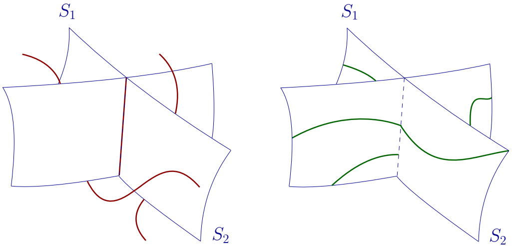
Calabi-Yau four wall-crossing

Arkadij Bojko
Enumerative Geometry in East Asia 2025



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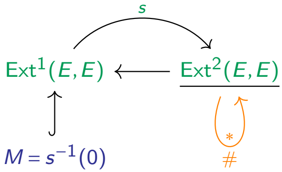
3. Locally around E, M looks like:



where $\#$ - the Serre isomorphism, $B : \mathrm{Ext}^2(E,E)^{\otimes 2} \to \mathbb{C}$ the induced pairing, and $B(s,s) = 0$.

Orientations

1. Set $V := \mathrm{Ext}^2(E,E)$ and choose a splitting $V = \Lambda \oplus \Lambda^*$ where Λ, Λ^* are Lagrangian subspaces.

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The argument for projective X consists of two parts - 1) prove orientability for $U(n)$-connections, 2) transport it to sheaves and complexes on X. 1) was corrected by Joyce-Upmeier under extra conditions, 2) is done by CGJ. For quasi-projective X, I reduced orientations for complexes to orientations in 1) on a compact manifold.

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Example

All X obtained as complete intersections in toric varieties satisfy this.

Compatibility of orientations

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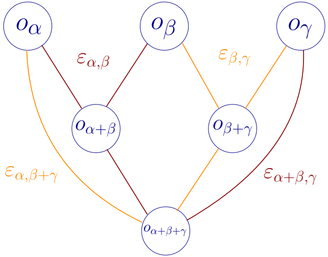
3. These signs satisfy associativity and more



Virtual integrals

1. When M is orientable, Borisov-Joyce(15') and later Oh-Thomas(20') construct

$$[M]^{\mathrm{vir}} \in H_{2-\chi(E,E)}(M,\mathbb{Z}),$$
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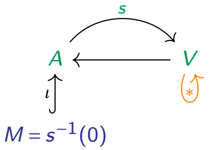
Example

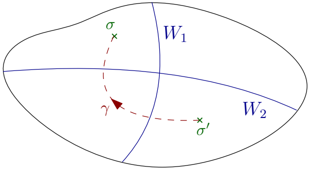
Assume that the local model holds globally for a projective A and a vector bundle $V \to A$:



where $B(s,s) = 0$ for the induced pairing B. If $V = \Lambda \oplus \Lambda^*$, then $\iota_*[M]^{\mathrm{vir}} = e(\Lambda)$.

Gross-Joyce-Tanaka conjecture

1. Consider two stability conditions in a stability manifold:



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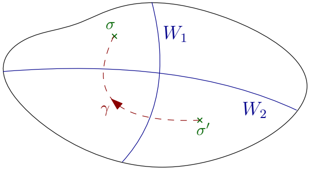

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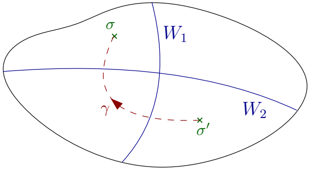

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3. For a pair of sheaves E_1, E_2, there can be short exact sequences

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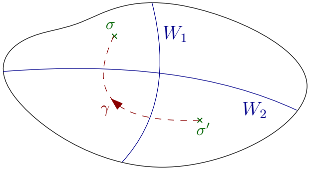

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4. Subtracting the first type and adding the second type gives

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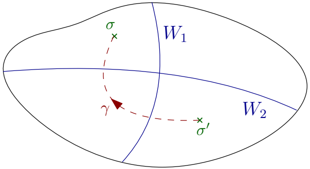

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5. Vertex algebras are refinements of Lie algebras.

Bicharacter construction

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3. There is an action of $H_* := H_*(B\mathbb{G}_m)$ (a Hopf algebra) on A_* induced by scaling automorphisms of objects. Using $(H_*)^* \cong \Bbbk[\![z]\!]$, the action induces a map

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Theorem (Borcherds(99'))

Let A_* be a bialgebra with a compatible action of the above Hopf algebra H_*. For a given

$$- \cap B_z \in A_* \otimes A_* \longrightarrow A_* \otimes A_*((z))$$

satisfying the axioms of a bicharacter, there is a vertex algebra given by

$$Y_z(v,w) = m\left[e^{zT} \otimes \mathrm{id}\left(v \otimes w \cap B_z\right)\right].$$

Joyce's bicharacter construction

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Theorem (Joyce (17'))

Let

$$\mathscr{E}\text{xt} = \mathrm{R}\mathscr{H}\text{om}_{\mathscr{M}_X \times \mathscr{M}_X}\left(\mathscr{E}_1, \mathscr{E}_2\right)$$

and $\varepsilon : K^0(X) \times K^0(X) \to \{\pm 1\}$ *be the signs* $\varepsilon_{\alpha,\beta}$ *determined by comparing orientations. The cap product*

$$\cap B_z = \cap \varepsilon\, z^{\mathrm{rk}} c_{z^{-1}}(\mathscr{E}\text{xt})$$

is a bicharacter and therefore gives rise to a vertex algebra.

Theorem (Borcherds' foundational paper)

Let A_* be a vertex algebra. Then $L_* = A_{*+2}/T(A_*)$ is a Lie algebra with

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1. In our case, there is a comparison (Joyce 17'):

$$L_* \sim H_{*+\mathrm{vdim}_{\mathbb{R}}}(\mathcal{M}_X^{\mathrm{rig}}),$$

so L_* contains all $[M_\alpha^\sigma]^{\mathrm{vir}}$ and their generalizations $\langle \mathcal{M}_\alpha^\sigma \rangle$.

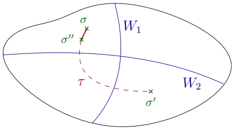
Conjecture (GJT(20'))

Often, the formula

$$\langle \mathcal{M}_\alpha^\sigma \rangle = \sum_{\underline{\alpha} \vdash_{\mathscr{A}} \alpha} \widetilde{U}(\underline{\alpha};\sigma',\sigma)\Big[\Big[\cdots\Big[\langle \mathcal{M}_{\alpha_1}^{\sigma'}\rangle, \langle \mathcal{M}_{\alpha_2}^{\sigma'}\rangle\Big],\ldots\Big],\langle \mathcal{M}_{\alpha_n}^{\sigma'}\rangle\Big]$$

holds in L_0.

1. The proof can be separated into two problems:



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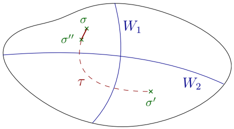

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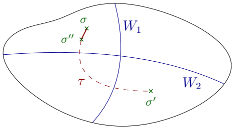

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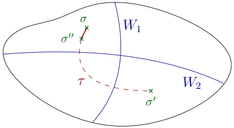

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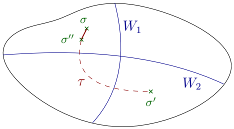

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2. New tools and conceptual framework for proving these two steps developed in B. (25').
3. Contains the first established proofs of wall-crossing in the CY4 setting.

Theorem (B. (25'))

Both (I) and (II) hold in the case of representations of CY4 quivers and sheaves or pairs on local Calabi-Yau fourfolds.

1. The invariants $\langle \mathscr{M}^{\tau}_{\alpha} \rangle$ counting stable sheaves are defined using

$$P^{\tau}_{L,\alpha} = \left\{ L \xrightarrow{s} F \text{ Joyce-Song/Bradlow stable } \right\}$$

and $[P^{\tau}_{L,\alpha}]^{\mathrm{vir}}$ for L a sufficiently positive line bundle.

Results II

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Theorem (B. (25'))

For a very ample D and $L' = L \otimes \mathscr{O}_X(-D)$, there is an embedding $\iota : P^\tau_{L,\alpha} \to P^\tau_{L',\alpha}$. It relates their virtual fundamental classes by

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3. The theorem from the previous slide is extended using the Jouanolou trick in

Theorem (B.-Kuhn-Liu-Thimm (in progress))

Both (I) and (II) hold for any CY 4-fold X.

First applications

Historically first application

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2. Starting point: For a line bundle L on X, construct

$$L^{[n]} = \pi_{H,*}({\pi_X}^*(L)\cdot \mathscr{F})$$

using the projections $X \xleftarrow{\pi_X} X \times \mathrm{Hilb}^n(X) \xrightarrow{\pi_H} \mathrm{Hilb}^n(X)$ and $\mathscr{I} \to \mathscr{O} \to \mathscr{F}$ the universal exact sequence. Define tautological invariants

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3. The wall-crossing formula for $\mathrm{Hilb}^n(X)$ becomes

$$\sum_{n\geq 0}[\mathrm{Hilb}^n(X)]^{\mathrm{vir}} q^n = \exp\Big\{\sum_{n>0}\Big[\langle \mathscr{M}_{np}\rangle, -\Big]q^n\Big\}e^{(1,0)}.$$

Cap product is a morphism

1. The action of $\cap c_n(L^{[n]})$ commutes with $[-,-]$ (roughly) because $L^{[n]}$ is a vector bundle of rank n.

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3. In fact, need invariants only for $L = \mathscr{O}_X(D)$ when D is a smooth divisor.
4. Conjectured by Cao–Kool (17') and proved by Park (21').

Nekrasov's conjecture

1. Consider the case when $X=\mathbb{C}^4$ with the natural action of the CY4 torus $T_3=\{t_1t_2t_3t_4=1\}\subset(\mathbb{C}^*)^4$.

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2. K-theoretic invariants $K_n(L,y)$ refining $C_n(L)$ were introduced by Nekrasov. Expressed using $\chi\big(\mathrm{Hilb}^n(X),\widehat{\mathcal{O}}^{\mathrm{vir}}\otimes -\big)$.

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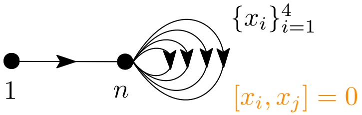
Example

Use

$$\mathrm{Hilb}^n(\mathbb{C}^4) = \mathrm{Rep}^{\sigma}_{(1,n)}(C_4)$$

where

$C_4 =$ 

and equivariant wall-crossing for CY4 quivers from B. (25').

Example continued

1. One limit of $K_n(L,y)$ is

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vanish.
4. Use
$$\left[\langle \mathscr{M}_{p}\rangle, -\right] = \frac{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

Example continued

1. One limit of $K_n(L,y)$ is
$$I_n = \int_{[\mathrm{Hilb}^n(\mathbb{C}^4)]^{\mathrm{vir}}} 1\,.$$
2. The generating series was conjectured to be:
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5. Other proofs in Kool-Rennemo or Cao–Zhao–Zhou.

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Stable pairs invariants

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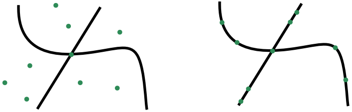

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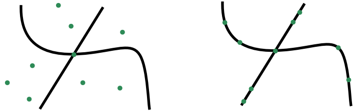

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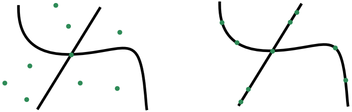

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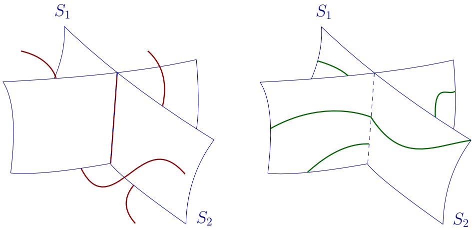

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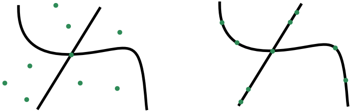

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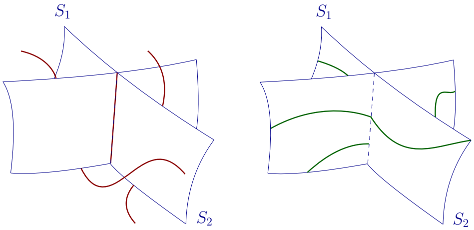

   2. DT - the usual Hilbert scheme of 2-dimensional subvarieties
4. I will denote the moduli spaces with $\mathrm{ch}(F) = (\gamma, \delta)$ by $\mathrm{PT}^{(i)}_{\gamma,\delta}$ even if $\gamma = 0$.

Refinements and tautological stable pair correspondences

1. For $\mathrm{PT}^{(i)}_{\gamma,\delta}$, define $L_{\gamma,\delta}$ the same way we did $L^{[n]}$.

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3. Conjecture of BKP:

$$\left\{ \begin{array}{c} \text{Integrals} \\ \text{over DT moduli spaces} \end{array} \right\} = \left\{ \begin{array}{c} \text{Integrals} \\ \text{over } \mathrm{PT}^{(0)} \text{ moduli spaces} \end{array} \right\} \cdot \left\{ \begin{array}{c} \text{Integrals} \\ \text{over } \mathrm{Hilb}^{n}(X) \end{array} \right\}.$$

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4. Would like to formalize it and extend it to $\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing.

$\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing

1. Without insertions get

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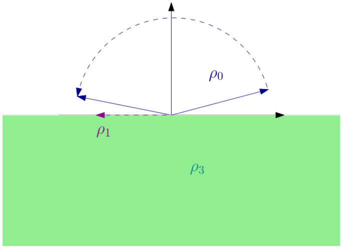

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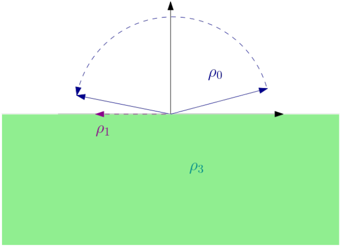

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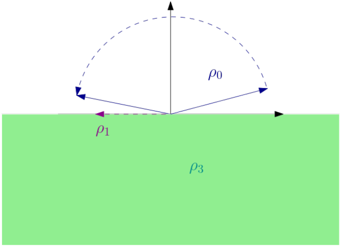

2. The operation $\cap c_{\mathrm{rk}}(L_{\gamma,\delta})$ no longer behaves well.
3. Instead, construct "vertex algebras" twisted $u^{\mathrm{rk}}c_{u^{-1}}(L_{\gamma,\delta})$

Additive deformations of vertex algebras

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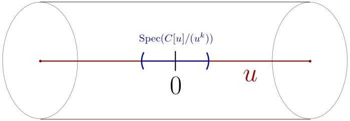
2. These are special nice deformations of VAs in the sense of H. Li (02') that depend on $(z-u)^{-1}$:



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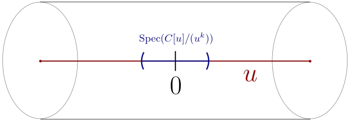
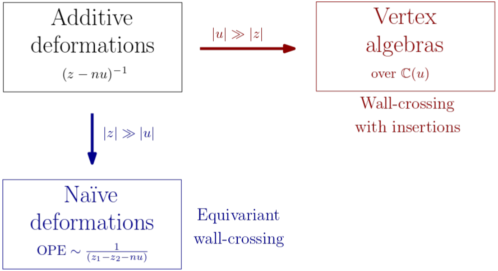


3. To get Lie algebras, need residues, so choose an expansion:



Applications: $\mathrm{PT}/\{\mathscr{O}_X\}$

1. Taking the appropriate coefficients of u in $[-,-]_u$ - the deformed Lie bracket, get:

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Example

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$$\begin{array}{ccc} X & \longleftarrow & L := \pi^* L_B \\ \downarrow \pi & & \downarrow \\ B & \longleftarrow & L_B \end{array} .$$

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Conjecture (Cao-Toda (21'))

$$\sum_{\pi_*(\delta)=0} P_\delta(L) q^\delta = \prod_{\substack{\beta:\ \pi_*(\beta)=0 \\ k>0}} \left(1-(-1)^k q^{(\beta,k)}\right)^{k\cdot n_{0,\beta}(L)} \prod_{\beta:\ \pi_*\beta=0} M\left(q^{(\beta,0)}\right)^{n_{1,\beta}(X)}.$$

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2. For now have

$$P_\delta(L) = \mathrm{Pol}\Big(\langle \mathcal{M}_{\beta,k}\rangle^L, \langle \mathcal{M}_{\beta,k}\rangle^{\mathcal{O}_X} : \pi_*(\beta) = 0, k > 0\Big), \quad \langle M^{ss}_\delta\rangle^L = \int_{\langle \mathcal{M}_\delta\rangle} c_1(L^{[\delta]}).$$

$\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing on elliptic fibrations

Example

Consider the situation $\pi : X \to B$ for an elliptic fibration and $L = \mathcal{O}_X$.

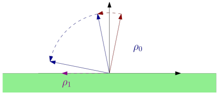
Conjecture (BKP (upcoming))

If $(\gamma,\delta) = \pi^*(\beta,n) + mp$ *for* $(\beta,n) \in H^{\geq 4}(B)$ *then*

$$\sum_{d\geq 0} \langle PT^{(0)}_{\gamma,\delta+dE}\rangle^{\mathcal{O}_X} q^d = \sum_{d\geq 0} \langle PT^{(1)}_{\gamma,\delta+d_0E}\rangle^{\mathcal{O}_X} q^{d_0} \sum_{\Delta\geq 0} \langle \mathrm{PT}_{\Delta E}\rangle^L q^\Delta$$

Claim (B. (upcoming))

The wall-crossing formula holds for the total path of



Will come back to prove the vanishing of the last segment of the arc.