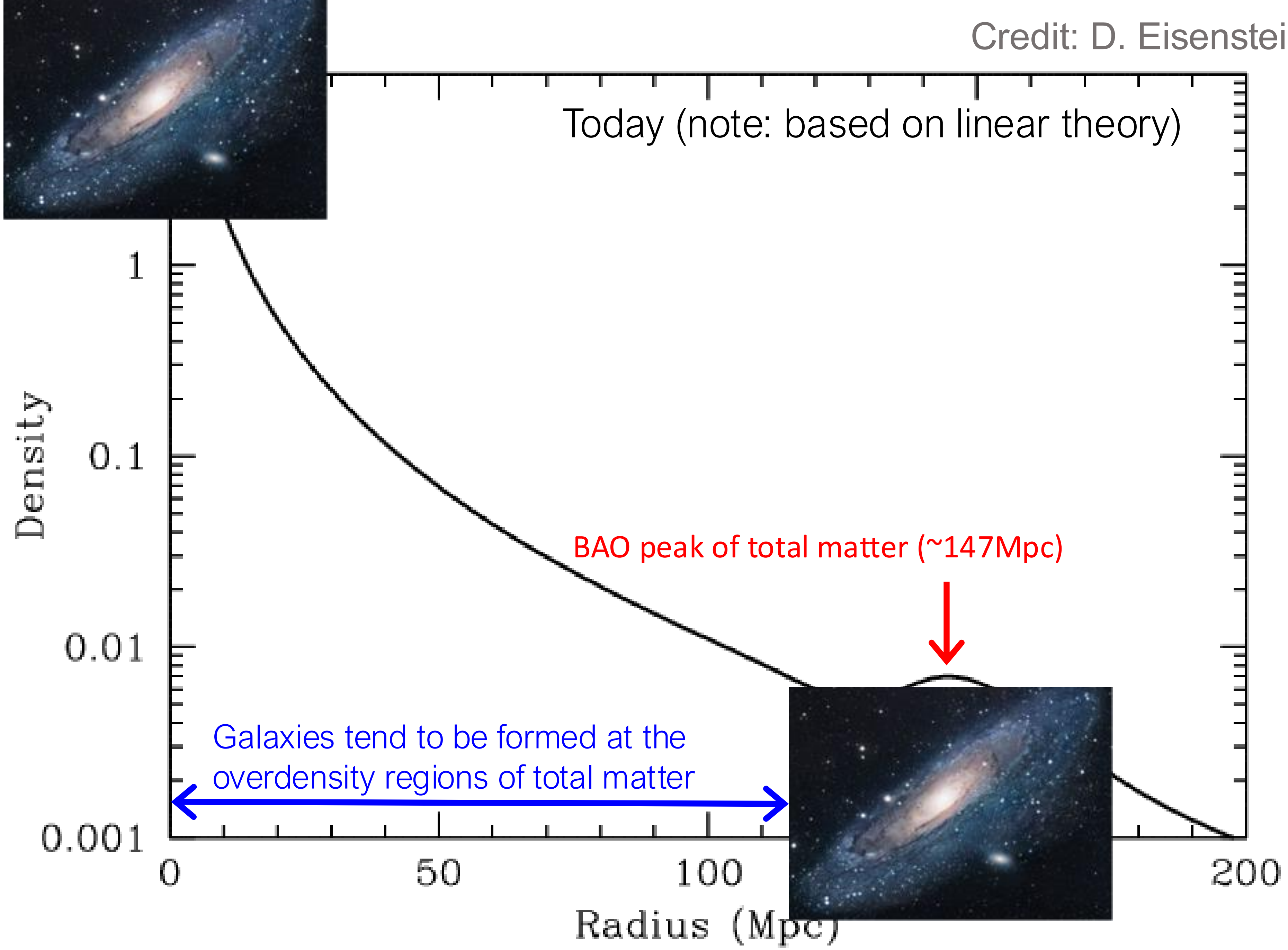
Credit: D. Eisenstein

Credit: D. Eisenstein



- CDM and baryon profiles are getting similar

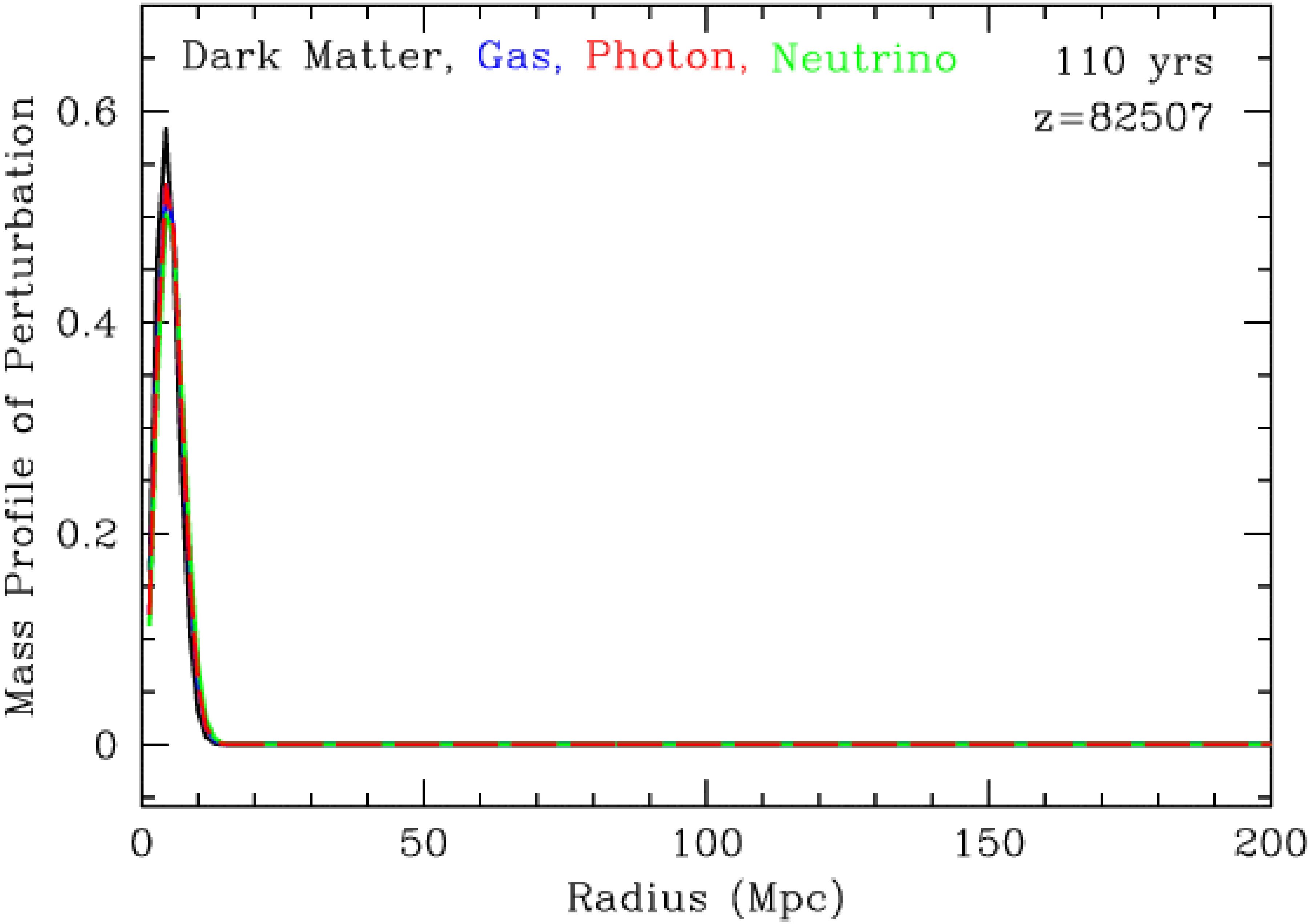
Credit: D. Eisenstein



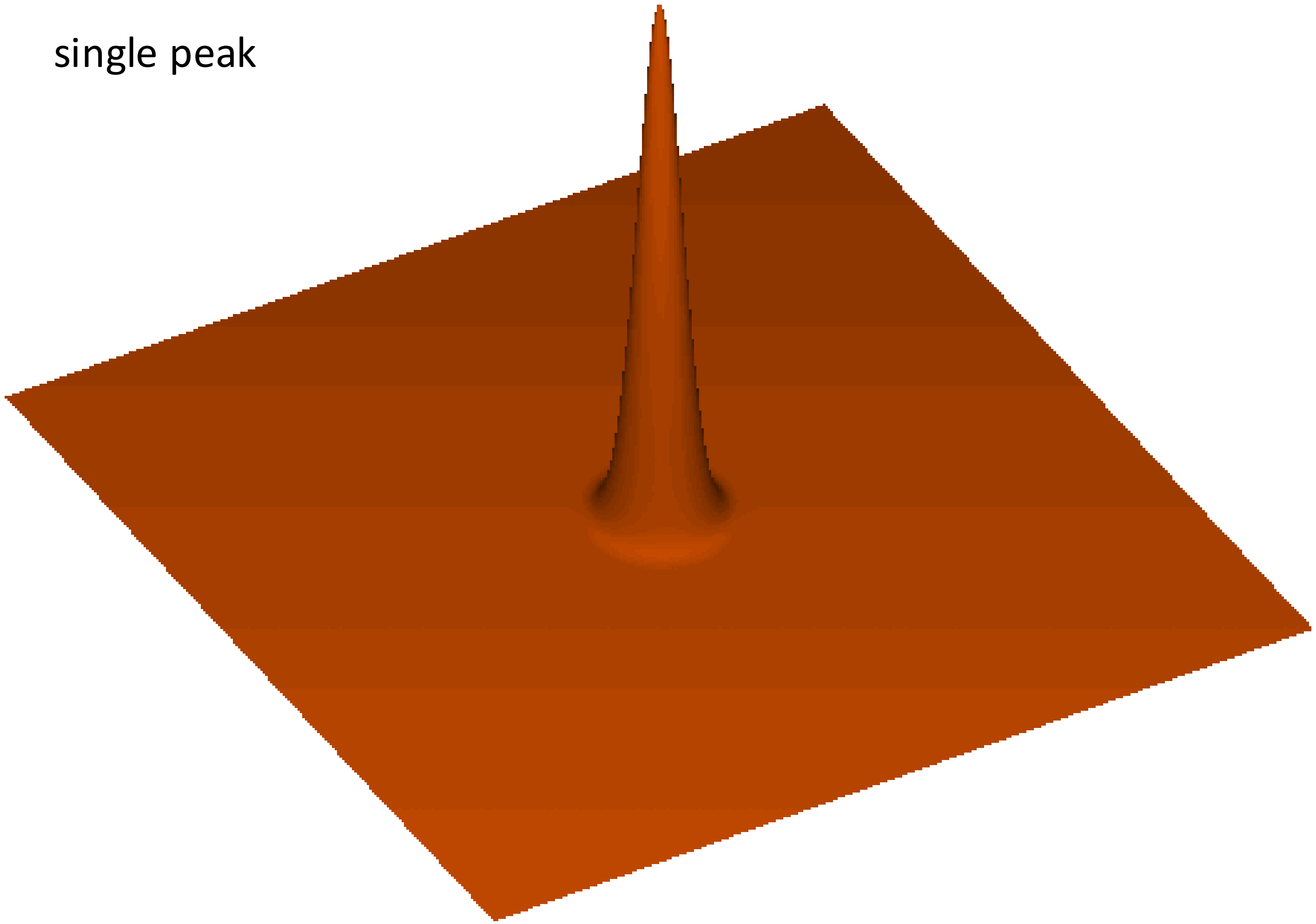
Today (note: based on linear theory)

BAO peak of total matter (~147Mpc)

Galaxies tend to be formed at the overdensity regions of total matter

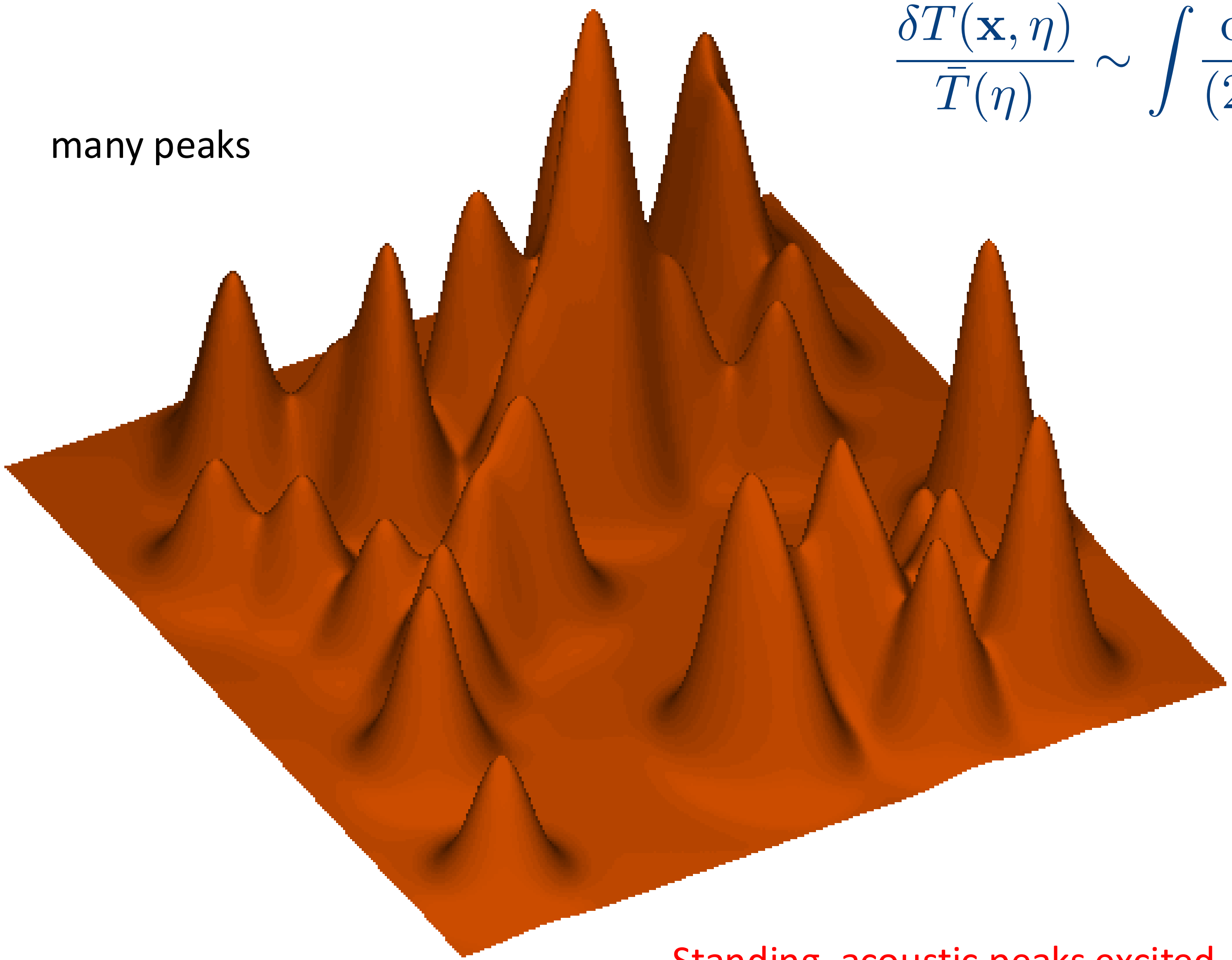

Dark Matter, Gas, Photon, Neutrino
110 yrs
z=82507
Mass Profile of Perturbation
0.6
0.4
0.2
0
0
50
100
150
200
Radius (Mpc)

single peak



$$\frac{\delta T(\mathbf{x},\eta)}{\bar{T}(\eta)} \sim \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} T_\gamma(\eta,k)\mathcal{R}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$$

many peaks



Standing, acoustic peaks excited

Illustration

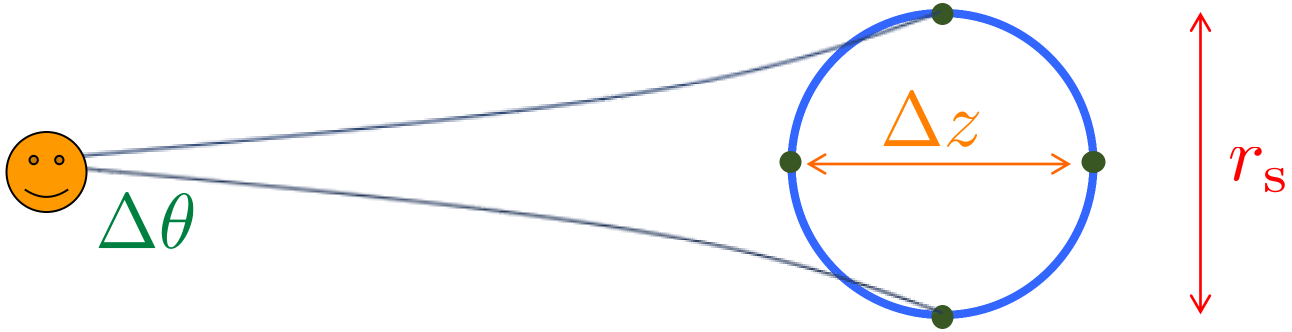
- Paired galaxies tend to have characteristic separation (rs)
- A robust method to measure the separation is the two-point correlation function
- The two-point correlation function gives a probability of finding paired of galaxies as a function of the separation

$$\mathrm{d}P = \bar{n}_\mathrm{g}^2 \left[1 + \xi_\mathrm{gg}(r)\right] \mathrm{d}V_1 \mathrm{d}V_2$$

where $\bar{n}_\mathrm{g}$ is the mean number density of galaxies

BAO geometrical test

- The sound horizon r_s is determined by physics in the early universe, and is precisely constrained by CMB (~0.2%)
- Paired galaxies have preferred separation (rs) – standard ruler, i.e. baryon acoustic oscillation (BAO)
- We can realize the BAO scale by the angular separation and redshift difference between paired galaxies



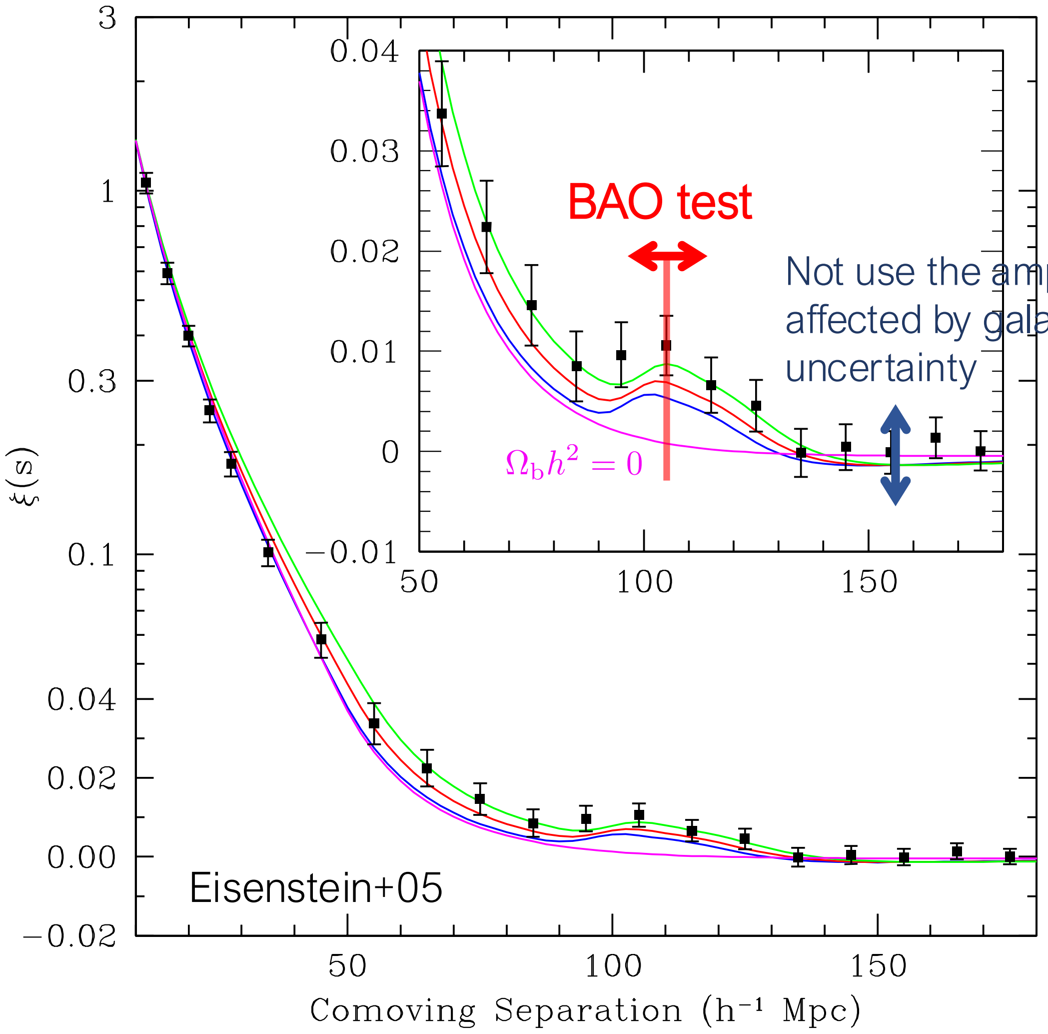
Redshift of center: z

- Because we a priori know r_s, we can infer the angular diameter distance and the Hubble expansion rate from the measured separations

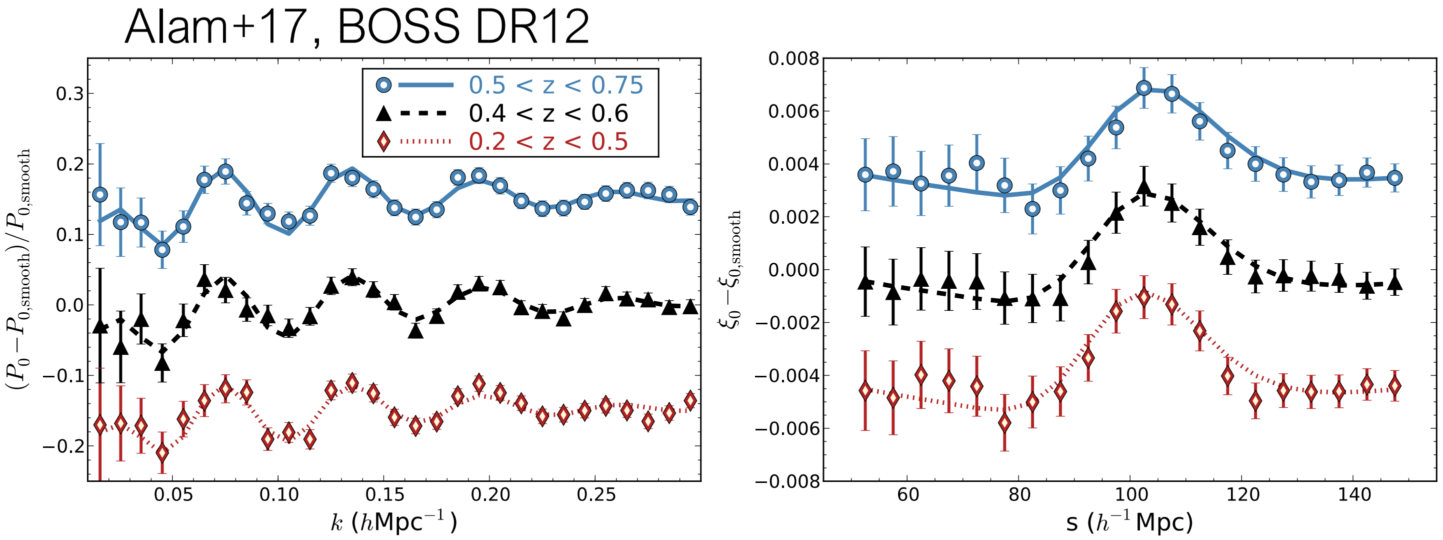
$$r_{\mathrm{s}} = \Delta\chi = \frac{\Delta z}{H(z)}, \quad r_{\mathrm{s}} = d_A(z)\Delta\theta$$

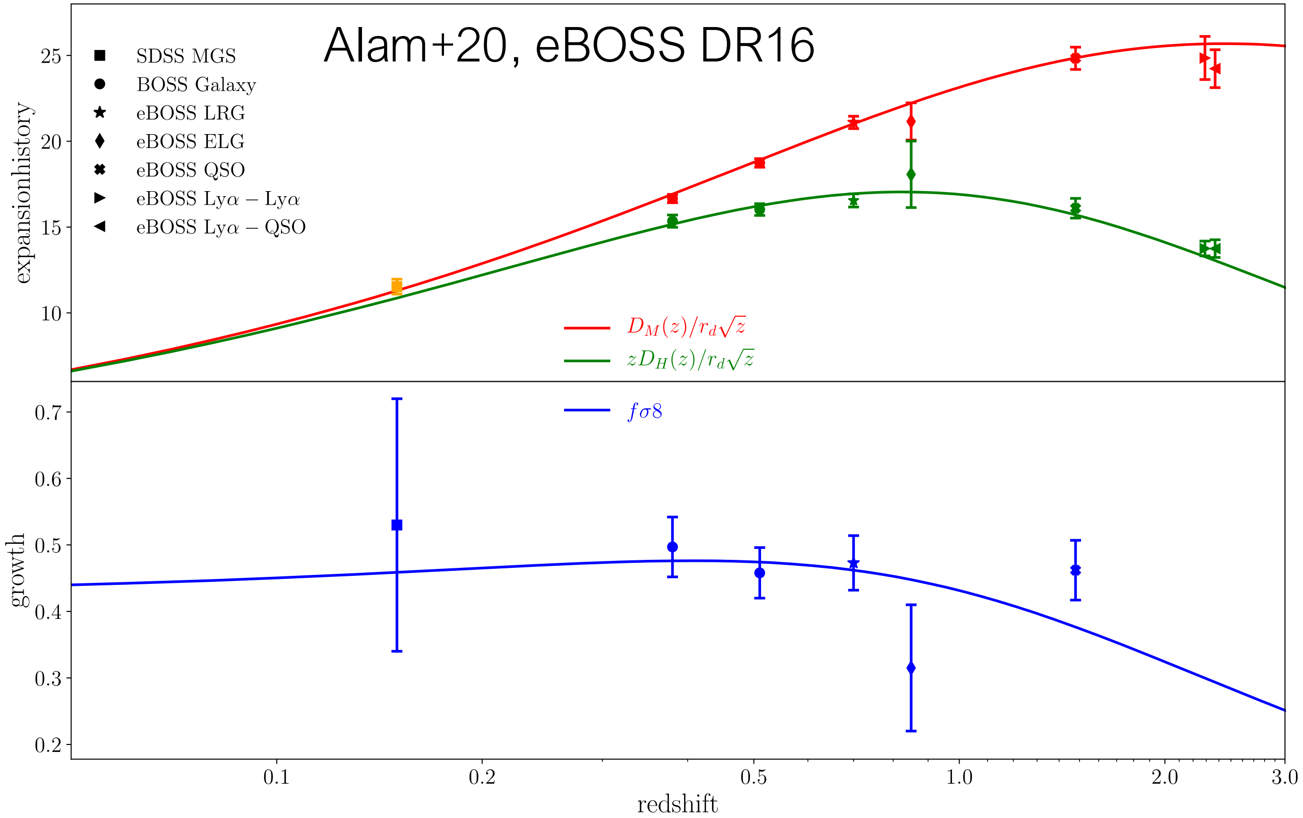
BAO test is free of (or not affected by) astrophysical systematic errors, and give a robust, powerful geometrical test of both the radial and angular distances

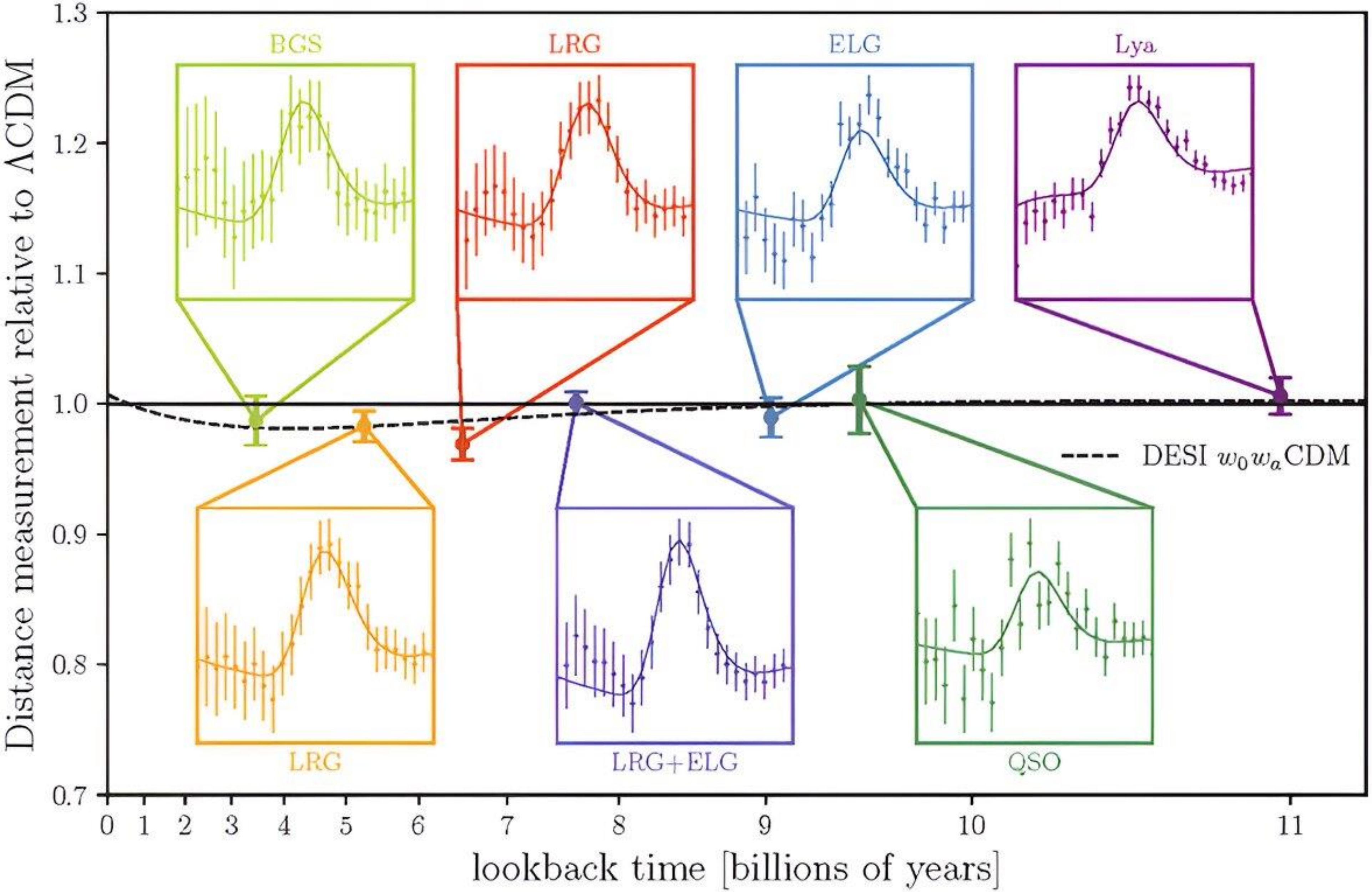
- Daniel Eisenstein made the first detection of BAO (~3σ) from real data, after some theoretical studies (Eisenstein+05)
- Free of galaxy bias uncertainty



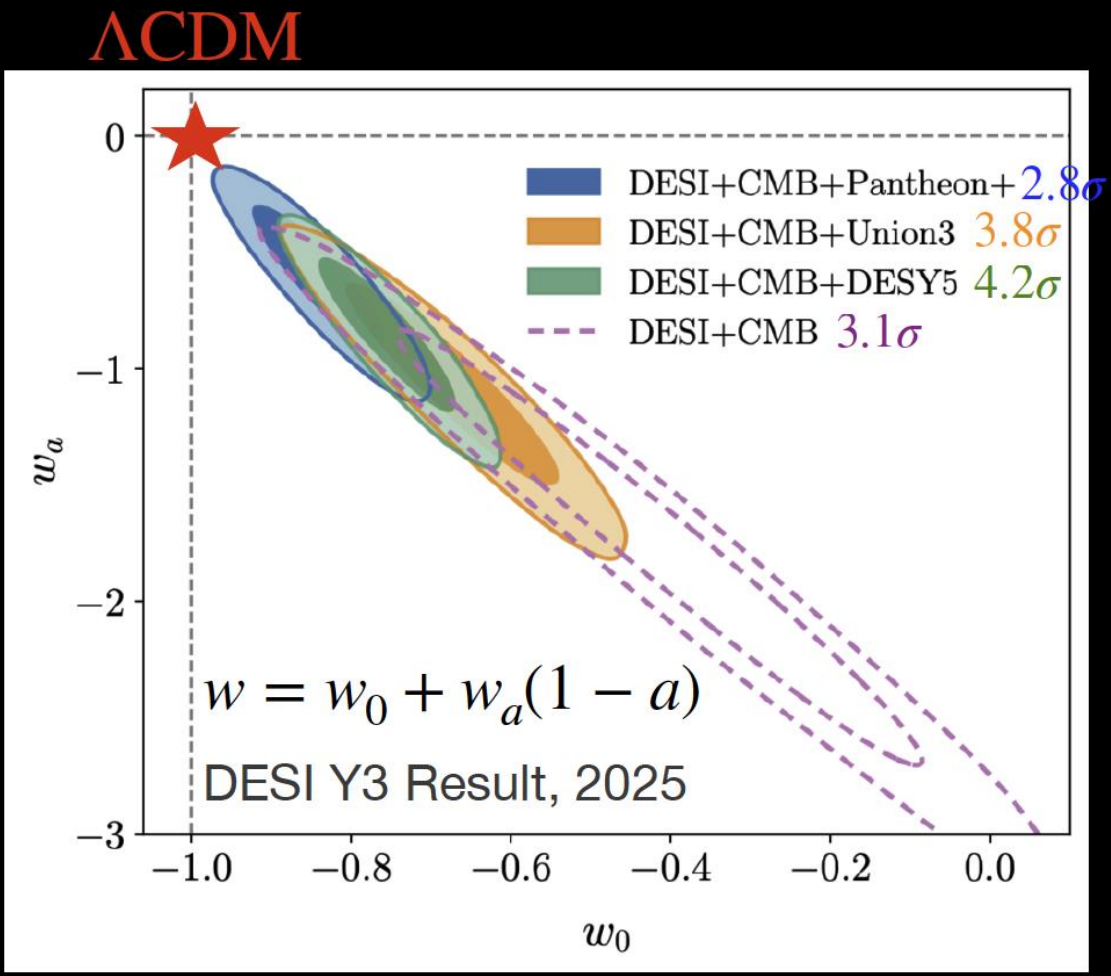
Not use the amplitude affected by galaxy bias uncertainty

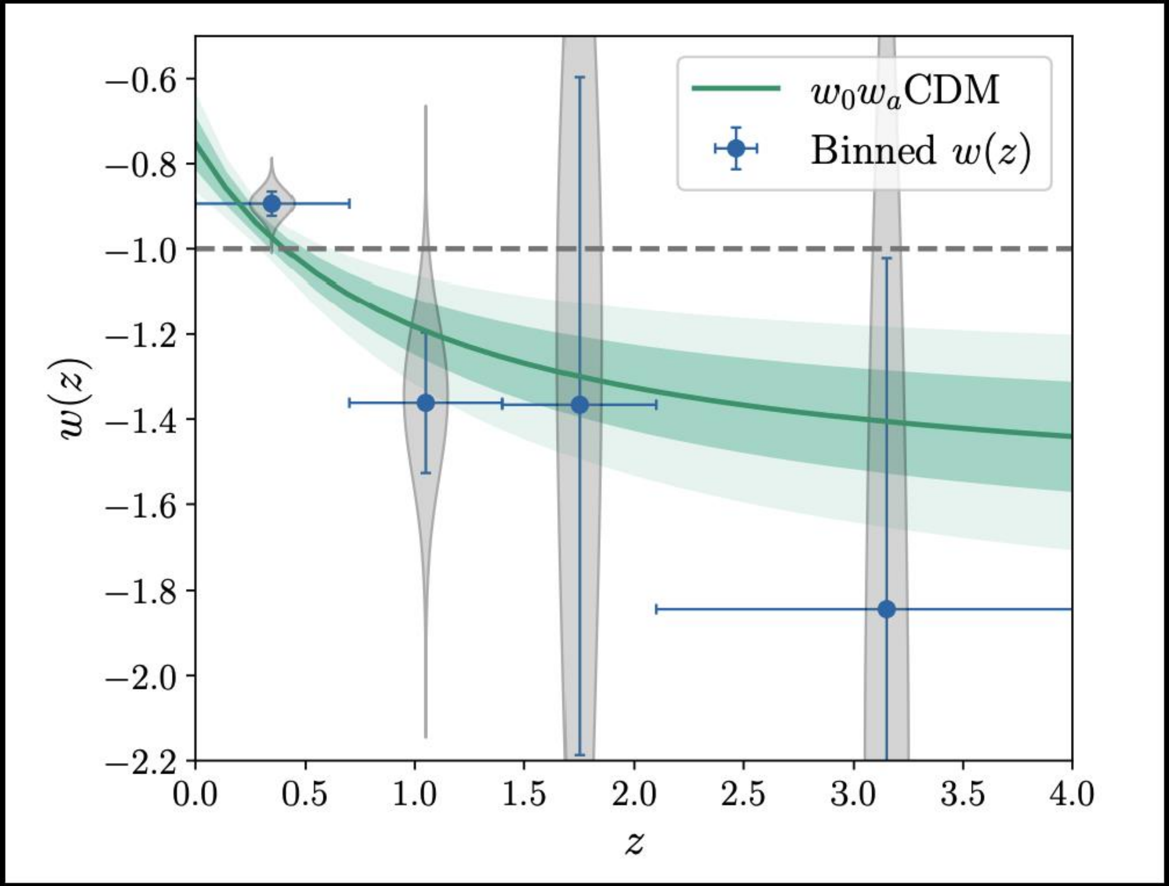





BGS
LRG
ELG
Lya
LRG
LRG+ELG
QSO
DESI w_0w_aCDM
Distance measurement relative to ΛCDM
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0
1
2
3
4
5
6
7
8
9
10
11
lookback time [billions of years]

Evolving Dark Energy?





Big questions in galaxy survey cosmology

- ΛCDM is really a correct model? Any inconsistency/tension or any new physics beyond ΛCDM?
- What are the initial conditions? A slow-roll, single-field inflation is sufficient?
  - Tensor mode, running index, primordial non-Gaussianity, … over all scales we see today?
- What is dark matter?
  - Axion, non-WIMP dark matter, PBH, …
- Dark energy is the cosmological constant Λ?
  - Is the time-varyng DE, the new DESI result, really correct? We need independent confirmation

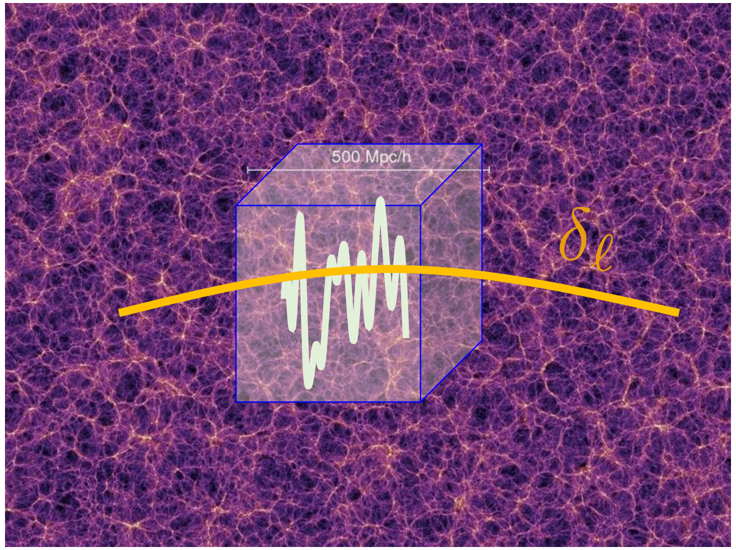
Many observables

- Baryonic acoustic oscillations (BAO): geometrical test
- Galaxy clustering and redshift-space distortion
- Weak lensing: galaxy shear and CMB lensing
- Thermal and kinetic SZ
- 21cm
- …

warm-up problem 1

- Suppose your finite volume is embedded into an over- or under-density region, characterized by delta_l (long-wavelength mode)
- The observer still sees a homogeneous and isotropic background in the volume, but the mean mass density is different from the global one

$$\bar{\rho}_{mW} = \bar{\rho}_m \left[1 + \delta_l(t)\right]$$



- The time evolution of sub-box linear fluctuations would obey (assuming the linear theory inside the box)

$$\ddot{\delta}_s(\mathbf{k},t) + 2H_W\dot{\delta}_s(\mathbf{k},t) - 4\pi G\bar{\rho}_{Wm}\delta_s(\mathbf{k},t) = 0$$

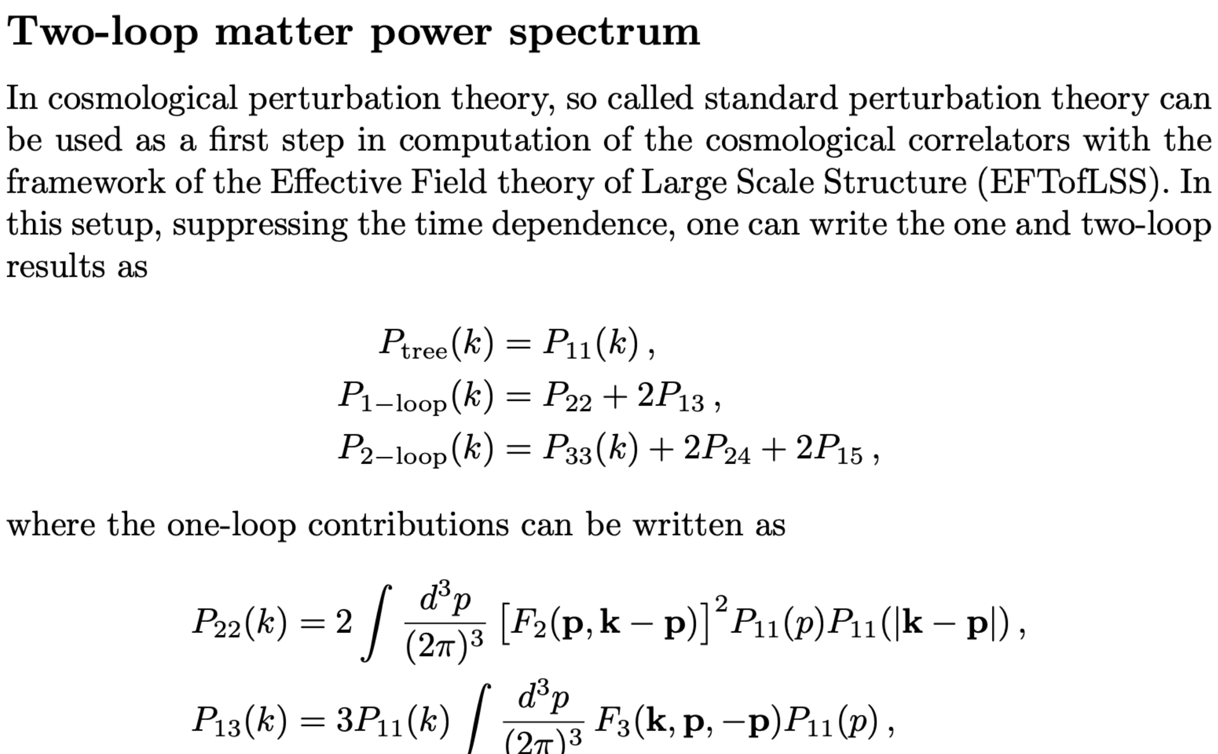
- Solve the time evolution of the sub-box mode up to the order of O(delta_s *delta_l) , assuming that the global background obeys the Einstein-de Sitter universe

$$\Omega_m = 1, \delta_l(t) \propto t^{2/3} \propto a(t), H_W \simeq H(t) - \frac{1}{3}\dot{\delta}_l(t)$$

- Any other use of the separate universe approach for cosmology?

problem 2 (from Zvonimir)

- On an analytical approach to modeling nonlinear structure formation
- Try to obtain two-loop matter power spectrum based on the cosmological perturbation theory: in more detail, see here.



Two-loop matter power spectrum

In cosmological perturbation theory, so called standard perturbation theory can be used as a first step in computation of the cosmological correlators with the framework of the Effective Field theory of Large Scale Structure (EFTofLSS). In this setup, suppressing the time dependence, one can write the one and two-loop results as

$$\begin{aligned} P_{\rm tree}(k) &= P_{11}(k)\,, \\ P_{1-{\rm loop}}(k) &= P_{22} + 2P_{13}\,, \\ P_{2-{\rm loop}}(k) &= P_{33}(k) + 2P_{24} + 2P_{15}\,, \end{aligned}$$

where the one-loop contributions can be written as

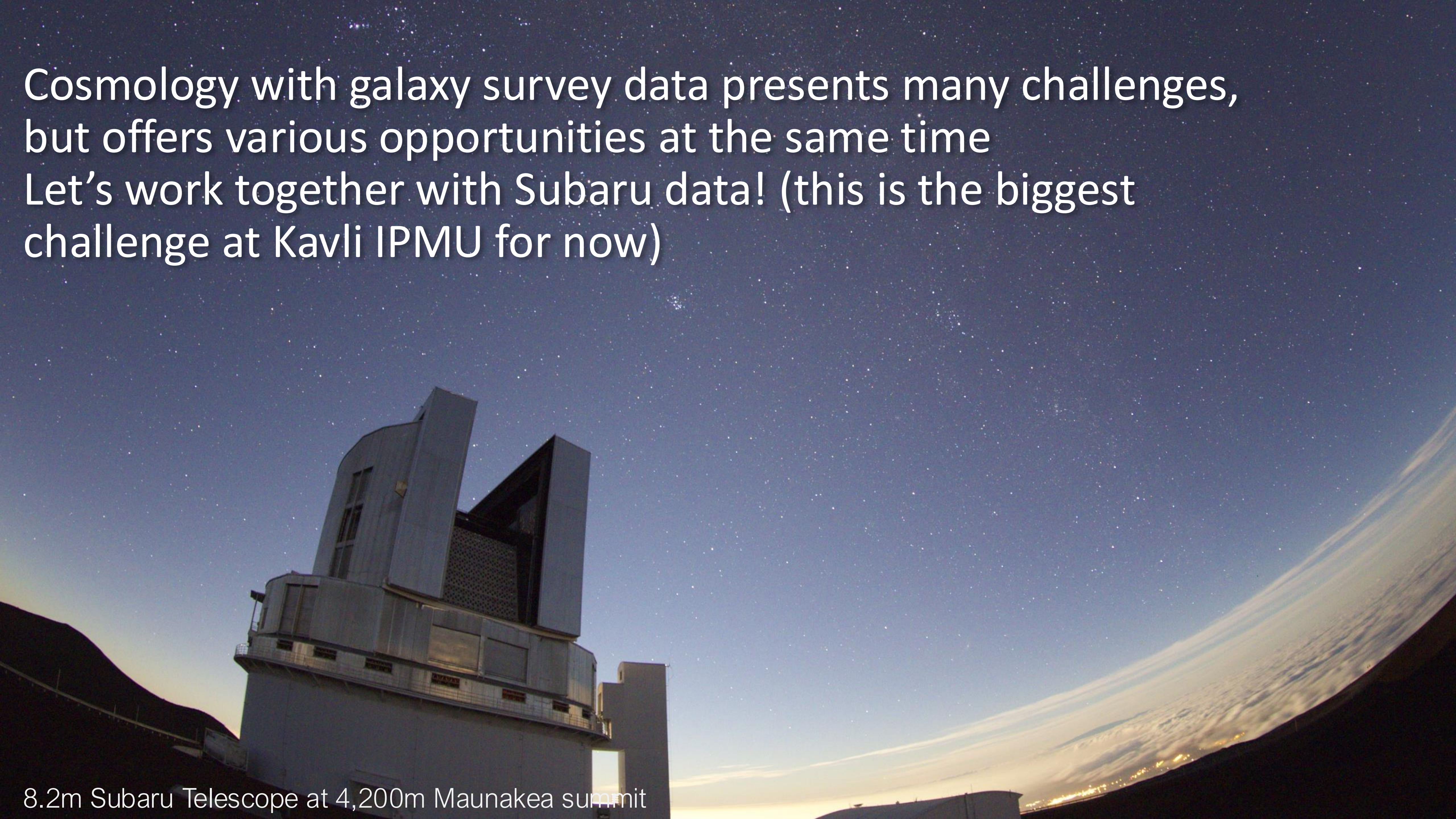
$$\begin{aligned} P_{22}(k) &= 2\int \frac{d^3p}{(2\pi)^3}\left[F_2(\mathbf{p},\mathbf{k}-\mathbf{p})\right]^2 P_{11}(p)P_{11}(|\mathbf{k}-\mathbf{p}|)\,, \\ P_{13}(k) &= 3P_{11}(k)\int \frac{d^3p}{(2\pi)^3} F_3(\mathbf{k},\mathbf{p},-\mathbf{p})P_{11}(p)\,, \end{aligned}$$

problem 3

- Suppose we aim to test ΛCDM model or exploit new physics beyond ΛCDM for anything. What will you do? (which new physics do you want to address? Which observable will you use for it? Why?) (H0, S8 tension, time-varying DE, …)

Cosmology with galaxy survey data presents many challenges, but offers various opportunities at the same time

Let’s work together with Subaru data! (this is the biggest challenge at Kavli IPMU for now)



8.2m Subaru Telescope at 4,200m Maunakea summit

References

- Modern Cosmology, S. Dodelson & F. Schmidt – CMB, inflation, large-scale structure
- Cosmology, D. Baumann, cosmology, inflation, large-scale structure
- 現代宇宙論, 松原隆彦 – 宇宙論全般
- 宇宙論の物理　上下, 松原隆彦 – 上の宇宙論より高度な内容
- Cosmology, S. Weinberg – CMB, inflation, large-scale structure
- Physical Foundations of Cosmology, V. Mukhanov – more on the early universe (inflation, particle physics cosmology, …)
- 宇宙マイクロ波背景放射, 小松英一郎 – focus on CMB physics
- PhD thesis by Wayne Hu -- http://background.uchicago.edu/~whu/Papers/thesis.pdf
- Extragalactic Astronomy and Cosmology, Peter Schneider
- Lecture notes by Daniel Baumann (U. Amsterdam) -- http://cosmology.amsterdam/education/
- Lecture notes by Chris Hirata (Ohio State U.) -- https://hirata10.github.io/ph8803/
- Lecture notes by Masamune Oguri (Chiba U.) – more on gravitational lensing, https://oguri.github.io/teach-j.html

useful codes

- astropy: a common core package for astronomy community (coordinate transformation, astrometry, …)
- CAMB: a public code for calculating CMB, lensing, galaxy counts, 21cm power spectra, matter power spectra and transfer functions
- CLASS: a public code for calculating CMB and large-scale structure observables
- CCL: the core cosmology library developed by DESC (Dark Energy Science Collaboration)
- Colossus: a python toolkit for calculating cosmology quantities, the LSS quantities, and the properties of halos
- DarkEmulator (Nishimichi, Takada, …): a python package for computing statistical quantities of halos, based on emulations of N-body simulation data
- Nbodykit: an open-source code for the analysis of N-body simulation data and large-scale structure data
- ….