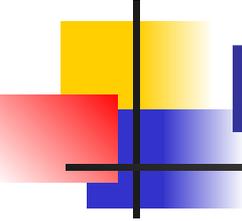# THE SANDPILE MODEL AND ITS EXTENSIONS: DYNAMICAL STRUCTURES and FIXED POINTS

PHAN Thị Hà Dương

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KAVLI: AOWM x IPMU - 03/2026

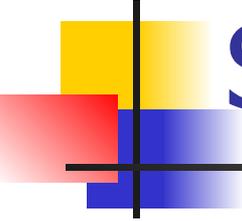# Self- Organized criticality phenomena (SOC)

- Large scale events which happen very rarely: earthquakes, avalanches or even stock market crashes.

- Clear cause: unusually rare events which normally never occur. Example: an avalanche may be triggered by an explosion.

- Other cause: apparently insignificant events of a type that occurs all the time.

- => self-organized criticality phenomena.
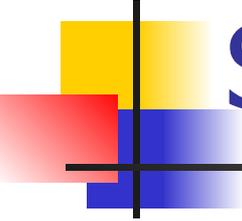
Sand Piles Model

- Consider grains of sand failing on a flat surface:
- Pile grows => system from steep, unstable slopes, which will collapse downwards in small avalanches.
- Steady state + avalanches => new Steady state. Grains land on it.
- Critical: in the steady state, there will be avalanches of all sizes up to a maximum scale set by the size of the system.
- Self-organizing: no external processes are needed for pile to form its steady states.

SPM

- Origin: Self-organized criticality problem (SOC): Bak, Tang and Wiesenfeld (1987).
- Mathematical model: Goles and Kiwi (1993).
- Language and algebraic approach: Bjorner, Lavasz and Shor (1991).
- Economic approach (Dollar Game): Biggs (1993)
- Graph theory approach: Dhar, Cori and Rossin (1993, -2003)
- Order and lattice approach: Goles, Morvan, and Phan (1998)
- Petri Net, Complex network: Phan, Le, Tran, Pham (2005 -)
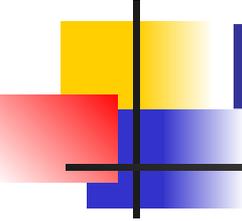
Discrete Dynamical Systems

- A Discrete Dynamical Systems is defined by its configurations and its rules of transitions.
- Convergence property. Fixed points.
- Optimal time for convergence.
- Characterization of accessible configurations
- Behavior of dynamical systems.
- Size of systems: number of configurations.
- Complexity of systems.
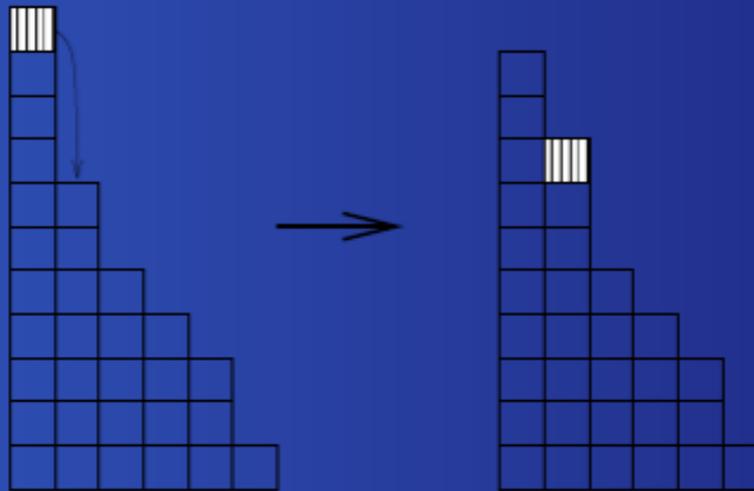- Relation between different systems.
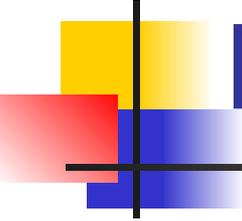
Mathematical tools

- Order and lattice structures: minima, maxima, longest path, suborder and sublattice relations.

- Theory of integer partitions: partitions, plane partitions, cardinality of some set of partitions.

- Algebra: commutative group, Laplace matrix, Tutte polynomial

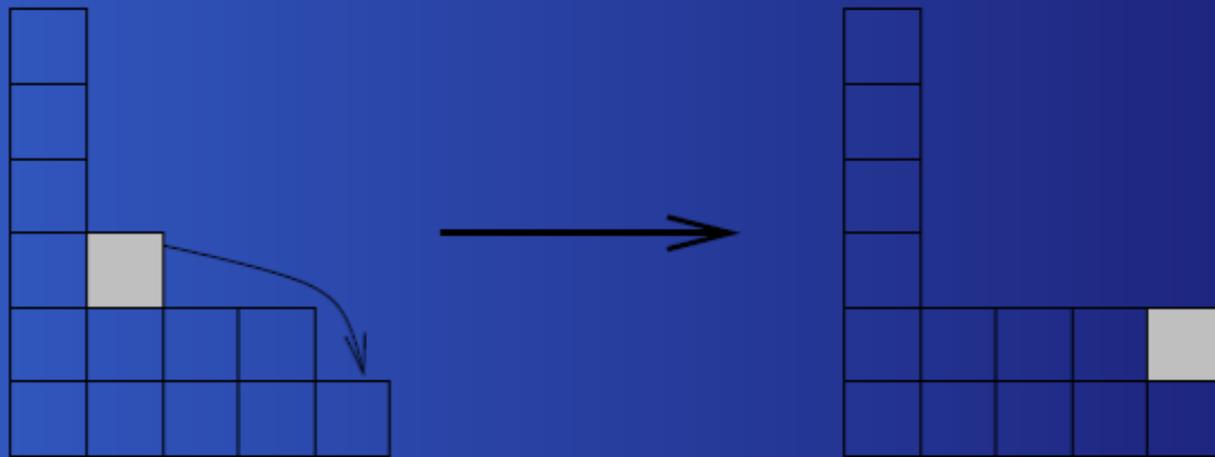- Graph theory: spanning tree, plane graphs

SPM rule

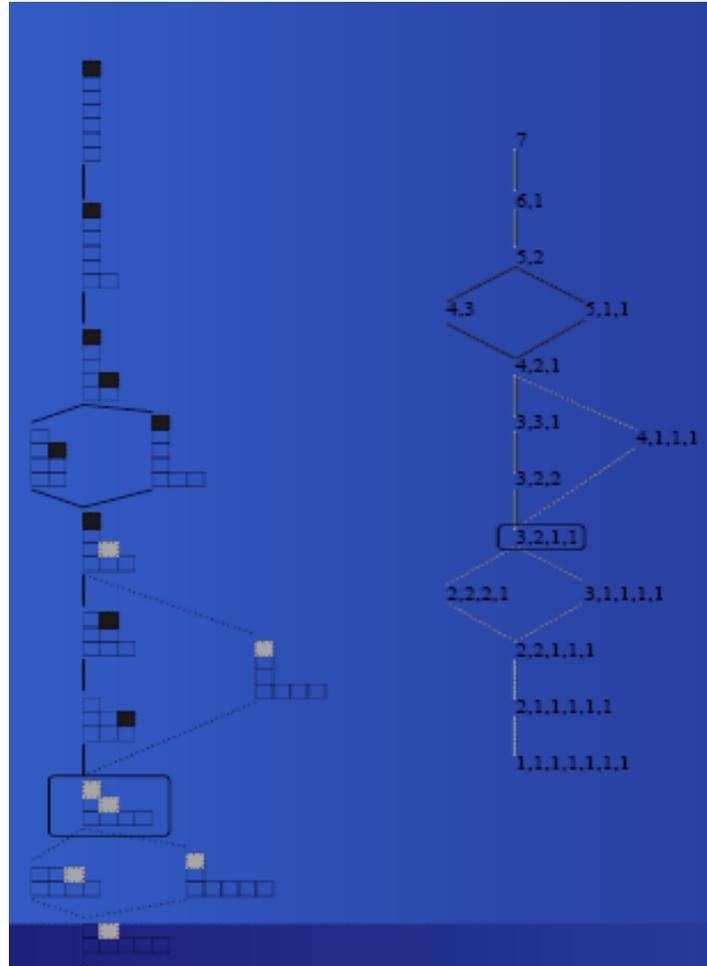$$(\ldots, a_i, a_{i+1}, \ldots) \rightarrow (\ldots, a_i - 1, a_{i+1} + 1, \ldots).$$
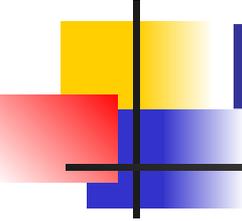
condition: $a_i - a_{i+1} \geq 2$.

Brylawski rule

$$(\ldots, p \ + \ 1, p, \ldots, p, p \ - \ 1, \ldots) \quad \rightarrow$$
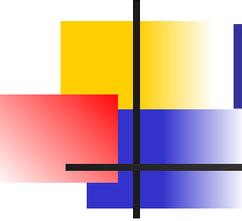$$(\ldots, p, p, \ldots, p, p, \ldots).$$
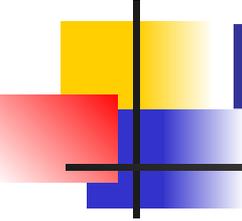
Small example

Bigger example (SPM(40))
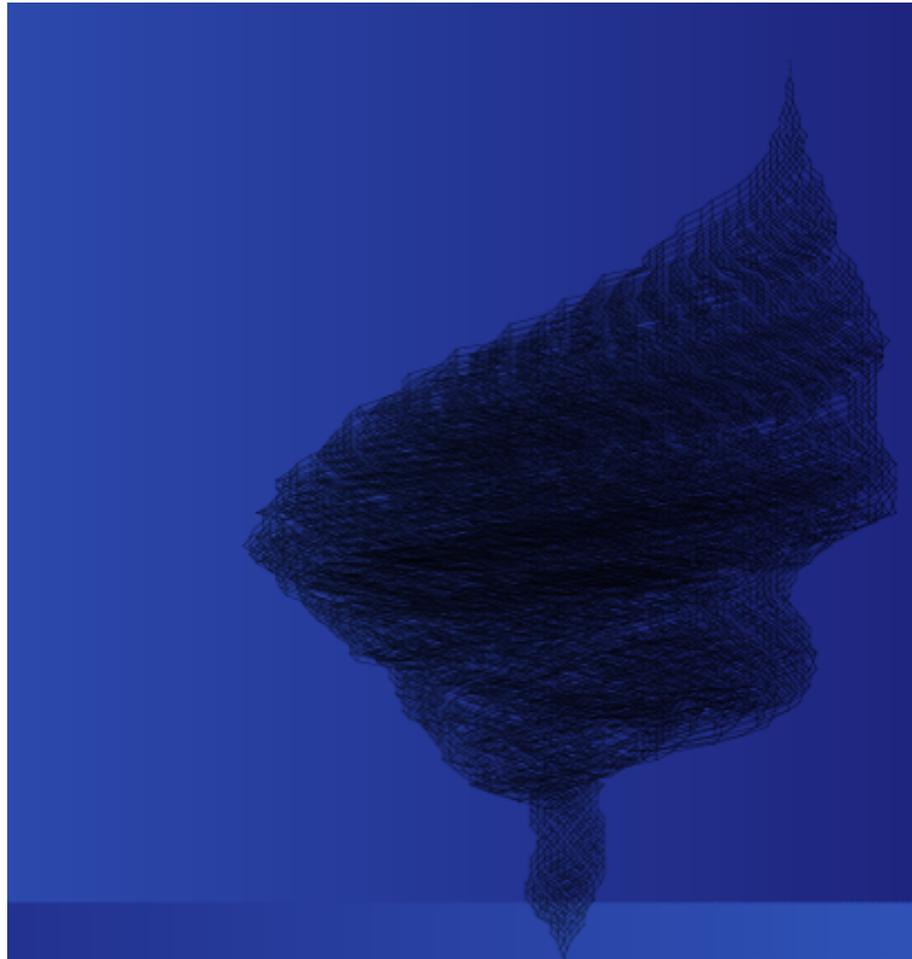
Notions

- Reachable configurations: configurations that can be reached from the initial configuration by evolution.

- Fixed point: configuration on which no rule can be applied.

- Configuration space: the set of reachable configuration.

- Behavior of the system: Structure of configuration space.

Behavior of the system

- Tools: Order and lattice theory.
- Order: a binary reflexive, antisymmetric and transitive relation.
- Lattice: an order T such that for all a, b of T, there exists sup(a,b) and inf(a,b).
- A finite lattice has a maximum and a minimum.
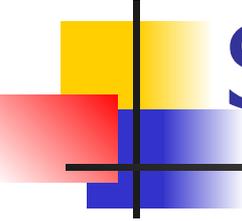
Structure of the system

- **Theorem** (Goles and Kiwi, 1993): The configuration space of SPM is an oder, and a suborder of Brylawski lattice

- **Theorem** (Morvan, Goles, Phan, 1998): the configuration space of extended SPM is a LATTICE and a suborder of Brylawski lattice

Chracterization of reachable configurations

Theorem (Goles, Morvan, Phan, 1998)

A configuration is reachable in SPM if it does not contain the following forbidden subsequence.
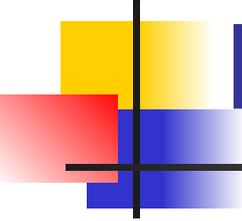
(p,p,p); (p,p,p-1,p-2, ..., q+1,q,q)
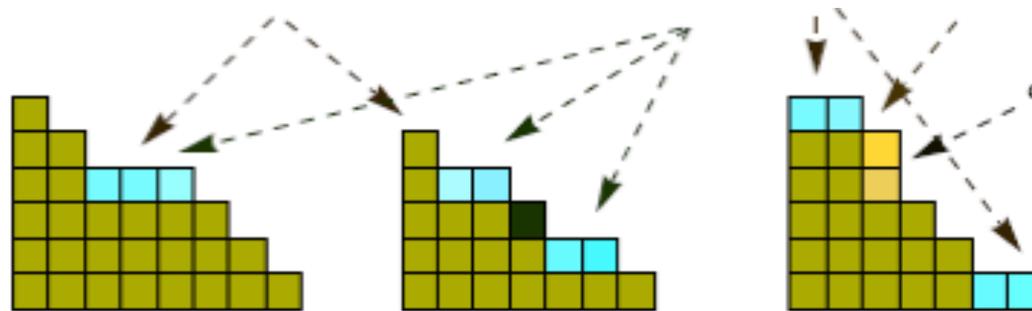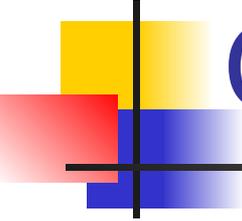
Corrolaries

- The system converges to an unique fixed point whatever the evolution.

- Characterization of reachable configuration.

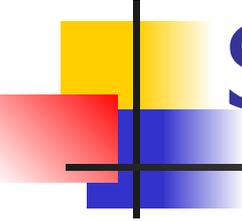- Chracterization of Reachability of the system.

Fixed point of SPM

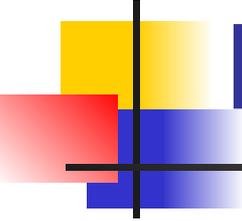- The fixed point of SPM is of the form:

(p,p-1, …,q+1, q, q, q-1, …, 2,1)
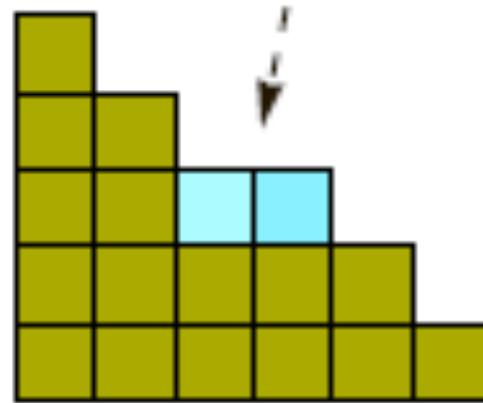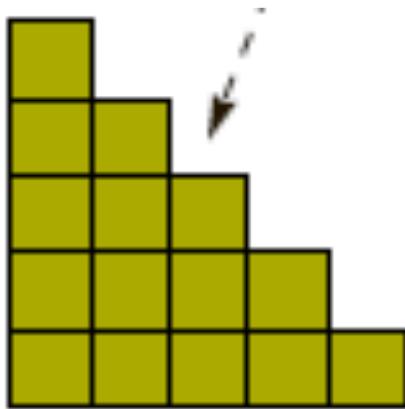
SPM in the nature

- SPM

- SPM in parallel: PSPM

- Two sides SPM (symmetric SPM): SSPM

- Symmetric SPM in parallel: PSSPM
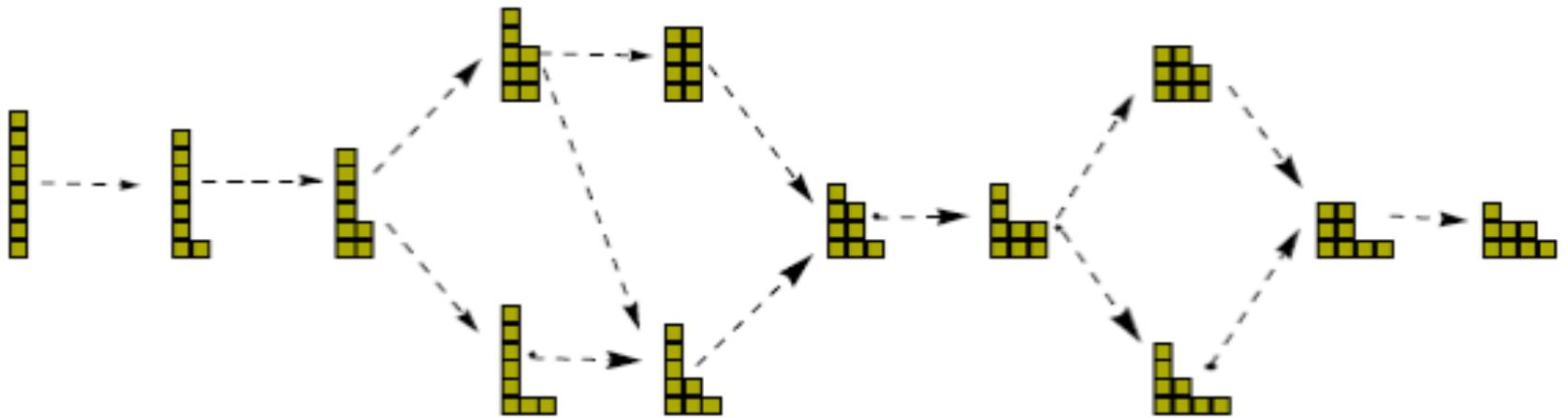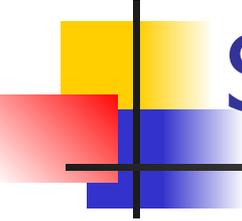
- We present results on the fixed points of these models

Fixed points of models

- Results on extensions of the SPM
- Fixed points of the Symmetric Sandpile Model (SSPM)
- Parallel Symmetric Sandpile Model (PSSPM): Questions on fixed points
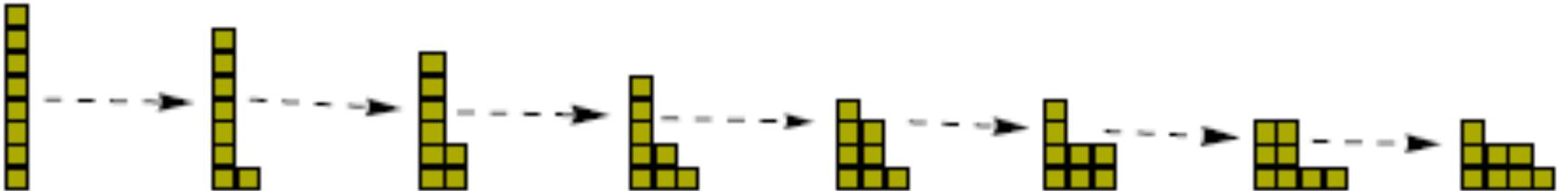- Result: All fixed points of the SSPM are reachable in the PSSPM (proof and algorithm).
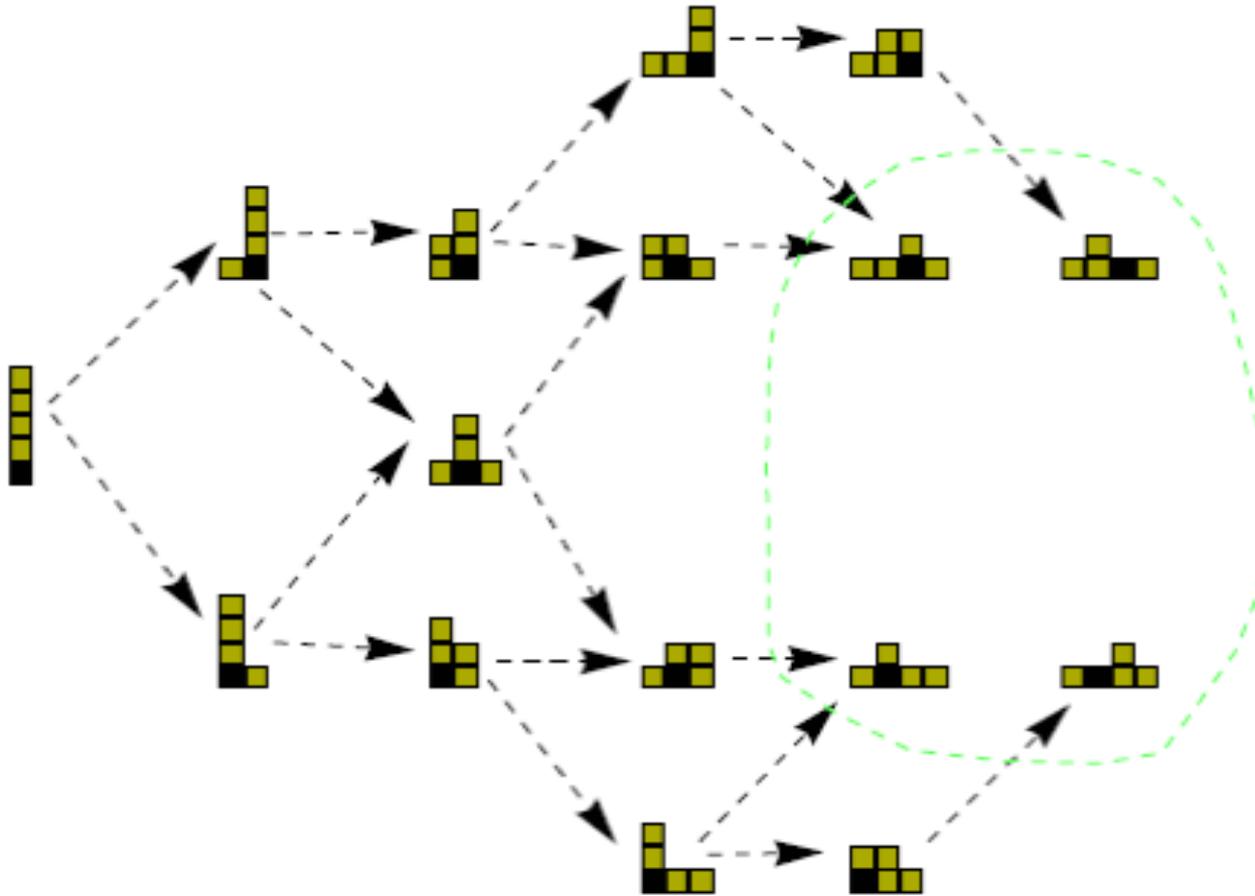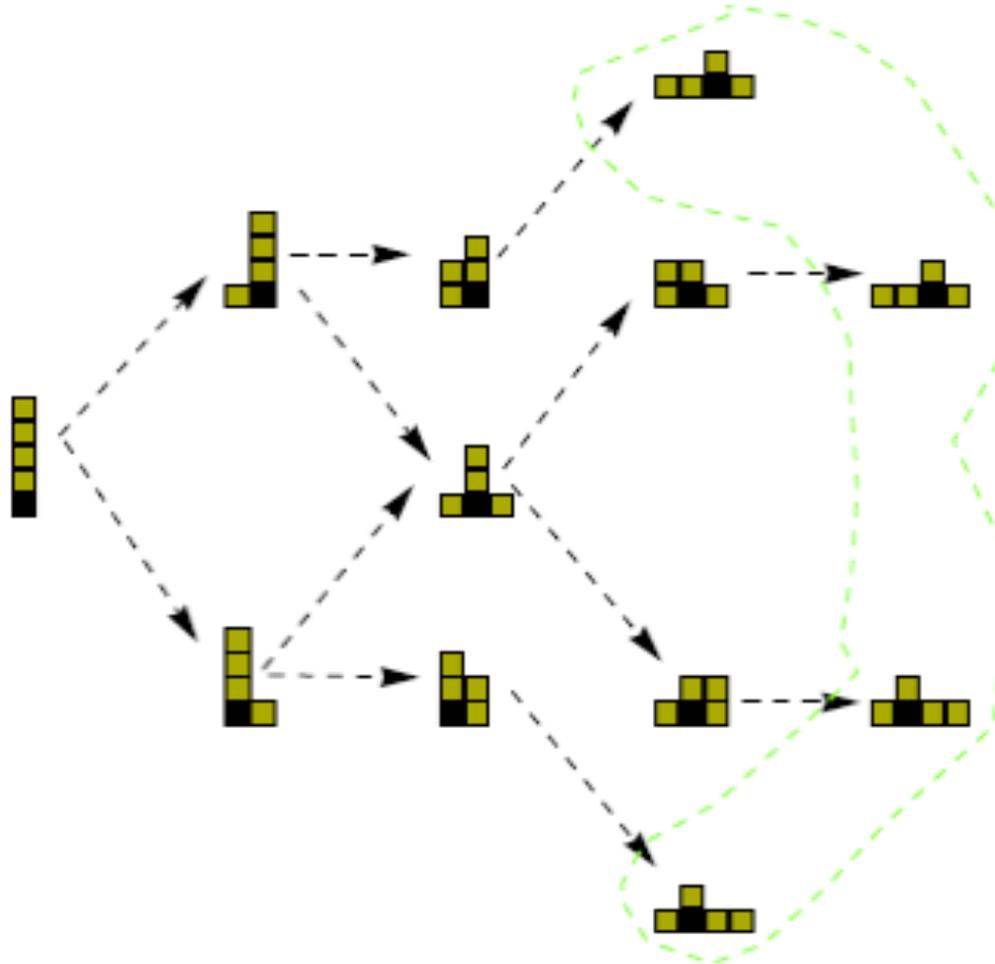
Small example of SPM

SPM in parallel (PSPM)

Symmetric SPM (SSPM)
(Phan, Formenti &Masson, 2005)

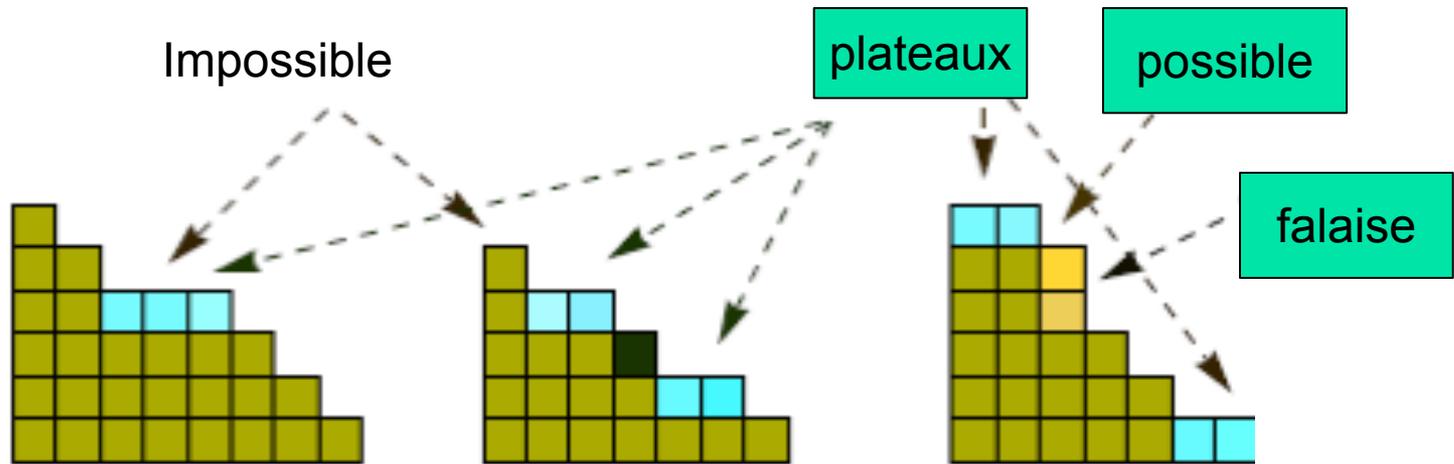# Symmetric SPM in parallel
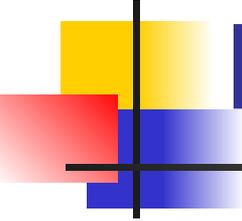(E. Formenti, V. T. Pham, T. H. D. Phan et T.T. H. Tran, 2006)
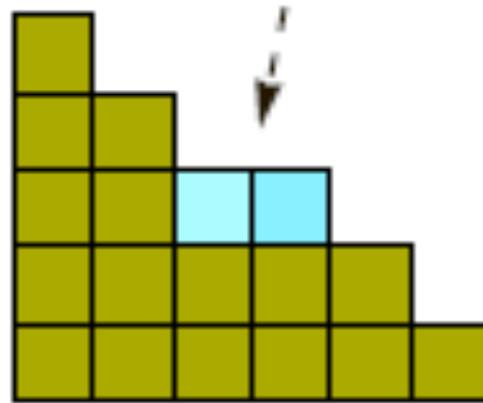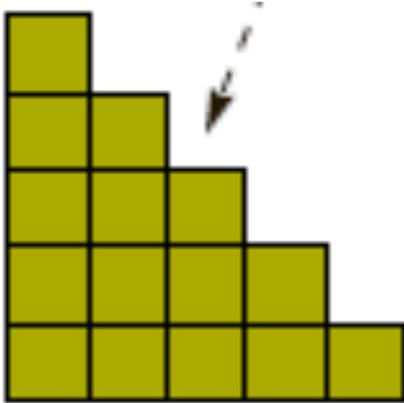
Characterization of Configurations of SPM
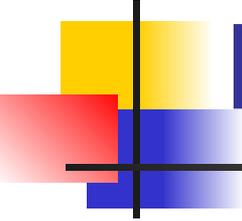

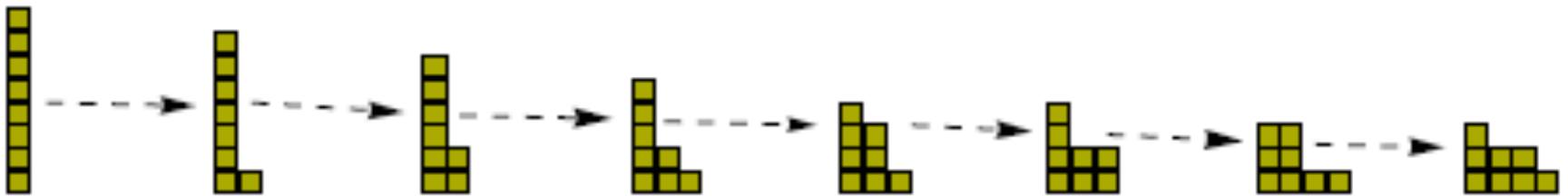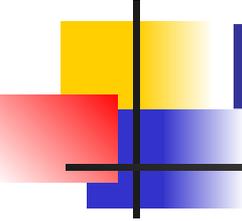Impossible

plateaux

possible

falaise

Fixed Point of SPM

Model PSPM

- Rule: all grains which can be fall fall in the same time
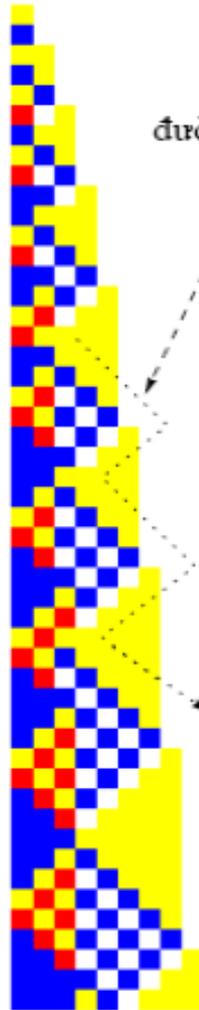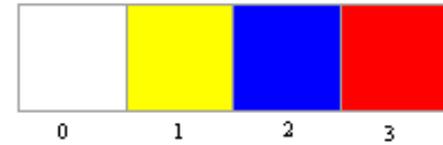
- Study domain: Cellular automata

Results of PSPM

- The unique fixed point is that of SPM.

- Computing the time for the convergence (Durand_Lose, 1996) :
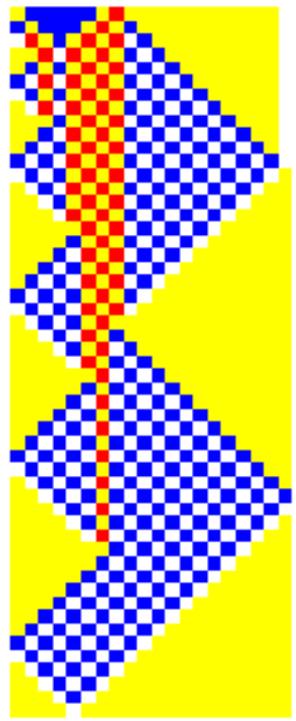 - There are two periods: column 1 changes and column 1 does not change.

Period 1

đưa
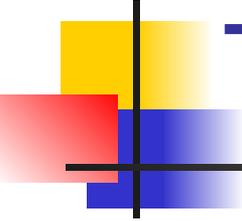

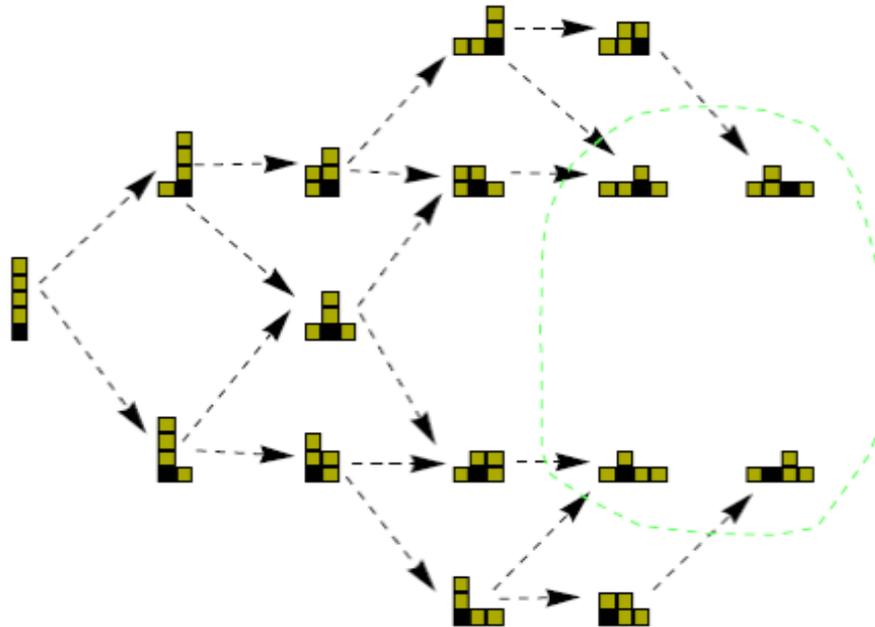|			
0	1	2	3

Period 2

Total time of PSPM

- Theorem (Durand-Rose, 1998): The total time to reach the fixed point in PSPM is

$$n + o(n) \leq T_{all}(n) \leq \left(1 + \frac{1}{2+\sqrt{2}}\right)n + o(n)$$

Model SSPM

- Definition : (Formenti, Masson & Pikosa, 2005 – Phan 2005)

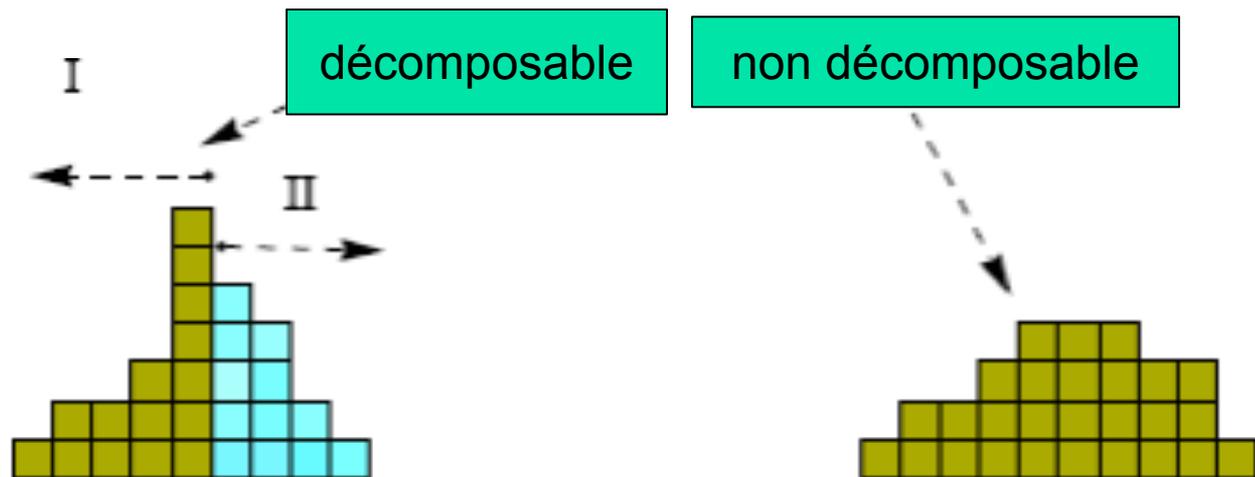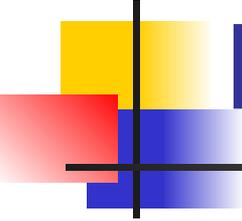- Rule: grains can fall to the left or to the right

Characterization of configurations of SSPM

Theorem (Phan, Formenti & Masson, 2005) An unimodal sequence is of SSPM if it can be decomposed into two partitions satisfying SPM condition.

Unimodal sequence is a sequence

$$a_0 \leq a_2 \leq \ldots \leq a_{p-1} \leq a_p \geq a_{p+1} \geq \ldots \geq a_n$$
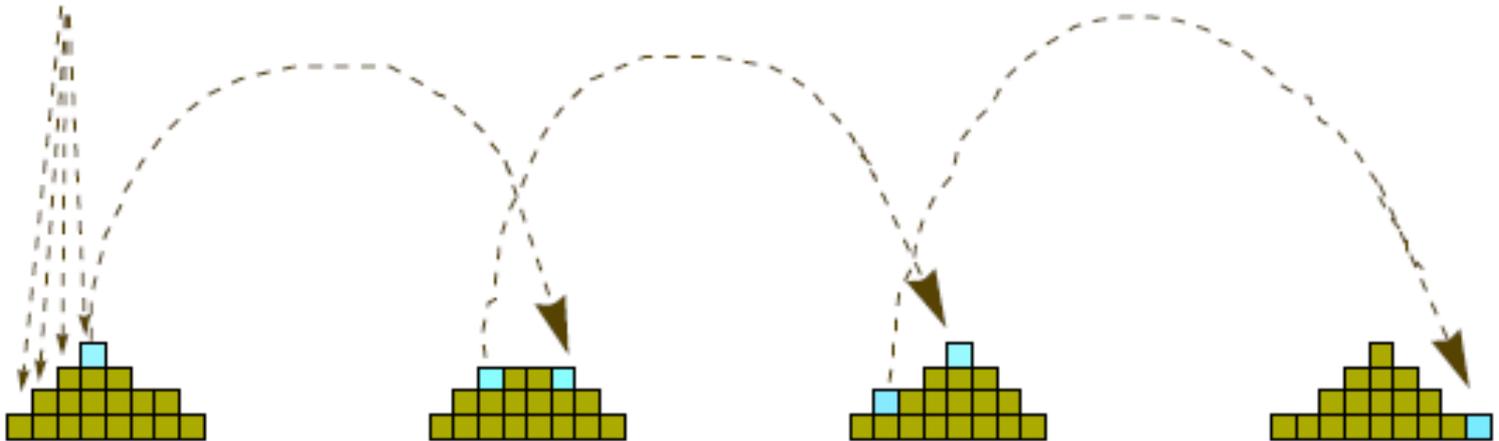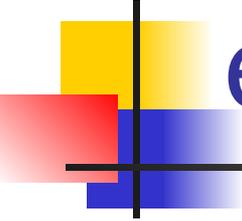

décomposable

non décomposable

Fixed points of SSPM

Theorem (Formenti & Masson, 2005) The number of forms of fixed points of SSPM is $\sqrt{n}$
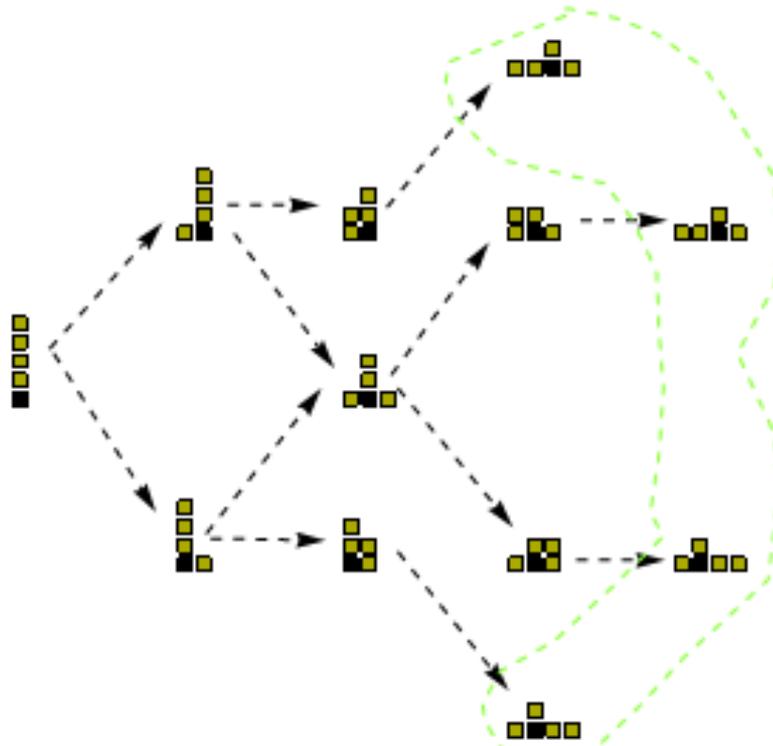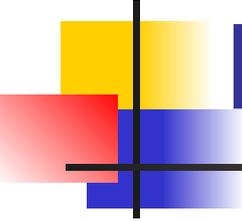
Minimal and maximal time of evolutions

- Explicite formulas for maximal and minima time of evolution to reach a fixed point.

- Explicite formulas for the longest and shortest evolutions.

Model PSSPM

- Grains can fall to the left or to the right
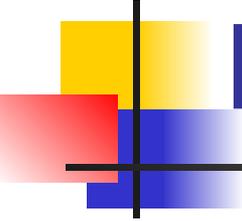- All grains which can fall fall in the same time

Remarks

- SPM has an unique fixed point. PSSPM is a sub space of SPM; so it has the same fixed point.
- SSPM has many fixed points. PSSPM is a sub space of SSPM.

- Question: relation between fixed points of SSPM and of PSSPM ?

- Simulation: PSSPM has all forms of fixed points of SSPM

Principal results on PSSPM

- **Theorem** (Formenti, Pham, Phan, Tran, 2008): The forms of fixed points of PSSPM and of SSPM are the same.
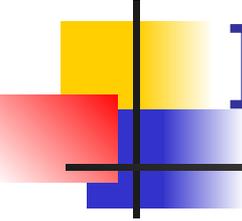
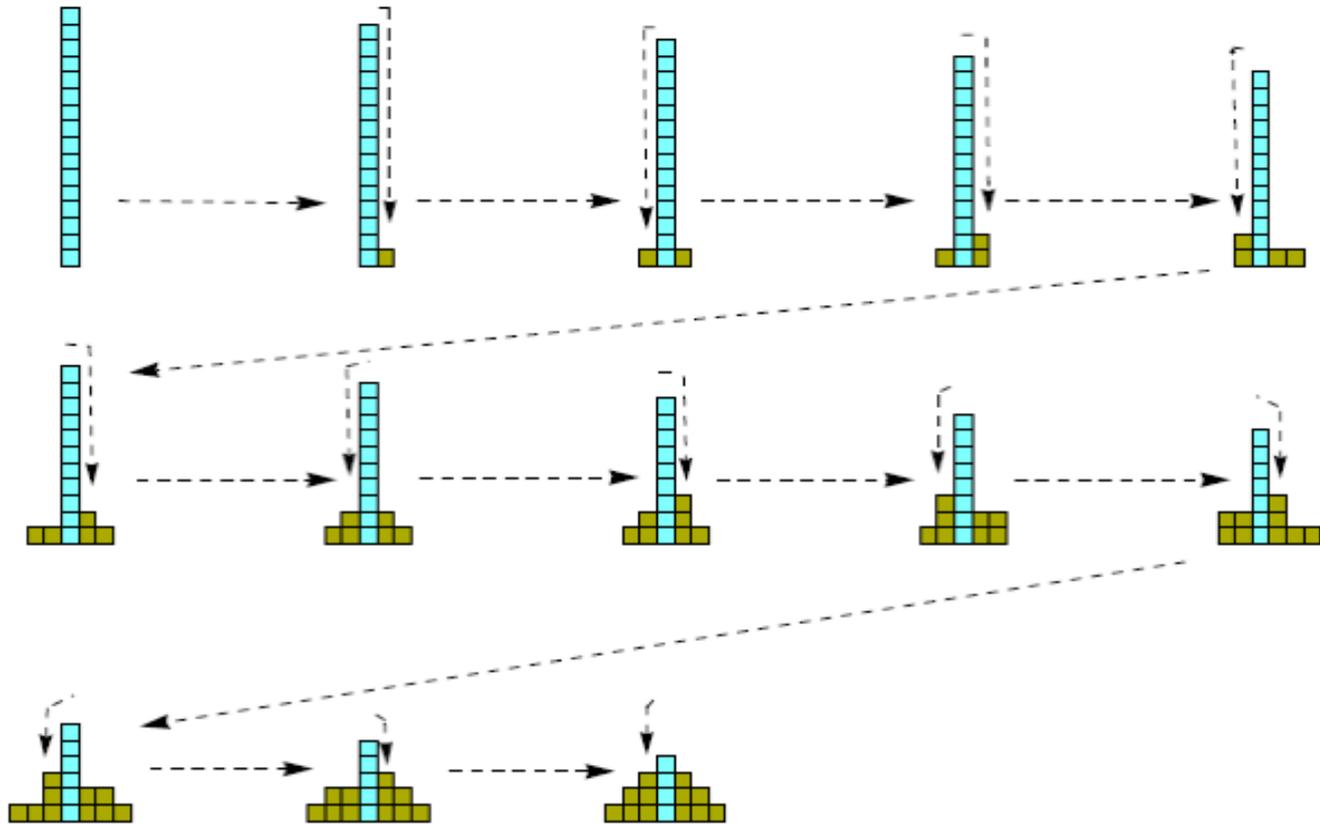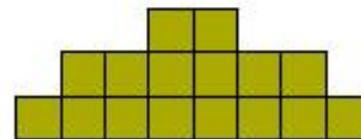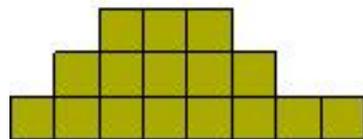# Idea of the proof

- Given a fixed point of SSPM, we show an algorithm by PSSPM rule to reach this fixed point.

Symmetrique procedure to reach a symmetric fixed point.
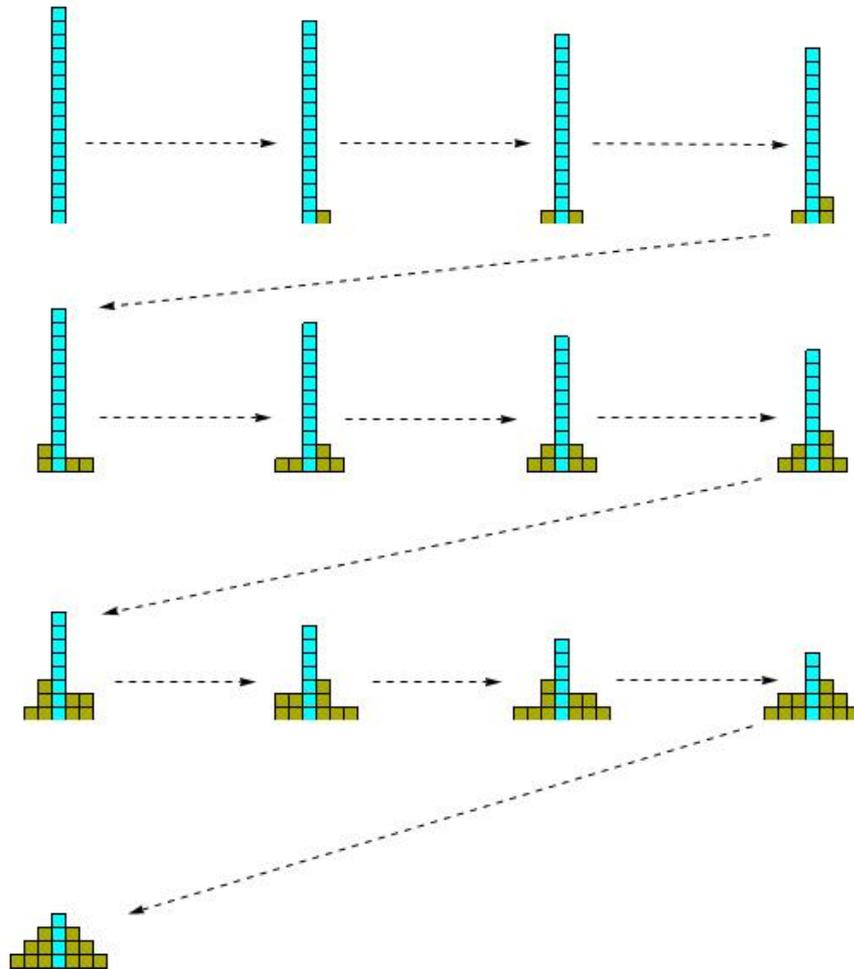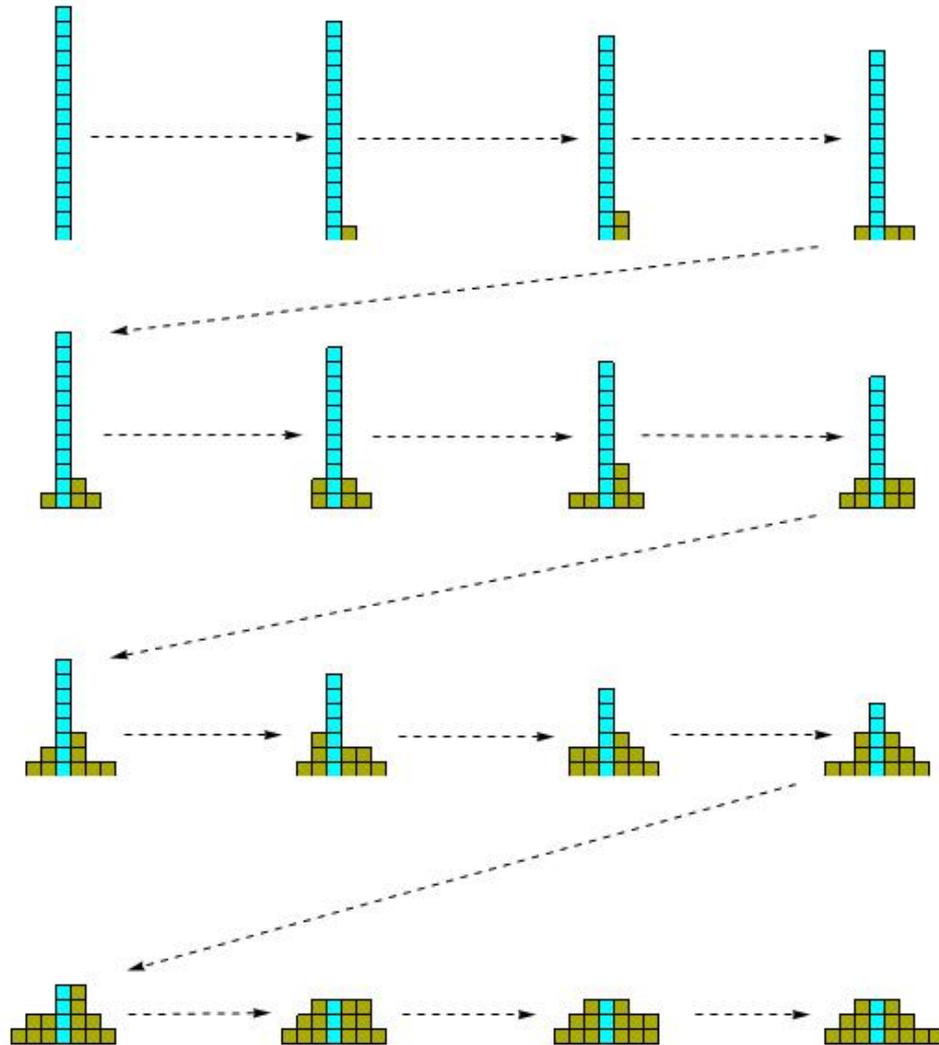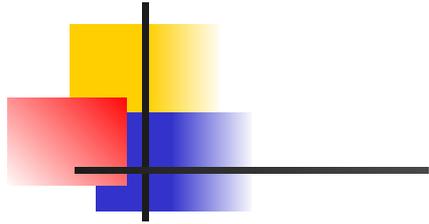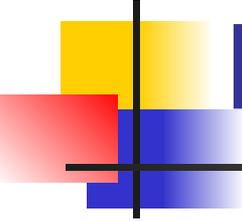
All fixed points

Symmetric fixed points
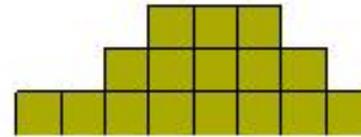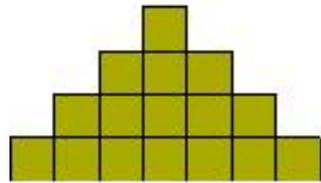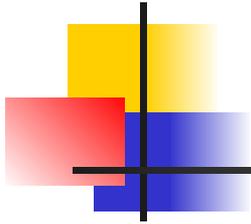
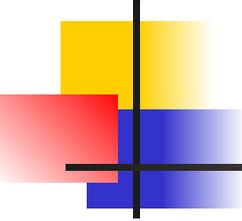# Non symmetric fixed points

Non symmetric fixed points

Perspectives

- Sand piles model on bidimension
- Sand piles model on an arbitrary graph: relation with complex network
- Simulation to calculate exact time of evolution in parallel symmetric SPM.

References

1. M. Latapy, R. Mantaci, M. Morvan, H. D. Phan, *Structure of some sandpile models*, Theoretical Computer Science 262 (2001), 525–556.

2. M. Latapy, H. D. Phan, *The lattice structure of chip firing games and related models*, Physica D 155 (2001), 69–82.

3. E. Goles, M. Morvan, H. D. Phan, *Sandpiles and order structure of integer partitions*, Discrete Applied Mathematics 117 (2002), 51–64.

4. E. Goles, M. Morvan, H. D. Phan, *The structure of a linear chip firing game and related models*, Theoretical Computer Science 270 (2002), 8?–841.

5. E. Duchi, R. Mantaci, H. D. Phan, D. Rossin, *Bidimensional sandpile and ice pile models*, Theoretical Computer Science (2007).

6. M. H. Le, P. T. H. Dương, *Strict partitions and discrete dynamical systems*, Theoretical Computer Science 389 (2008), 82–90.

7. P. T. H. Dương, *Two-sided sandpile model and unimodal sequences*, RAIRO – Theoretical Informatics and Applications 42 (2008), 631–646.

8. P. T. H. Dương, T. T. T. Hương, *On the stability of sand piles model*, Theoretical Computer Science 411 (2010), 594–601.

9. K. Perrot, P. T. H. Dương, P. V. Trung, *On the set of fixed points of the Parallel Symmetric Sandpile Model*, DMTCS Proc., Automata 2011.

10. P. V. Trung, P. T. H. Dương, *Lattices generated by Chip Firing Game models: Criteria and recognition algorithms*, European Journal of Combinatorics 34 (2013), 812–832.

11. E. Formenti, P. V. Trung, P. T. H. Dương, T. T. T. Hương, *Fixed-point forms of the parallel symmetric sandpile model*, Theoretical Computer Science 533 (2014), 1–14.

Thank you for your attention!