Partitions, Eisenstein Series, and Their Applications

Soon-Yi Kang

March 6, 2026

Kangwon National University

2026 AOWMxIPMU Workshop

Kavli Institute for the Physics and Mathematics of the Universe

Integer Partitions

Definition

A *partition* of a positive integer n is a way of writing n as a sum of positive integers, where the order of the summands does not matter:

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \quad \text{with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0.$$

We denote this as $\lambda \vdash n$.

Example: $n = 4$

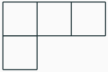
There are $p(4) = 5$ partitions of 4:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Young Diagrams

Each partition λ is visually represented by a Young diagram (or Ferrers diagram).

$\lambda = (3, 1) \vdash 4$



Partitions in Representation Theory

The General Theorem for Finite Groups

For any finite group G, the number of irreducible representations of G over $\mathbb{C}$ is equal to the number of conjugacy classes of G.

The Special Case of S_n: The Role of Partitions

For the symmetric group S_n, the correspondence is explicitly mediated by partitions $\lambda \vdash n$:

Conjugacy Classes of S_4

The group S_4 has $p(4) = 5$ conjugacy classes, each determined by its cycle structure:

(1234)	$(123)(4)$	$(12)(34)$	$(12)(3)(4)$	$(1)^4$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	(1^4)

Partitions in Representation Theory

Partitions and Irreps of S_n

$$\begin{aligned} p(n) &= \#\{\text{Distinct Young diagrams for } n\} \\ &= \#\{\text{Conjugacy classes of } S_n\} \\ &= \#\{\text{Inequivalent irreducible representations of } S_n \text{ over } \mathbb{C}\} \end{aligned}$$

Dimensions via Standard Young Tableaux (SYT)

The dimension d_λ of an irrep V^λ equals the number of ways to fill the diagram λ with $1, \ldots, n$ such that entries strictly increase along rows and down columns.

Example: $d_{(3,1)} = 3$

There are exactly 3 ways to fill the shape $(3, 1) \vdash 4$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Burnside's Identity: $\sum_{\lambda \vdash 4} d_\lambda^2 = 1^2 + 3^2 + 2^2 + 3^2 + 1^2 = |S_4|$.

Partition Functions in Physics

The generating function for the partition function $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m} = \frac{q^{1/24}}{\eta(\tau)}$$

where $q = e^{2\pi i\tau}$ and $\eta(\tau)$ is the *Dedekind eta function*.

- **In physics,** this product $Z(q)$ is the *canonical partition function* for a collection of independent bosonic modes. Setting $q = e^{-\beta\epsilon}$, the coefficient $p(n)$ counts the possible "states" at energy level n.
- $q^{1/24}$ **Factor** relates to the *Casimir energy* in Quantum Field Theory.
- **Modularity of** $\eta(\tau)$ is tied to *consistency* in 2D QFT.

Log-derivative: In thermodynamics, derivatives of $\log Z$ encode physical quantities (e.g. internal energy). Taking the logarithmic derivative,

$$q\frac{d}{dq}\log\left(\prod_{m\geq 1}\frac{1}{1-q^m}\right) = \sum_{n\geq 1}\sigma_1(n)q^n$$

which is essentially the q-expansion of the weight 2 Eisenstein series.

Modular Forms

Modular Forms of Weight k

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form* of weight $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$ if for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

and $f(\tau)$ is holomorphic as $\tau \to i\infty$. We denote this space as $\mathcal{M}_k$.

Any $f \in \mathcal{M}_k$ has a q-**expansion**: $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i \tau}$.

Structure of $\mathcal{M}_k$

The space $\mathcal{M}_k$ is a finite-dimensional $\mathbb{C}$-vector space:

- $\mathcal{M}_k = \{0\}$ for $k < 0$ and for odd k.
- $\mathcal{M}_0 = \mathbb{C}$ (only constant functions).
- $\mathcal{M}_2 = \{0\}$ (there are no non-zero modular forms of weight 2).

A non trivial modular form

The first non-trivial space appears at $k = 4$, where $\dim \mathcal{M}_4 = 1$.

Eisenstein Series

Eisenstein Series $E_k(\tau)$

For even $k > 2$, the *Eisenstein series* is the smallest non-trivial modular form of weight k, defined by:

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where B_k are Bernoulli numbers and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

The Weight 4 and 6 Cases

As $\dim \mathcal{M}_4 = 1$ and $\dim \mathcal{M}_6 = 1$, the Eisenstein series E_4 and E_6 are the unique generators:

- $E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \in \mathcal{M}_4$
- $E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \in \mathcal{M}_6$

Graded ring of modular forms

$$\mathcal{M}(SL_2(\mathbb{Z})) = \bigoplus \mathcal{M}_k(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

The Weight 2 Eisenstein Series

The Weight 2 Eisenstein Series

Recall that $\mathcal{M}_2 = \{0\}$. However, we can still define the series:

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

$E_2(\tau)$ fails to be a modular form due to an additional "drift" term:

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{6c(c\tau + d)}{\pi i}.$$

Completion

However, its *completion* $E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi y}$ (where $y = \mathrm{Im}(\tau)$) restores the weight 2 modularity, though at the cost of holomorphicity.

Ramanujan's Differential Equations

The Eisenstein series E_2, E_4, E_6 form a closed system under differentiation $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$:

$$DE_2 = \frac{1}{12}(E_2^2 - E_4), \quad DE_4 = \frac{1}{3}(E_2 E_4 - E_6), \quad DE_6 = \frac{1}{2}(E_2 E_6 - E_4^2).$$

Arithmetic of $p(n)$: Ramanujan's Congruences

Multiplicative Properties of $p(n)$
Although $p(n)$ is a purely additive combinatorial object, it exhibits striking arithmetic properties modulo $5, 7$, and 11:

Ramanujan's partition congruences (1919)

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

The Role of Eisenstein Series and Differential Equations
Many proofs of these identities rely on the algebraic structure of the ring $\mathbb{C}[E_2, E_4, E_6]$ and Ramanujan's Differential Equations.

The Algebra of Quasi-modular Forms

Quasi-modular Forms

A function f is a *quasi-modular form* if it is a polynomial in E_2, E_4, E_6.
The algebra of quasi-modular forms is $\widetilde{\mathcal{M}}(SL_2(\mathbb{Z})) = \mathbb{C}[E_2, E_4, E_6]$.

Structure of the Graded Rings

- **Modular Forms:** For a fixed weight k, $E_4^a E_6^b$ with $a, b \in \mathbb{Z}_{\geq 0}$ and $4a + 6b = k$ form a basis for $\mathcal{M}_k(SL_2(\mathbb{Z}))$.

$$\mathcal{M}(SL_2(\mathbb{Z})) = \bigoplus \mathcal{M}_k(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

- **Quasi-modular Forms:** For a fixed weight $k \geq 0$, $E_2^a E_4^b E_6^c$ with $a, b, c \in \mathbb{Z}_{\geq 0}$ an $2a + 4b + 6c = k$ form a basis for the space of quasi-modular forms of weight k.

$$\widetilde{\mathcal{M}}(SL_2(\mathbb{Z})) = \mathbb{C}[E_2, E_4, E_6].$$

Prime Detecting Quasi-modular Forms

Consider the following quasi-modular form (of mixed weights):

$$\begin{aligned} f &= -396 + 360E_2 - 30E_2^2 + 66E_4 + 5E_2^3 - 15E_2E_4 + 10E_6 \\ &= -36\left(11 - E_4 - 10(D^2 - D + 1)E_2\right) \\ &= 240\sum_{n=1}^{\infty}\left[(n^2 - n + 1)\sigma_1(n) - \sigma_3(n)\right] q^n \end{aligned}$$

The Fourier coefficients of f are zero at $n = 0, 1$ and all primes, and strictly positive for all composite n.

Lelièvre (2004) For distinct non-negative integers $k \neq \ell$, the function

$$g_{k,\ell}(n) := (n^\ell + 1)\sigma_k(n) - (n^k + 1)\sigma_\ell(n)$$

vanishes if and only if $n = 1$ or n is prime.

Generalization and the Identities of E_2

Lelièvre (2004) Let $G_k(q) := \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n = -\frac{B_k}{2k}(E_k(\tau) - 1)$. Then for any two distinct nonnegative odd integers k and ℓ, the quasi-modular form

$$f_{k,\ell} := (D^{\ell} + 1)G_{k+1} - (D^k + 1)G_{\ell+1}$$

has Fourier coefficients that are zero exactly at $n = 1$ and at prime numbers.

The Fourier and Lambert Series of E_2

$$\begin{aligned} E_2(\tau) &= 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n \\ &= 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n \\ &= 1 - 24\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \end{aligned}$$

MacMahon's Partition Function

MacMahon's partition function, $M_k(n)$ $(k, n \in \mathbb{N})$ is defined by

$$M_k(n) := \sum_{\substack{0<m_1<m_2<\cdots<m_k \\ n=m_1d_1+m_2d_2+\cdots+m_kd_k}} d_1 d_2 \cdots d_k.$$

Note that $M_1(n) = \sum_{d|n} d$ is the divisor sum.

This represents the sum of the products of the heights across all possible arrangements of n squares into k rectangular blocks with varying widths.

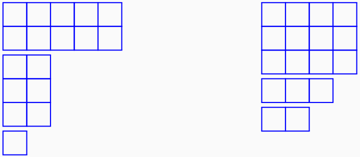


Figure 1: Two examples of partitions of 17 squares into 3 rectangular blocks: 1×1, 2×3, 5×2 and 2×1, 3×1, 4×3. In these cases, the products of the heights are given by $1 \cdot 3 \cdot 2$ and $1 \cdot 1 \cdot 3$, respectively.

MacMahon's Partition Generating Function

MacMahon(1920), Andrews-Rose (2013)

$$A_k(q) := \sum_{n=1}^{\infty} M_k(n) q^n$$
$$= \sum_{0<m_1<m_2<\cdots<m_k} \frac{q^{m_1+m_2+\cdots+m_k}}{(1-q^{m_1})^2(1-q^{m_2})^2\cdots(1-q^{m_k})^2}.$$

is a (mixed weight) quasimodular form.

Craig–van Ittersum–Ono (2024)

For $n \in \mathbb{N}$, we have

- $(n^2 - 3n + 2)M_1(n) - 8M_2(n) \geq 0$,
- $(3n^3 - 13n^2 + 18n - 8)M_1(n) + (12n^2 - 120n + 212)M_2(n) - 960M_3(n) \geq 0$,
- For $n \geq 2$, these expressions vanish if and only if n is prime.

MacMahon's level 2 Partition Function

$$A_{2,k}(q) := \sum_{n=1}^{\infty} M_k^{(2)}(n) q^n$$
$$:= \sum_{\substack{0<m_1<m_2<\cdots<m_k \\ m_i \not\equiv 0 \ (2), \ (1\le i\le k)}} \frac{q^{m_1+m_2+\cdots+m_k}}{(1-q^{m_1})^2(1-q^{m_2})^2\cdots(1-q^{m_k})^2}.$$

K.-Matsusaka-Shin (2025) For positive integers $n \geq 2$, we have

$$(n^2-4n+3)M_1^{(2)}(n) - 24M_2^{(2)}(n) \begin{cases} = 0 & \text{if } n \text{ is odd prime,} \\ < 0 & \text{if } n = 2^\ell \text{ for } \ell \in \mathbb{Z}_{\geq 1}, \\ > 0 & \text{otherwise.} \end{cases}$$

Generalized Lelièvre's Criteria

Restricted Eisenstein Series For an integer $N \geq 1$, we define the level N Eisenstein series of weight k as:

$$G_k^{(N)}(q) := \sum_{n=1}^{\infty} \sigma_{k-1}^{(N)}(n) q^n,$$

where the restricted divisor sum is given by:

$$\sigma_{k-1}^{(N)}(n) := \sum_{\substack{d \mid n \\ \gcd(n/d, N)=1}} d^{k-1}$$

K.-Matsusaka-Shin (2025) Let k, ℓ be positive integers with $\ell > k$. For $n \geq 2$, the n-th Fourier coefficient of

$$f_{k,\ell}^{(N)}(q) := (D^\ell + 1) G_{k+1}^{(N)}(q) - (D^k + 1) G_{\ell+1}^{(N)}(q)$$

satisfies

$$\begin{cases} = 0 & \text{if } n \text{ is prime with } n \nmid N, \\ < 0 & \text{if all prime factors } p \text{ of } n \text{ satisfy } p \mid N, \\ > 0 & \text{otherwise.} \end{cases}$$

Partition Eisenstein Series

For each $j \in \mathbb{N}$, we let $G_{2j}(\tau)$ be the classical weight $2j$ holomorphic Eisenstein series. To be precise, for $k > 1$:

$$G_{2k}(\tau) := -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

and for $k = 1$, $G_2(\tau)$ is the quasi-modular Eisenstein series of weight 2.

Partition Eisenstein series

For a partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k}) \vdash k$,

$$G_\lambda(\tau) := G_2(\tau)^{m_1} G_4(\tau)^{m_2} \cdots G_{2k}(\tau)^{m_k}$$

is the partition Eisenstein series of weight $2k$ associated to λ.

- Here, m_j is the multiplicity of j in the partition λ.
- The classical series $G_{2k}(\tau)$ corresponds to the single-part partition $\lambda = (k^1)$.
- These functions are elements of the quasi-modular polynomial ring $\mathbb{C}[G_2, G_4, G_6, \ldots]$.

Partition Eisenstein Traces

Partition Eisenstein Trace

Let $\phi : \mathcal{P} \to \mathbb{C}$ be a function on the set of partitions $\mathcal{P}$. For each $k \in \mathbb{N}$, the *weight $2k$ partition Eisenstein trace* $Tr_k(\phi; \tau)$ is defined as:

$$Tr_k(\phi; \tau) := \sum_{\lambda \vdash k} \phi(\lambda) E_\lambda(q)$$

Amdeberhan–Ono–Singh (2025)

- $Tr_k(\phi; \tau)$ is a **quasi-modular form** of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z})$.

Motivation

- **Solving Ramanujan's Claim:** Ramanujan asserted the quasi-modularity of certain q-series (U_{2t}, V_{2t}) in his "Lost Notebook." This trace provides the *first explicit proof* by identifying their exact partition weights.

The Crank of a Partition

Dyson (1944) famously hypothesized the existence of a "crank" that would combinatorially explain Ramanujan's congruence:

$$p(11n+6) \equiv 0 \pmod{11}$$

Andrews–Garvan (1988)
For a partition λ, let $\ell(\lambda)$ be the largest part, $n_1(\lambda)$ be the number of 1s, and $\mu(\lambda)$ be the number of parts strictly larger than $n_1(\lambda)$. The **crank** $c(\lambda)$ is:

$$c(\lambda) := \begin{cases} \ell(\lambda) & \text{if } n_1(\lambda) = 0 \\ \mu(\lambda) - n_1(\lambda) & \text{if } n_1(\lambda) > 0 \end{cases}$$

Generating Function for the Crank The number of partitions of n with crank m, denoted $M(m,n)$, is given by:

$$C(z;q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}$$

Crank Moments as Partition Eisenstein Traces

Crank Moments $C_{2k}(q)$ The $2k$-th power moments of the Andrews–Garvan crank are

$$C_{2k}(q) := \sum_{n=0}^{\infty} \left(\sum_{m=-\infty}^{\infty} m^{2k} M(m,n) \right) q^n.$$

Amdeberhan, Griffin, Ono, Singh (2025)

$$(q;q)_\infty C_{2k}(q) = \sum_{n=0}^{k} \frac{(2k)_{2k-2n}}{4^n \cdot (2n+1)} \cdot Tr_{k-n}(\phi_c; \tau),$$

where

$$\phi_c(\lambda) = \frac{1}{\prod_{j=1}^{k} m_j!((2j)!)^{m_j}}$$

is a combinatorial weight related to the elementary symmetric functions of the parts of λ.

Crank Moment Representation in Eisenstein Series

R. C. Rhoades (2013)

- **Atkin–Garvan (2003)** provided a recursive method to show $C_{2k}(q)$ is quasi-modular.
- **Rhoades** transformed this *recursive construction* into an *explicit closed form* using Jacobi theta function properties.
- The expressions involved complex multiple sums and Jacobi form Taylor expansions.

Amdeberhan, Griffin, Ono, Singh (2025)

- Reinterpreted these closed forms as *Partition Eisenstein Traces*.
- Used *Pólya's cycle index polynomials* to provide a new structural proof, linking them to the basis E_λ.

K. and Lee (in Progress) We provide a *simplified proof* by applying *Faà di Bruno's formula* to the logarithmic derivative of the crank generating function

Thank You!