AOWM x IPMU

Minimal submanifolds in higher codimension

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Manifold M : locally like $\mathbb{R}^n$. using transition maps to patch together ex: S^2, $\mathbb{R}P^n$. $G(2.4)$

can be given abstractly

For $x \in M$, can define Tangent space $T_x M$

Riemannian manifold (M, g). at every pt x. give a symmetric positive definite inner product $g(x)$. which varies differentiably (smoothly)

With g, we can define length, angle, volume, curvature

can do calculus, PDE on M Geometric Analysis

For a curve $\gamma : [a,b] \longrightarrow M$, use g to define $|\gamma'(t)|$

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle_g^{\frac{1}{2}}\, dt$$

On $\mathbb{R}^n$, the shortest curve connecting two points are straight lines.

What are the situation in (M, g)?

Suppose γ is parametrized by arc length. i.e $|\gamma'(t)| = 1$ denote $\gamma'(t) = T$

and γ_s a variation of γ, fixing end pts.

$$\frac{d\,\gamma_s(t)}{ds}\Big|_{s=0} = V \qquad \alpha : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$$

$$\qquad\qquad\qquad t \quad . \quad s$$

a variational field along γ denote $\gamma_s(\cdot) = \alpha(\cdot, s)$

Can compute $\frac{d}{ds} L(\gamma_s)\Big|_{s=0}$ like the directional derivative of a fun in $\mathbb{R}^n$

$= -\int_a^b \langle V, \nabla_T T \rangle \, dt.$

the covarian derivative of T w.r.t T

If $\frac{d}{ds} L(\gamma_s) = 0$ for $\forall V$, $\Rightarrow \nabla_T T = 0$. for $\forall t \in [a, b]$

γ is called a geodesic

ex: The geodesics on S^2 are part of great circles

Similarly for a submanifold $\Sigma^n \subset M^{n+k}$, we can define its area $A(\Sigma) = \int_\Sigma dv_\Sigma = \int_\Sigma \sqrt{\det h_{ij}} \, dx^1 \cdots dx^n$

From (M, g), it induces a Riemanian metric h on Σ

We can consider a deformation Σ_s of Σ, with variational field V.

$i : \Sigma \hookrightarrow M$

$\varphi_s : \Sigma \hookrightarrow M$ $\quad \left.\frac{d\varphi_s(x)}{ds}\right|_{s=0} = V(x)$

$$\left.\frac{d A(\Sigma_s)}{ds}\right|_{s=0} = -\int_\Sigma \langle V, H \rangle \, dV_\Sigma$$

$\varphi_0 = i$, require V cpt support if Σ is noncompact

H: called the mean curvature vector on Σ

when in o.n basis $\{e_i\}_{i=1}^n$. $\quad H = \sum_{i=1}^n \nabla^\perp_{e_i} e_i$

Σ is called minimal if $H = 0$

(Note: just a critical pt. not exactly minimal.

geodesics are 1-dimal minimal subjects

Plateau problem:

Given an imbeded curve Γ in $\mathbb{R}^3$, try to find minimal surface Σ. with boundary Γ Soap film

(i) Existence.

absolutely minimizer

(ii) uniqueness ?

(iii) more general Γ, and in M

(iv) higher dimension and codimension

(v) Geometric measure theory (GMT).

Take minimizing seq. problem & difficulties

(vi) regularity (hypersurface, v. higer codimension

Minimal surface eq :

① $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. $\Sigma_f = (x, f(x))$ minimal

$\Longleftrightarrow \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0$

$\Longleftrightarrow \sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}\right) f_{ij} = 0$

If Ω with boundary

Dirichlet problem

② $f : \Omega \subset \mathbb{R}^n \longmapsto \mathbb{R}^k$, $\Sigma_f = (x, f^1(x), \dots f^k(x))$ minimal

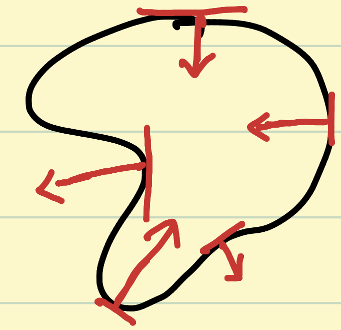
$\Longleftrightarrow \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f^\alpha}{\partial x^j}\right) = 0$, $\alpha = 1 \dots k$

$g_{ij} = \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j}$. induced metric

- (g^{ij}) : inverse matrix of (g_{ij})
- $g = \det(g_{ij})$

$m = 1$ v. $m \geq 2$

Mean curvature flow.



① curve shorten flow (CSF) $\left(\frac{d\gamma_t}{dt}\right)^{\perp} = \vec{k}$

Embedded closed plane curves shrink to round pts

(Gage - Hamilton - Grayson)

Application: use CSF. to find closed geodesic in M

(S^2, g) has at least 3 simple closed geodesics

(i.e no self-intersecting) optimal

② $X_t : \Sigma^n \hookrightarrow (M^{n+k}, g)$ satisfy $\left(\frac{\partial X_t}{\partial t}\right)^{\perp} = H(X_t(x))$ $\forall t$

(if sol exists for long time and converges. $\Longrightarrow$ give minimal submfd)

Embeded convex closed hypersurface in $\mathbb{R}^n$ shrink to round pts (Huisken)

- short time existence, singularity . .
- $m=1$. v. $m>1$

other methods of constructing minimal submfd

(i) Weierstrass representation

(ii) reflection

(iii) critical pts of Energy functional

($n=2$. $\Leftrightarrow$ harmonic & conformal)

(iv) gluing method

⋮

Application of minimal submanifolds

- the boundary of Black hole events horizon
- to prove Postive Mass theory, Penrose ineq in GR
- ~~Interface~~
- Cristal structure
- model in biology
- To study topology
- in material
- in image

⋮

a curve in plane $C' \subset \mathbb{R}^2$.

① one direction of generalization: to hypersurface

② can be considered as of middle dim

$dX(T) = \cos\theta$. $\quad T = \gamma'(s)$

$\frac{d\theta}{ds} = k$ curvature

or $dZ(T) = e^{i\theta}$ $\quad z = x + iy$

Consider $L^n \subset \mathbb{R}^{2n}$, $\quad dZ = dz^1 \wedge \cdots \wedge dz^n$

require $dZ(e_1 \wedge \cdots \wedge e_n) = e^{i\theta}$

$\{e_i\}$ o.n on $T_x L$

L is called Lagrangian & θ Lag angle we have $H_L = J\nabla\theta$

on $\mathbb{R}^2$. $\theta = 0 \Rightarrow \gamma = x$-axis, $\theta = c \Rightarrow \gamma$ straight line

on $\mathbb{R}^{2n}$. if $\theta = 0$, or $dZ|_L = d\mathrm{Vol}_L$, L is called special Lagrangian. if $\theta = \theta_0$, L is called sLag of phase θ_0, They are minimal submanifolds and in fact volume minimizing, which are calibrated by $\mathrm{Re}\, dZ$ (or $\mathrm{Re}\,(e^{-i\theta} dZ)$)

The notion can also be defined in a Calabi-Yau manifold, when a parallel $(n,0)$ form Ω can be defined.

SLag plays an important role in String theory and mirror symmetry. SYZ conjecture

- There are lots of studies in SLag, but its general existence is still quite open
- We will come methods to study the problem later

The Notion of a Lagrangian submanifold can also be defined in a symplectic manifold (M^{2n}, ω) as follows:

ω: a closed non-degenerated 2-form.

$L^n \subset M^{2n}$. and $\omega|_L \equiv 0$

(appear naturally in physics.

ex: phase space

T^*N

$\mathbb{R}^{2n}$.

Kähler mfd

- In a Kähler manifold.

g. and ω are related by

$$\omega(u, v) = \langle Ju, v \rangle$$

we can study objects that are both minimal and Lagrangian.

nice proproperty on immersed minimal Lagragian

$$\frac{d^2 A_s}{ds^2}\Big|_{s=0} = \int_L (|d\alpha_v|^2 + |\delta \alpha_v|^2 - \overline{Ric}(V, V))\, dvol_L$$

$\alpha_V(\cdot) = \omega(V, \cdot)$ 1-form

(i) so if M has negative Ric

L is always strictly stable

(ii) If $Ric_M = 0$ (CY). $\left.\frac{d^2 L}{ds^2}\right|_{s=0} \geq 0$

and Jacobi field $\Leftrightarrow$ harmonic 1-form

they can be realized by the deformation of sLag

(Mclean)

(iii) If $Ric_M > 0$. L is always unstable

But in general, it is hard to expect the existence of min. Lag in a Kähler mfd.

$\because$ on a Lag L in a Kähler mfd, we have

$$d\alpha_H = \text{Ric form}\big|_L$$

$\Rightarrow$ on a min. Lag $\omega|_L \equiv 0$

$Ric|_L \equiv 0$ over-determined

On a Kähler-Einstein mfd ($Ric = c\omega$), the obstruction disappear. So we will study min Lag on Kähler-Einstein manifolds

① For MCF in KE, if initial L is Lag. then Lag condition will be preserved for smooth sol of MCF (Lag mean curvature flow LCMF)

Thmoas-Yau-Joyce conjecture.

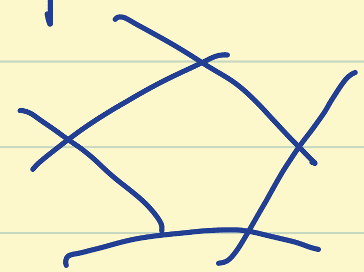
② Taking area minimizer among Lagranian submfd

minimizer not necessarily minimal

③ Construct examples with special antaza in highly symmetric spaces (D. Joyce, ⋯)

④ PDE   SLag equation

⑤ Gluing method   resolve self-intersection pts
SLag loops with banlancing condition



⑥ check directly

⑦ deformation

⋮

Thm: Lee 1998

Suppose (N, ω, g_0) KE surface with $c_1 < 0$.

and $[A] \in H_2(N, \mathbb{Z})$ can be presented by a finite union of branched minimal Lag surface. w.r.t (ω, g_0)

For any KE metric g. in the same connected component of g_0 in the moduli space of KE on N
(complex structure)

Assume g is connected to (g_0, ω). by (g_t, ω_t)
by composing a diffeomorphism, we can assume $\omega_t = \omega$

Then $[A]$ can also be presented by a finite union of branched min. Lag surfaces w.r.t. (g, ω)

aim to generalize the results

① To complete non-compact branched minimal Lag.
② To asymptotic Lagrangian Plateau problem
in complex hyperbolic surfaces.

(problem asked by Jacob Bernstein)

Suppose we have a complete minimal Lag surface in $\mathbb{C}H^2$,
with asymptotic boundary value Γ_0 in $S^3(\infty) \approx \partial B^4(1)$
Γ Legendrian in $\partial B^4 = S^3$ and Γ. Legendrian homotopic to Γ_0
try to solve the asymptotic Lagrangian Plateau.
problem for Γ. by continuity

ideal of proof for the cpt case

Difference with other works

① Works not only for local deformation.

just need a path connecting the metrics

② Deal with the deformation of singular objects

It is proved by continuity method

Suppose the Thm holds for g_t. $t \in A \subset [0,1]$

Show openness, and closeness

In the closeness, when taking the limit of a seq of min

Lag, singularity will occur. and ② is needed.

For $n=2$. we can study the problem in the map setting and the singularities of minimal surfaces are branched pts

- We can study Eenergy functional instead of Area
  branched minimal Lag are strictly stable
  (for defomation with supports away from branched pts)
  Strictly critical for $E(\varphi, [h]. g_0)$
  also has stable critical pts near φ_0 w.r.t g_t
  for small t. $\Rightarrow$ obtain branched minimal surface L_t
  w.r.t g_t

For minimal surface Σ in a Kähler surface

if not hol. or anti hol. the complex pts & anti complex

pts are isolated, and all of negative index

(Webster, Wolfson)

$$a(\Sigma) = \frac{1}{2\pi}\int_\Sigma (K_T + K_N)\, dA = \sum_{\text{complex}} \text{index}(p_i) + \sum_{\text{anti complex}} \text{index}(q_i)$$

(Webster, Wolfson, Tian-Chen)

If Σ_i stable branched minimal & $\Sigma_i \to \Sigma_\infty$,

then $a(\Sigma_\infty) = \lim_{i\to\infty} a(\Sigma_i)$ $\qquad a(L_0) = \lim_{t\to 0} a(L_t)$

"

0

$\because$ $a(L_t)$ integer value

$\therefore$ $a(L_t) = 0$ $\Longrightarrow$ totally real

$\because$ index $(p_i) < 0$

index $(q_i) < 0$

For a totally real branched minimal surface in KE with $R<0$. it must be Lag

$\Rightarrow$ the openness of deformation for branched min Lags

$$a(\Sigma) = \int_\Sigma \tfrac{1}{4}\left(R\alpha^2 - |\nabla J|^2\right) dA$$

Closeness follows from bd area

For complete case, first work on compact case with boundary

- Eeach step and related theorems need to modify and rederive

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Thank you!