The Scalar-Induced Gravitational-Wave Background:

Diagrammatic Approach for Primordial Non-Gaussianity up to Arbitrary Order

Based on arXiv:2505.16820

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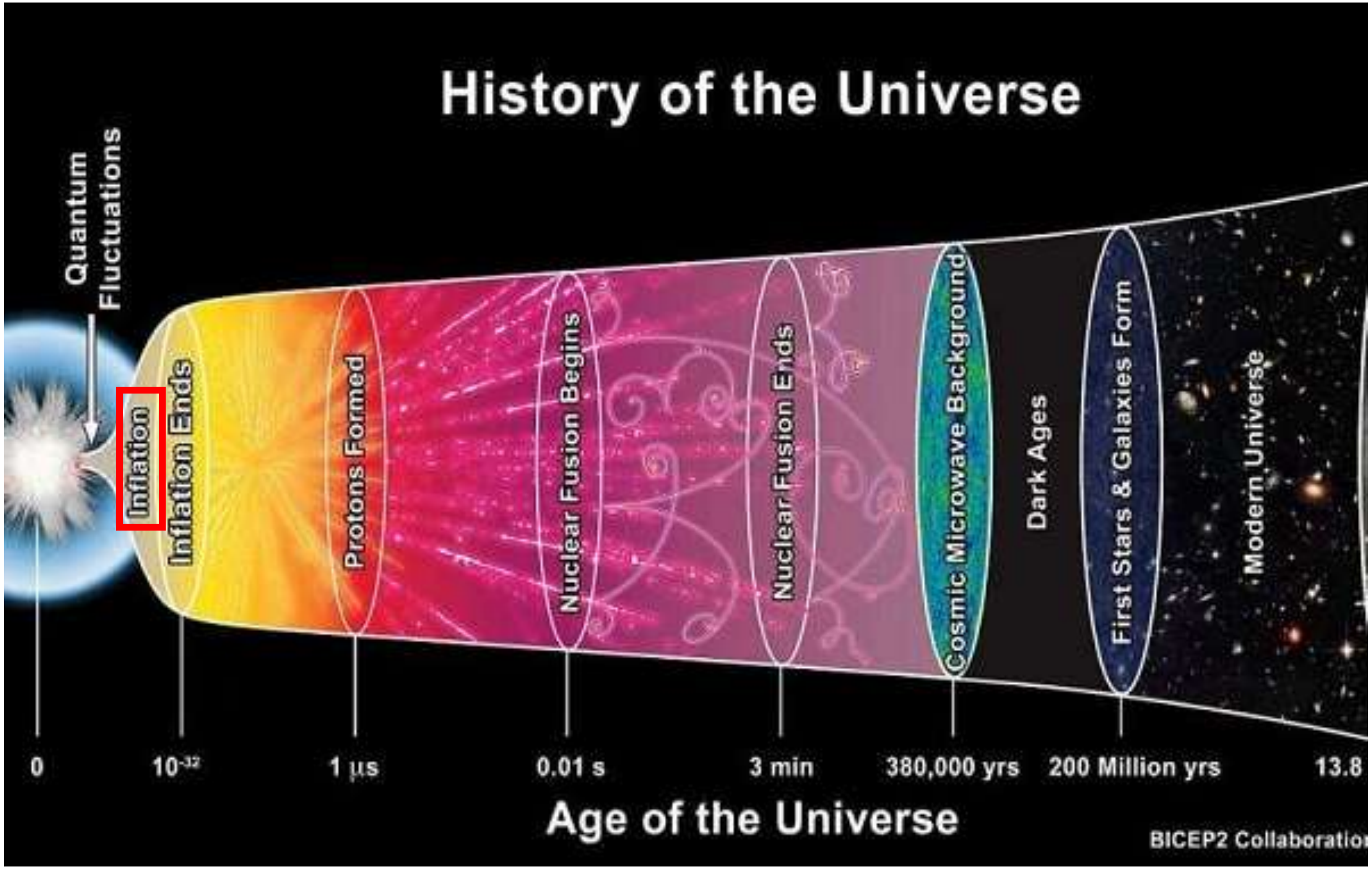
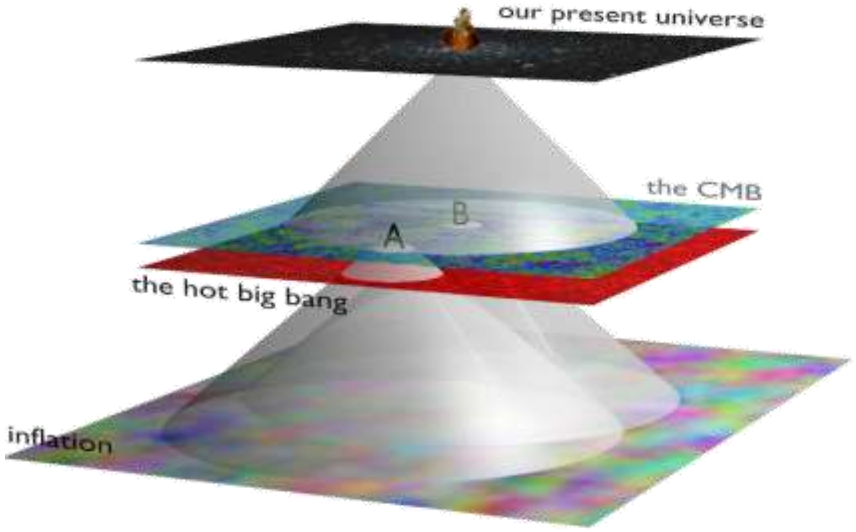
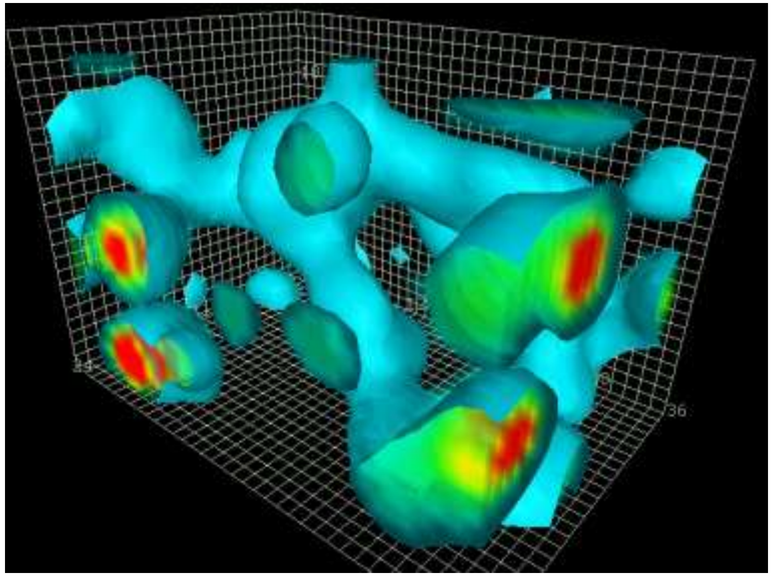
IPMU, Feb. 10th, 2026

OUTLINE

Motivation: Primordial Curvature Perturbations

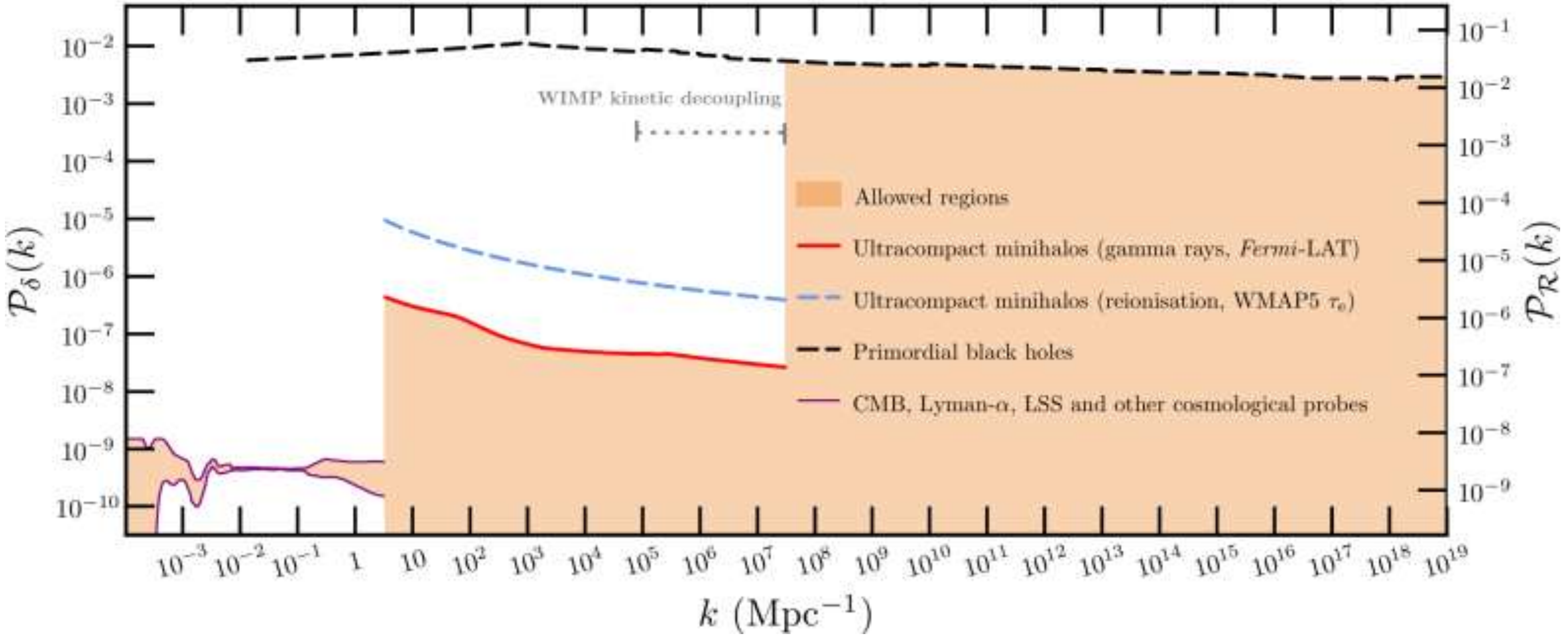


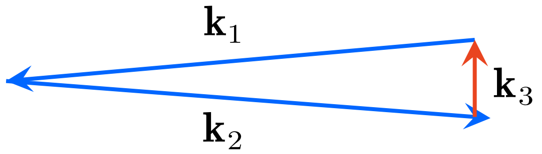




primordial curvature perturbations $\longleftrightarrow$ quantum fluctuations

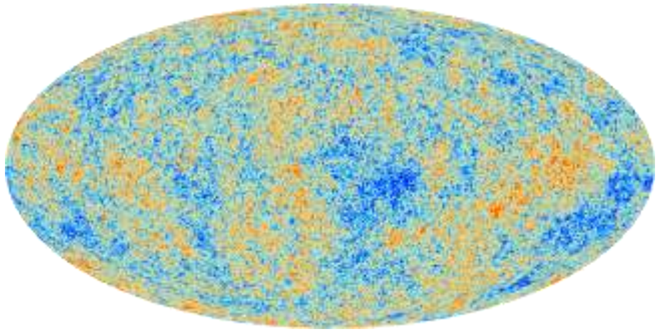
Motivation: Primordial Curvature Perturbations





local-type

$$\zeta = \zeta_g + \frac{3}{5} f_{NL}\left[\zeta_g^2 - \langle\zeta_g\rangle^2\right] + \frac{9}{25} g_{NL}\zeta_g^3 + \frac{27}{125} h_{NL}\left[\zeta_g^4 - 3\langle\zeta_g\rangle^4\right] + \cdots$$



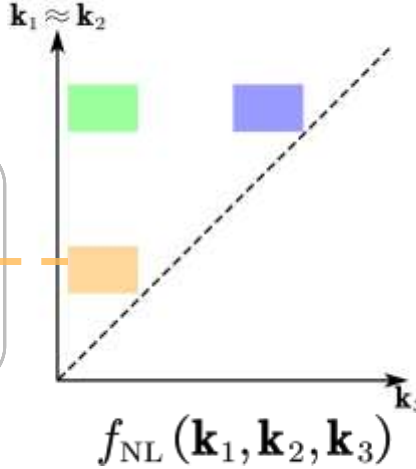
CMB: ~ 10^{-16}Hz

$$\Delta_L(k) = A_L\left(\frac{k}{k_*}\right)^{n_s-1}$$

$$A_L = (2.10 \pm 0.03) \times 10^{-9}$$
$$n_s = 0.965 \pm 0.004$$
$$k_* = 0.05\text{Mpc}^{-1}$$

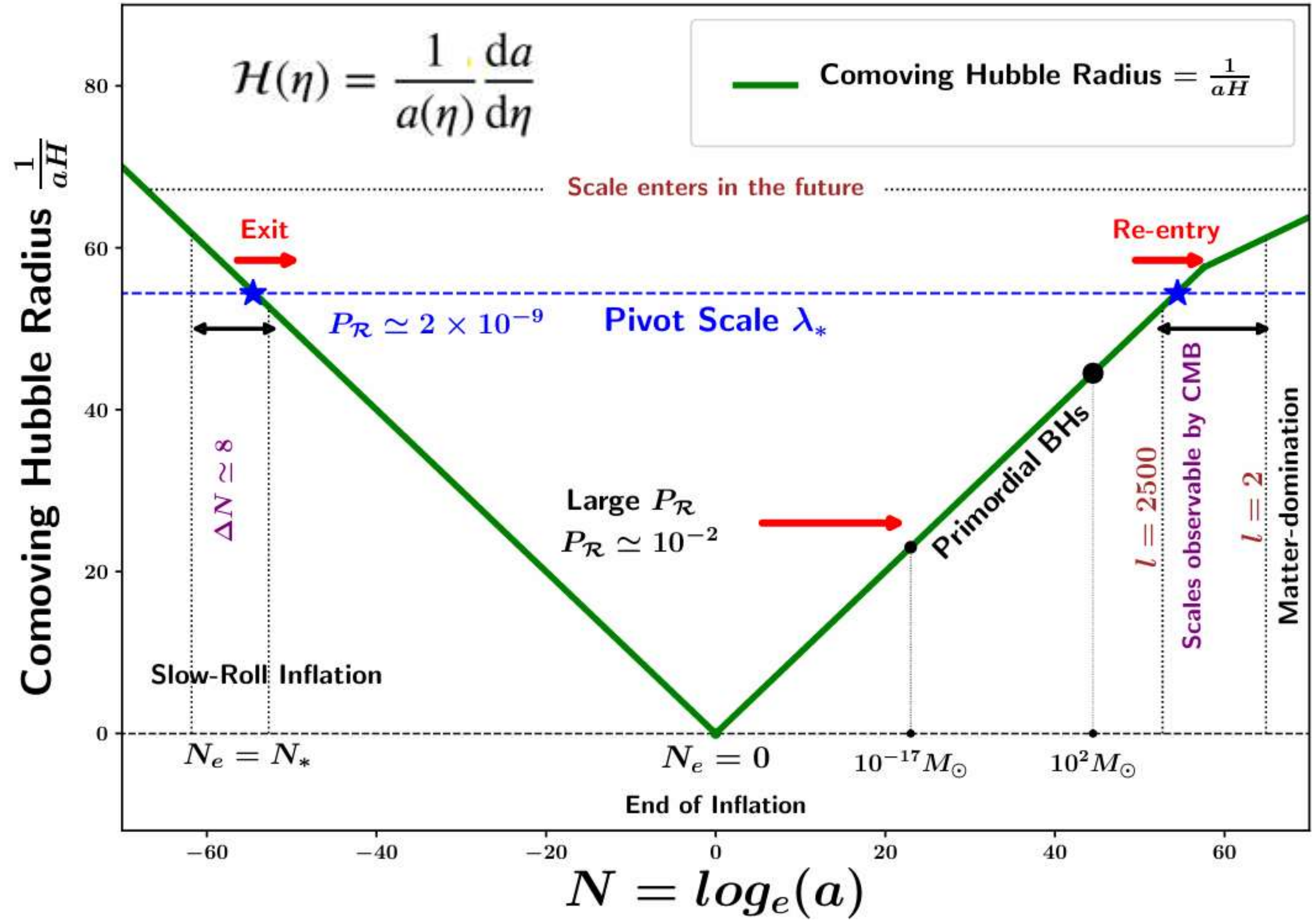
Planck 2018 Results

$$f_{NL} = -0.9 \pm 5.1 \text{ (68\% CL)}$$
$$g_{NL} = (-5.8 \pm 6.5) \times 10^4 \text{ (68\% CL)}$$



10^{-9} Hz ~ 1 Hz ? ?

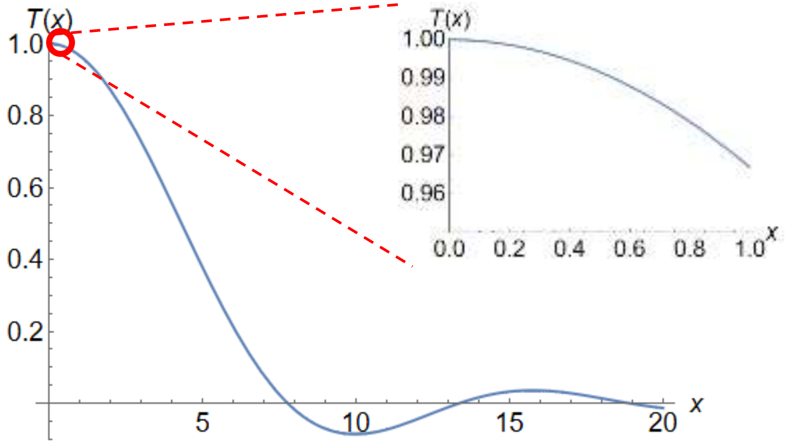
Motivation: Scalar-Induced Gravitational Waves



$$ds^2 = a^2[-e^{\Phi}d\eta^2 + (e^{-\Phi}\delta_{ij} + \frac{h_{ij}}{2})dx^i dx^j]$$

Perfect Fluid: $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + P g_{\mu\nu}$

$$\Phi(\eta, q) = \left(\frac{3+3w}{5+3w}\right)T(q\eta)\zeta(q)$$



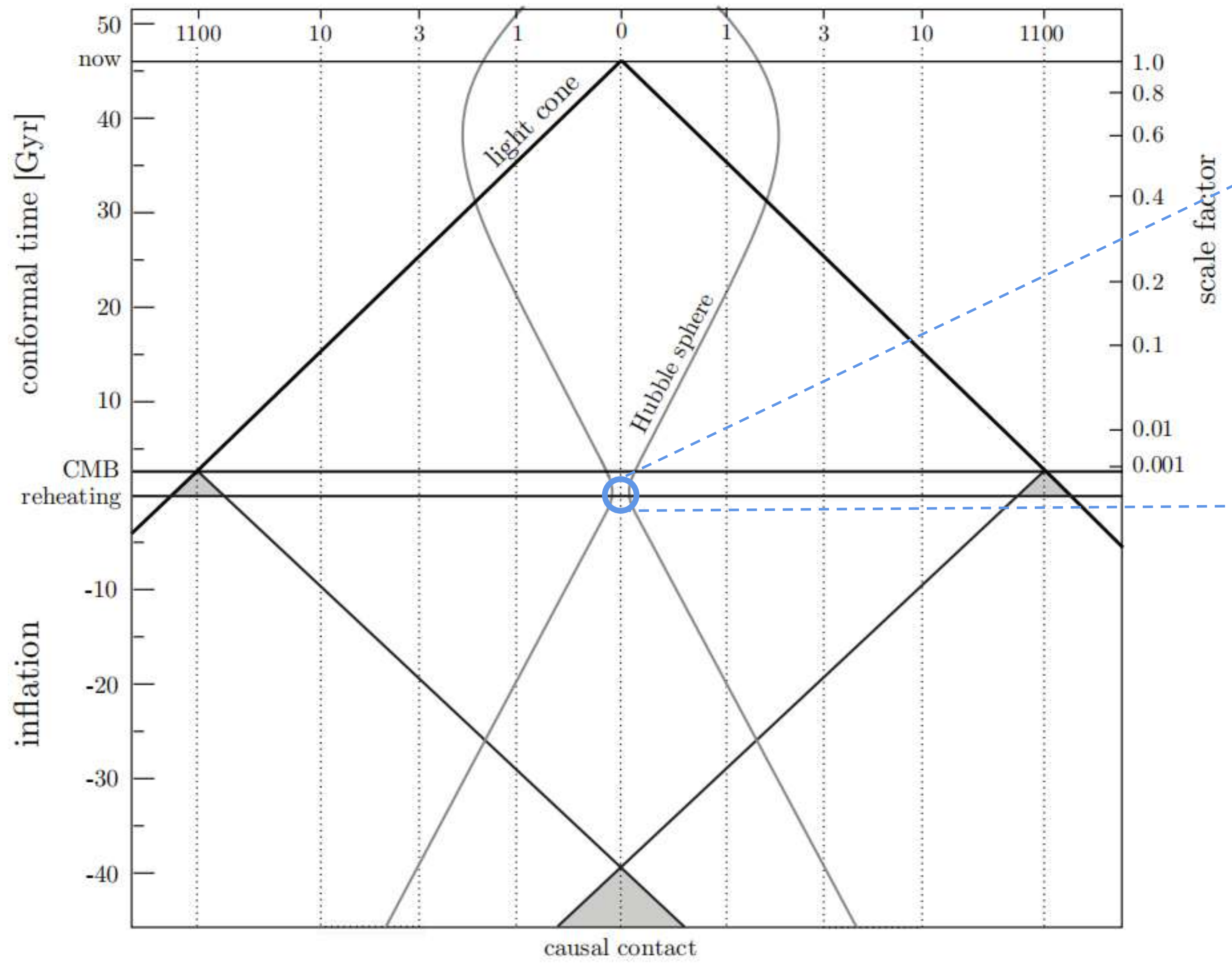
SIGW Motion Equation

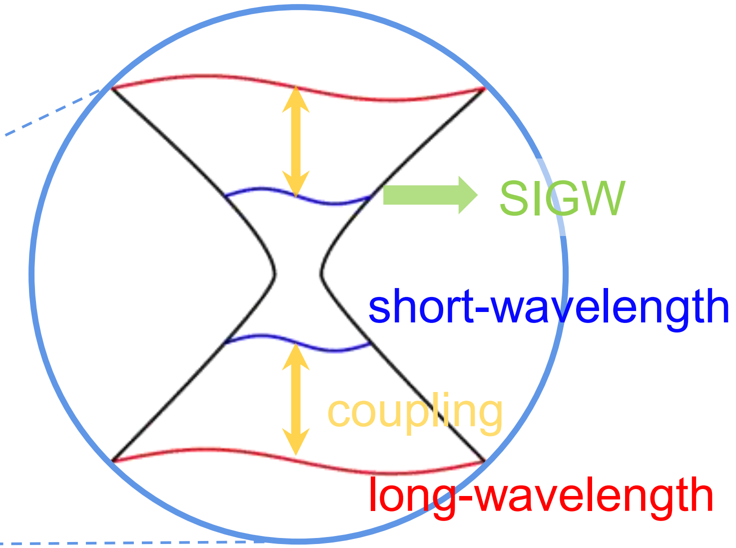
$$\partial_\eta^2 h_\lambda(\eta, q) + 2\mathcal{H}\partial_\eta h_\lambda(\eta, q) + q^2 h_\lambda(\eta, q) = 4S_\lambda(\eta, q)$$

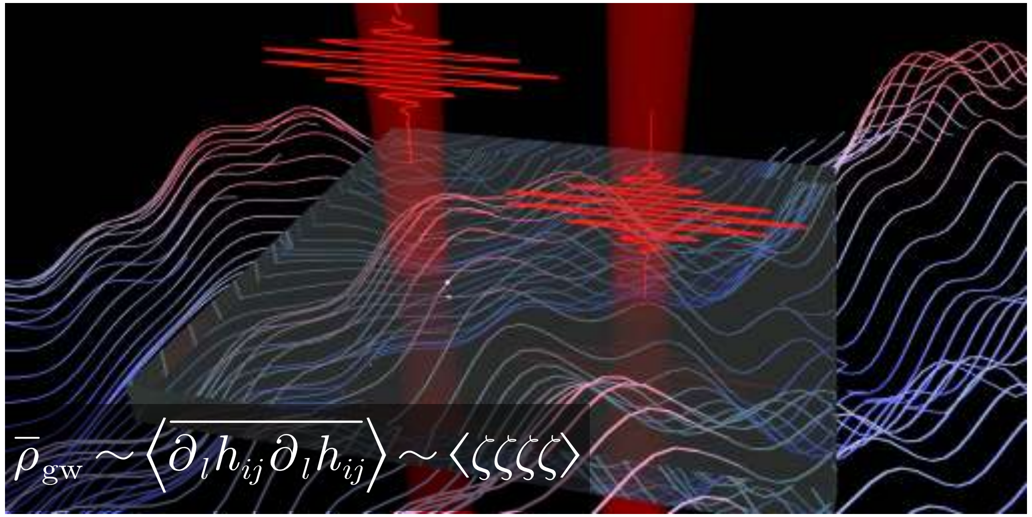
$$S_\lambda(\eta, q) \sim Q_\lambda(q, q_a)\zeta(q_a)\zeta(q - q_a)$$

$$h_\lambda(\eta, q) = 4\int \frac{d^3 q_a}{(2\pi)^{3/2}} \zeta(q_a)\zeta(q - q_a) Q_\lambda(q, q_a) \hat{I}(|q - q_a|, q, \eta)$$

Motivation: SIGW & PNG

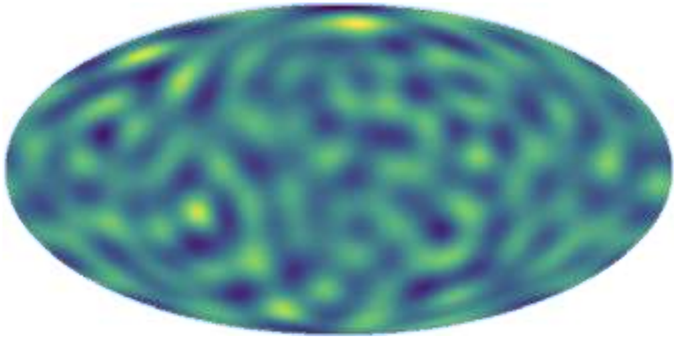






$$h_\lambda(\eta, \boldsymbol{q}) = 4 \int \frac{\mathrm{d}^3 \boldsymbol{q}_a}{(2\pi)^{3/2}} \zeta(\boldsymbol{q}_a)\zeta(\boldsymbol{q} - \boldsymbol{q}_a) Q_\lambda(\boldsymbol{q}, \boldsymbol{q}_a) \hat{I}(|\boldsymbol{q} - \boldsymbol{q}_a|, q, \eta)$$

Motivation: Statistics of SIGW Energy-Density



How to extract SIGW signals from SGWB?

$$\omega_{\rm gw}(\eta,\boldsymbol{x},\boldsymbol{q}) = \frac{\bar{\Omega}_{\rm gw}(\eta,q)}{4\pi}\left(1+\delta_{\rm gw}(\eta,\boldsymbol{x},\boldsymbol{q})\right)$$

$$\langle h_\lambda(\eta,\boldsymbol{q}_1)h_{\lambda'}(\eta,\boldsymbol{q}_2)\rangle = \delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{q}_2)P_{h_\lambda}(\eta,q_1)$$

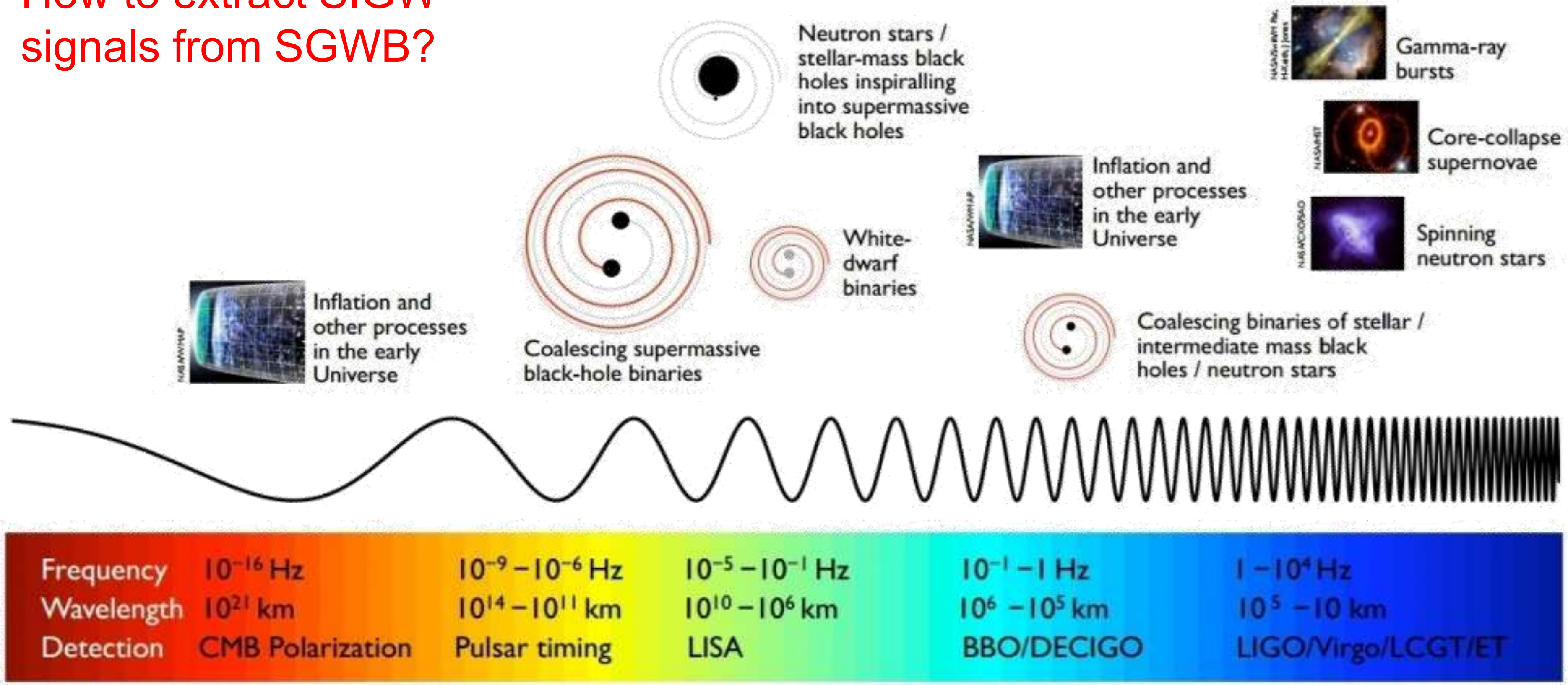
Isotropic Background

$$\bar{\Omega}_{\rm gw}(\eta,q) = \frac{q^5}{96\pi^2a^2H^2}\sum_\lambda \overline{P_{h_\lambda}(\eta,q)}$$

$$\sim \langle\zeta\zeta\zeta\zeta\rangle$$

Anisotropic and non-Gaussian Background

Angular correlation functions of $\delta_{\rm gw}$ (or $\omega_{\rm gw} \sim \langle\zeta\zeta\zeta\zeta\rangle$)



Diagrammatic Approach

$$\omega_{\rm gw} \sim \bar{\rho}_{\rm gw} \sim \left\langle \overline{\partial_k h_{ij} \partial_k h_{ij}} \right\rangle \sim \langle \zeta\zeta\zeta\zeta \rangle$$

$$\zeta = \zeta_{\rm g} + F_{\rm NL}\left(\zeta_{\rm g}^2 - \langle \zeta_{\rm g}^2 \rangle\right) + G_{\rm NL}\zeta_{\rm g}^3 + H_{\rm NL}\left(\zeta_{\rm g}^4 - 3\langle \zeta_{\rm g}^2 \rangle^2\right) + \cdots$$

$$\begin{aligned}\langle \zeta\zeta\zeta\zeta \rangle = {} & \langle \zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g} \rangle + F_{\rm NL}^2 \langle \zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g} \rangle + G_{\rm NL} \langle \zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g} \rangle \\ & + F_{\rm NL}^4 \langle \zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g} \rangle + F_{\rm NL}H_{\rm NL} \langle \zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g}\zeta_{\rm g} \rangle + \cdots\end{aligned}$$

Wick's theorem

$$\left\langle \zeta_g(\boldsymbol{p_1})\zeta_g(\boldsymbol{p_2})\zeta_g(\boldsymbol{q}-\boldsymbol{p_1}-\boldsymbol{p_2})\zeta_g(\boldsymbol{k}-\boldsymbol{q})\zeta_g(\boldsymbol{p_3})\zeta_g(\boldsymbol{p_4})\zeta_g(\boldsymbol{q'}-\boldsymbol{p_3}-\boldsymbol{p_4})\zeta_g(\boldsymbol{k'}-\boldsymbol{q'}) \right\rangle$$

Diagrammatic Approach

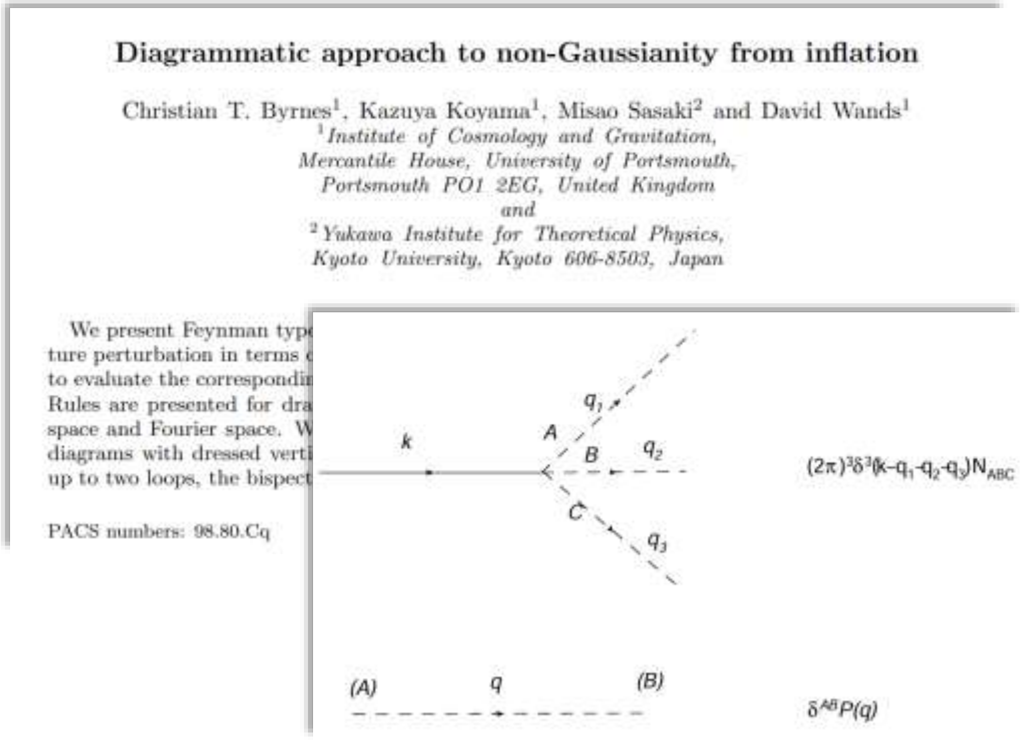
$$\omega_{\mathrm{gw}} \sim \bar{\rho}_{\mathrm{gw}} \sim \left\langle \overline{\partial_k h_{ij} \partial_k h_{ij}} \right\rangle \sim \langle \zeta\zeta\zeta\zeta \rangle$$

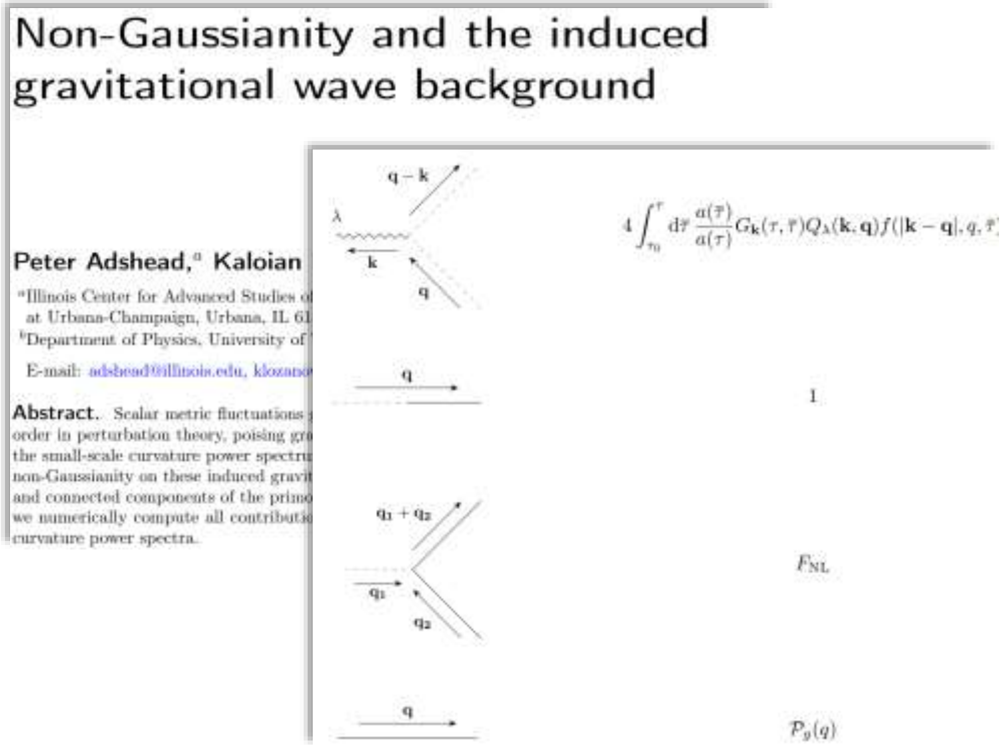
$$\zeta = \zeta_{\mathrm{g}} + F_{\mathrm{NL}}\left(\zeta_{\mathrm{g}}^2 - \langle \zeta_{\mathrm{g}}^2 \rangle\right) + G_{\mathrm{NL}} \zeta_{\mathrm{g}}^3 + H_{\mathrm{NL}}\left(\zeta_{\mathrm{g}}^4 - 3\langle \zeta_{\mathrm{g}}^2 \rangle^2\right) + \cdots$$

$$\langle \zeta\zeta\zeta\zeta \rangle = \langle \zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}} \rangle + F_{\mathrm{NL}}^2 \langle \zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}} \rangle + G_{\mathrm{NL}} \langle \zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}} \rangle$$
$$+ F_{\mathrm{NL}}^4 \langle \zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}} \rangle + F_{\mathrm{NL}} H_{\mathrm{NL}} \langle \zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}}\zeta_{\mathrm{g}} \rangle + \cdots$$

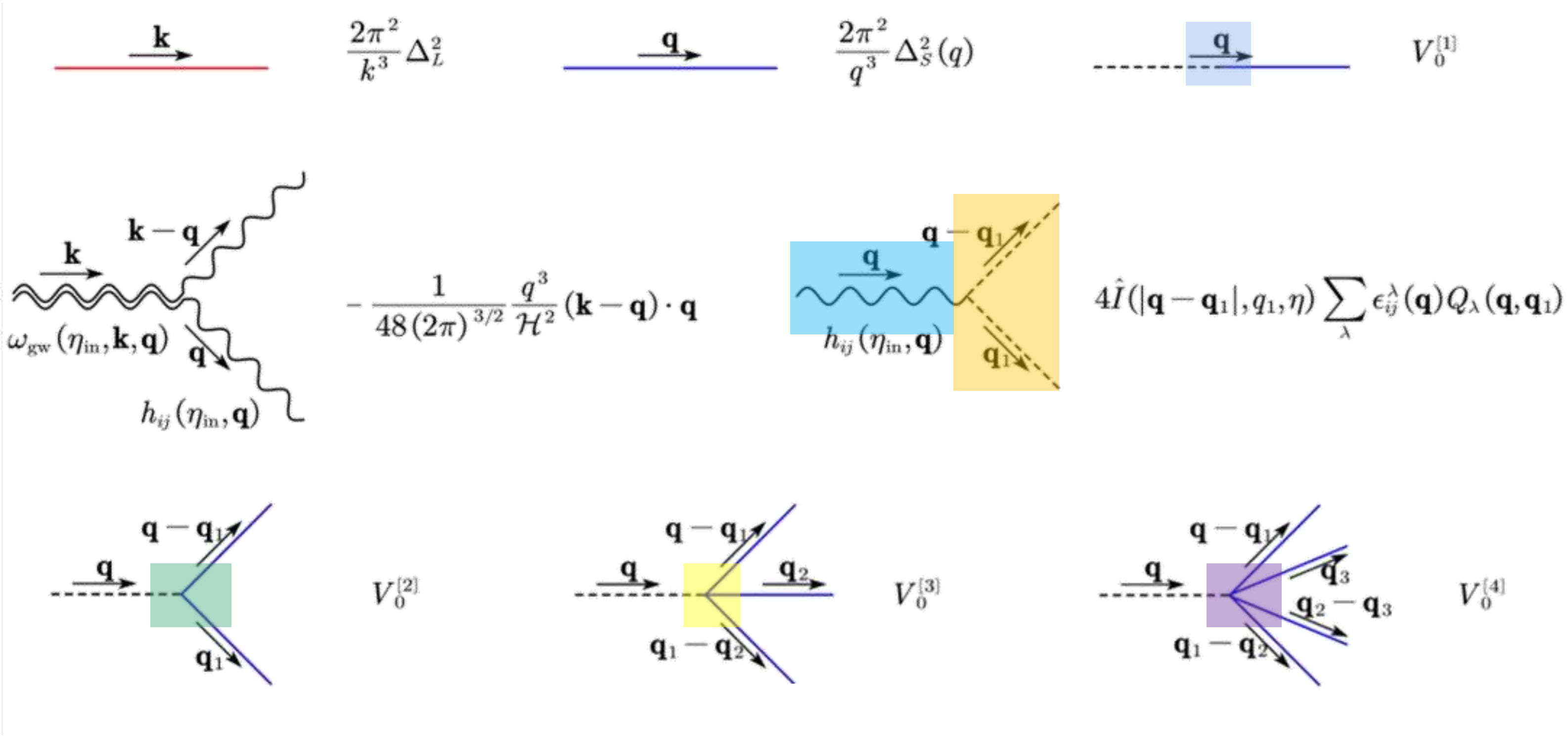
C. T. Byrnes, K. Koyama, M. Sasaki, D. Wands, JCAP 11 (2007), 027
P. Adshead, K. D. Lozanov, Z. J. Weiner, JCAP 10 (2021), 080

Wick's theorem





Diagrammatic Approach: Feynman-like Rules



R.-G. Cai, S. Pi, M. Sasaki, PRL 122 (2019) 201101
P. Adshead, K. D. Lozanov, Z. J. Weiner, JCAP 10 (2021), 080
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C. Yuan, D.-S. Meng, Q.-G. Huang, JCAP 12 (2023), 036
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 06 (2024) 039
Y.-H. Yu, S. Wang, PRD 109 (2024) 083501
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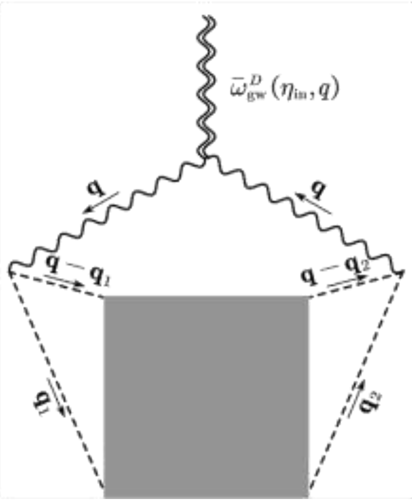
$$\zeta = \zeta_{\rm g} + F_{\rm NL}(\zeta_{\rm g}^2 - \langle\zeta_{\rm g}^2\rangle) + G_{\rm NL}\zeta_{\rm g}^3 + H_{\rm NL}(\zeta_{\rm g}^4 - 3\langle\zeta_{\rm g}^2\rangle^2)$$

$$\zeta_{\rm g} = \zeta_{\rm gS} + \zeta_{\rm gL}$$

$$h_\lambda(\eta,\boldsymbol{q}) = 4\int\frac{d^3\boldsymbol{q}_a}{(2\pi)^{3/2}}\zeta(\boldsymbol{q}_a)\zeta(\boldsymbol{q}-\boldsymbol{q}_a)Q_\lambda(\boldsymbol{q},\boldsymbol{q}_a)\hat{I}(|\boldsymbol{q}-\boldsymbol{q}_a|,q,\eta)$$

$$\bar{\rho}_{\rm gw} \sim \left\langle\overline{\partial_l h_{ij}\partial_l h_{ij}}\right\rangle \sim \langle\zeta\zeta\zeta\zeta\rangle$$

$$\bar{\Omega}_{\rm gw}(\eta,q) = 4\pi\bar{\omega}_{\rm gw}(\eta,q)$$

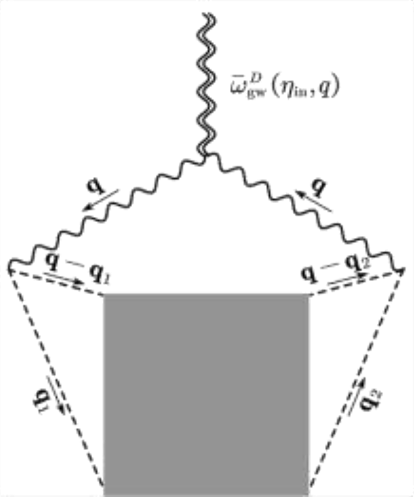


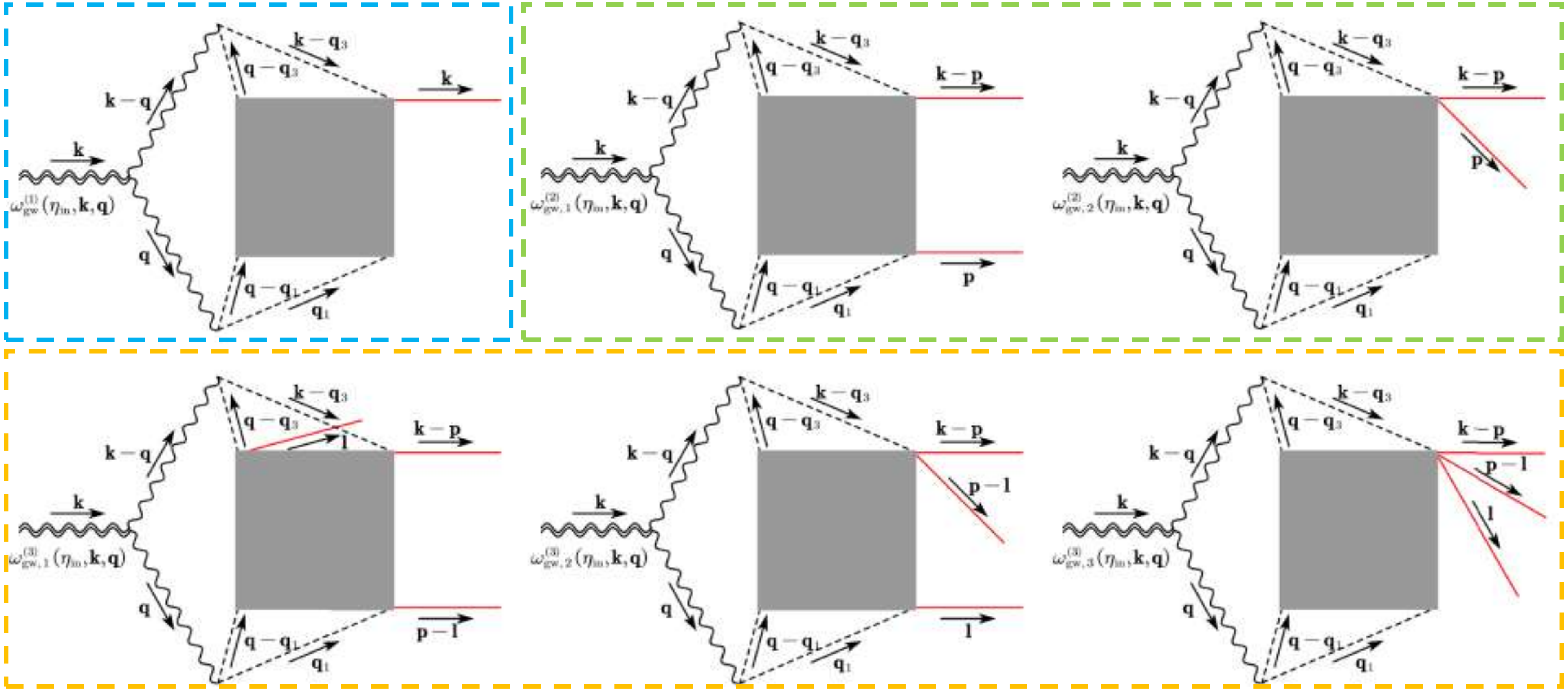
$$\bar{\omega}_{\rm gw}^D = \frac{q^5}{3(2\pi)^3\mathcal{H}^2}\left\{\prod_{i=1}^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\right\}\left\{\sum_{\lambda=+,\times}Q_\lambda(\mathbf{q},\mathbf{q}_1)Q_\lambda(\mathbf{q},\mathbf{q}_j)\overline{\hat{I}(|\mathbf{q}-\mathbf{q}_1|,q_1,\eta_{\rm in})\hat{I}(|\mathbf{q}-\mathbf{q}_j|,q_j,\eta_{\rm in})}\right\}\mathcal{P}^D$$

Diagrammatic Approach: Feynman-like Rules

$$\omega_{gw}(\eta,\boldsymbol{x},\boldsymbol{q}) = \overline{\omega}_{gw}(\eta,q) + \omega_{gw}^{(1)}(\eta,\boldsymbol{x},\boldsymbol{q}) + \omega_{gw}^{(2)}(\eta,\boldsymbol{x},\boldsymbol{q}) + \omega_{gw}^{(3)}(\eta,\boldsymbol{x},\boldsymbol{q}) + \cdots$$

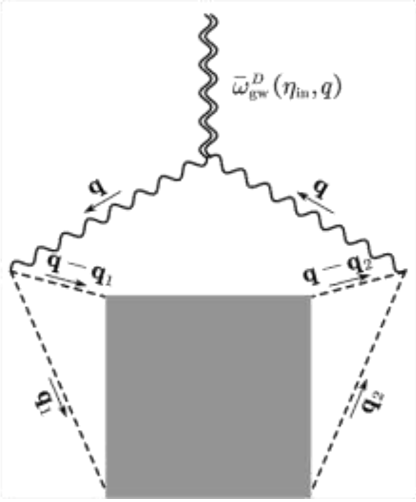
$$\zeta_g = \zeta_{gS} + \zeta_{gL}$$

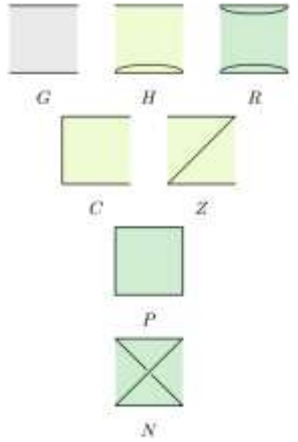




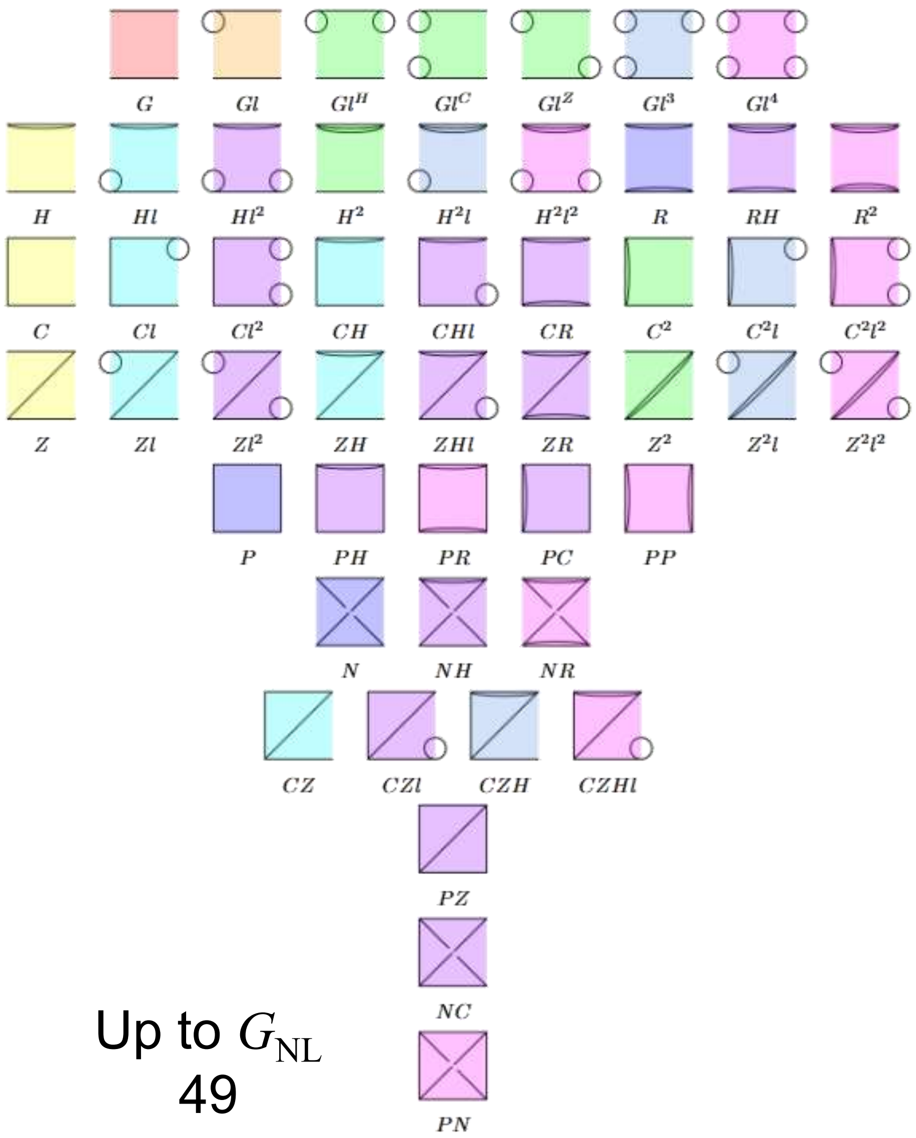
Diagrammatic Approach: Diagrams

JL, S. Wang, Z.-C. Zhao, and K. Kohri, arXiv:2309.07792

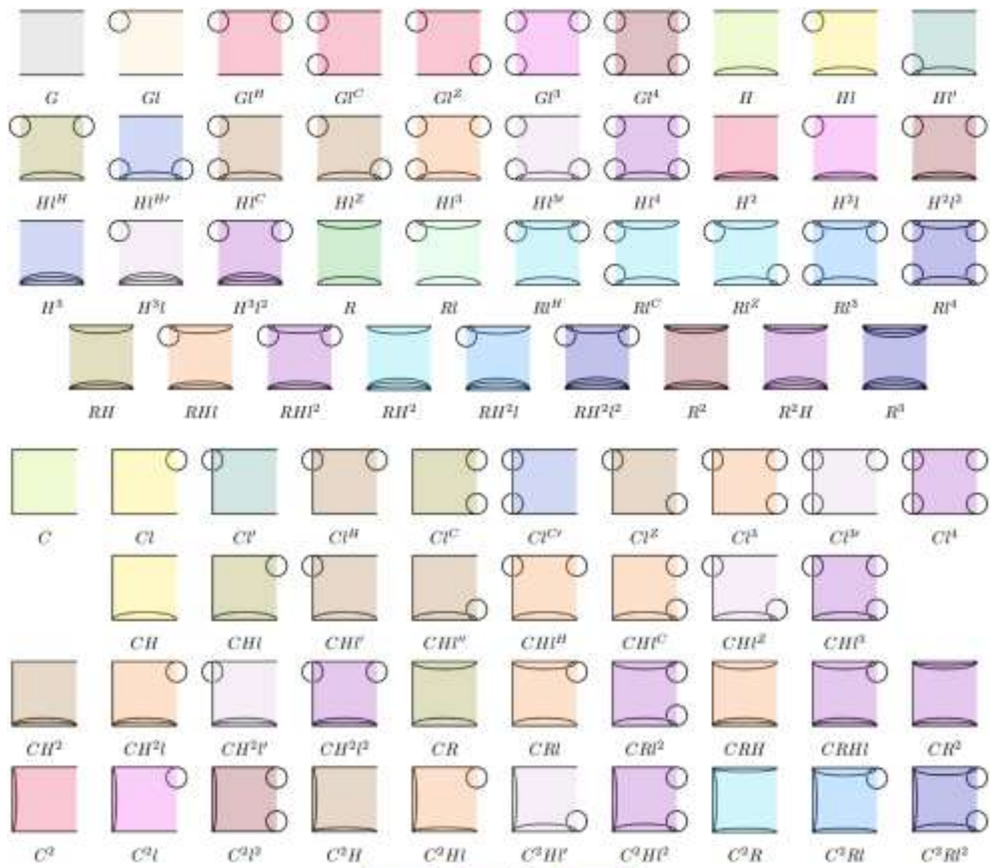


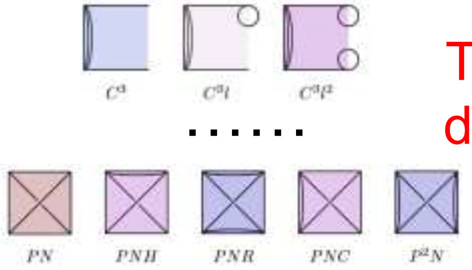


Up to F_{NL}
7



Up to G_{NL}
49



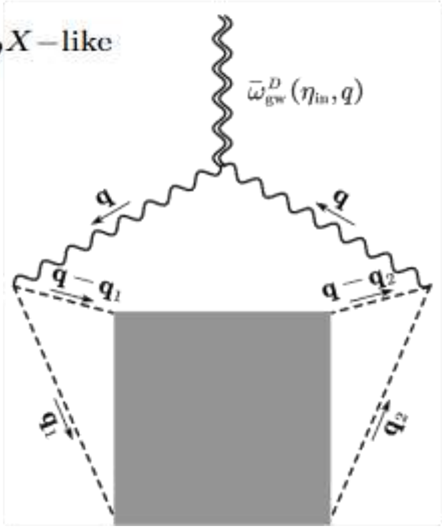


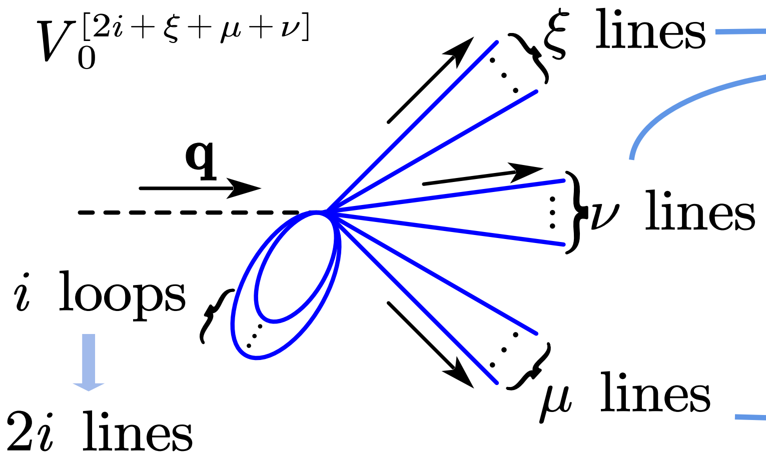
Up to H_{NL}
236

Too many diagrams!

Diagrammatic Approach: “Renormalized” Feynman-like Rules

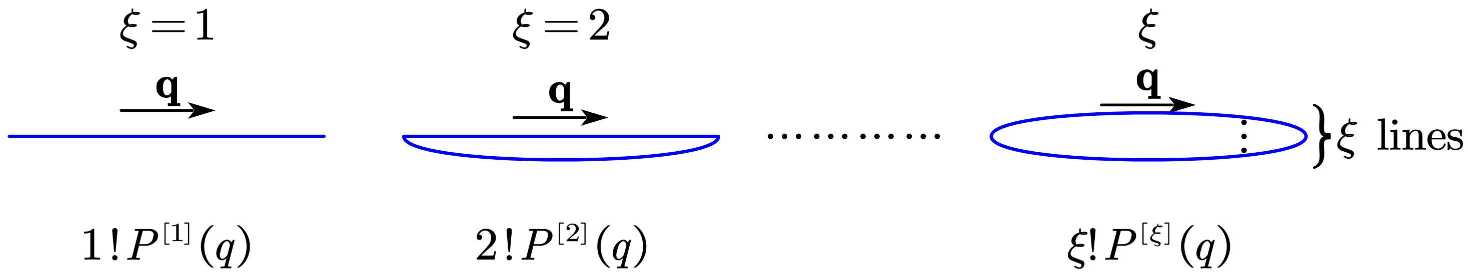
$$\bar{\omega}_{\mathrm{gw}}^{X-\mathrm{like}} = \frac{q^5}{3(2\pi)^3\mathcal{H}^2}\left\{\prod_{i=1}^{n}\int\frac{\mathrm{d}^3\mathbf{q}_i}{(2\pi)^3}\right\}\left\{\sum_{\lambda=+,\times} Q_\lambda(\mathbf{q},\mathbf{q}_1)Q_\lambda(\mathbf{q},\mathbf{q}_j)\overline{\hat{I}(|\mathbf{q}-\mathbf{q}_1|,q_1,\eta_{\mathrm{in}})\hat{I}(|\mathbf{q}-\mathbf{q}_j|,q_j,\eta_{\mathrm{in}})}\right\}\mathcal{P}^{X-\mathrm{like}}$$





$$\frac{(2i)!}{2^i i!}\binom{2i+\xi+\mu+\nu}{\nu}\binom{2i+\xi+\mu}{\mu}\binom{2i+\xi}{\xi}A_{\mathrm{S}}^{i}V_0^{[2i+\xi+\mu+\nu]}$$

All loops are interchangeable, and likewise the two lines in every loop.



$$P^{[1]}(q) = \frac{2\pi^2}{q^3}\Delta_S^2(q)\ ,$$

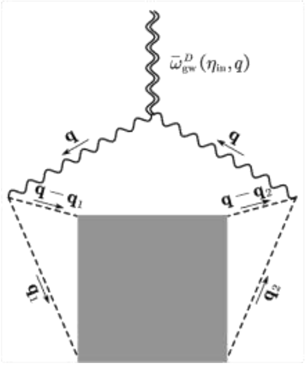
$$P^{[\xi]}(q) = \int\frac{\mathrm{d}^3\tilde{\mathbf{q}}}{(2\pi)^3}P^{[\xi-1]}(\tilde{q})P^{[1]}(|\mathbf{q}-\tilde{\mathbf{q}}|)$$

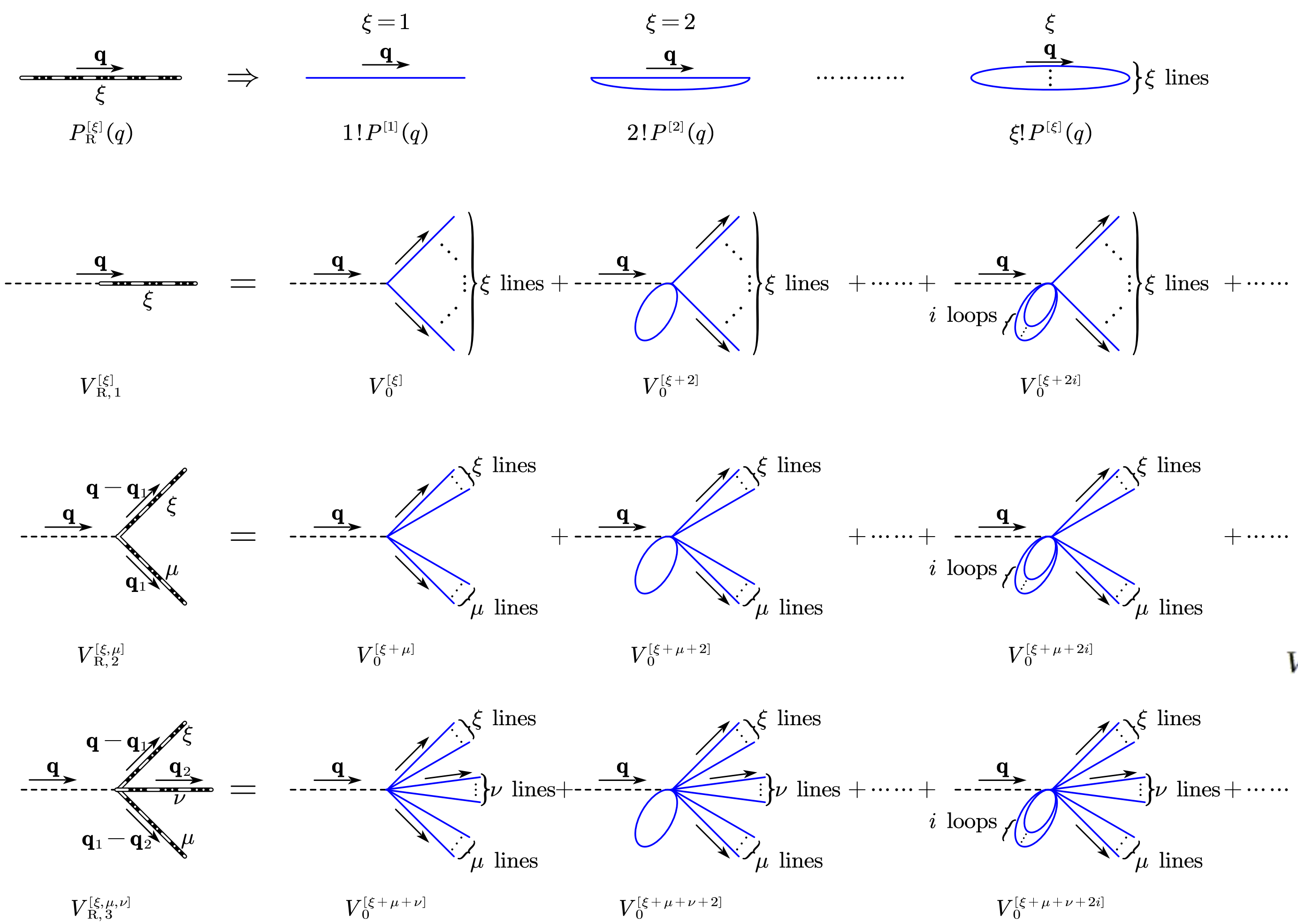
where $\xi\geq 2, \xi\in\mathbb{N}_+$.

$$P_{\mathrm{R},a}^{[\xi]} = \xi! P_a^{[\xi]}$$

Diagrammatic Approach: “Renormalized” Feynman-like Rules

$$\bar{\omega}_{\mathrm{gw}}^{X-\mathrm{like}} = \frac{q^5}{3(2\pi)^3\mathcal{H}^2}\left\{\prod_{i=1}^{n}\int\frac{\mathrm{d}^3\mathbf{q}_i}{(2\pi)^3}\right\}\left\{\sum_{\lambda=+,\times} Q_\lambda(\mathbf{q},\mathbf{q}_1)Q_\lambda(\mathbf{q},\mathbf{q}_j)\overline{\hat{I}(|\mathbf{q}-\mathbf{q}_1|,q_1,\eta_{\mathrm{in}})\hat{I}(|\mathbf{q}-\mathbf{q}_j|,q_j,\eta_{\mathrm{in}})}\right\}\mathcal{P}^{X-\mathrm{like}}$$





$$P^{[1]}(q) = \frac{2\pi^2}{q^3}\Delta_S^2(q)\ ,$$

$$P^{[\xi]}(q) = \int\frac{\mathrm{d}^3\tilde{\mathbf{q}}}{(2\pi)^3}P^{[\xi-1]}(\tilde{q})P^{[1]}(|\mathbf{q}-\tilde{\mathbf{q}}|)$$

where $\xi \geq 2, \xi \in \mathbb{N}_+$.

$$P_{\mathrm{R},a}^{[\xi]} = \xi! P_a^{[\xi]}$$

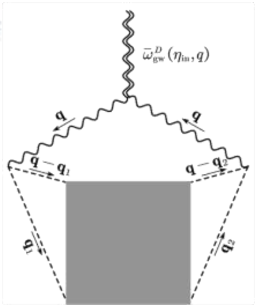
$$V_{\mathrm{R},1}^{[\xi]} = \sum_{i=0}^{\lfloor(o-\xi)/2\rfloor}\frac{(2i)!}{2^i i!}\binom{2i+\xi}{\xi}A_S^i V_0^{[2i+\xi]}\ ,$$

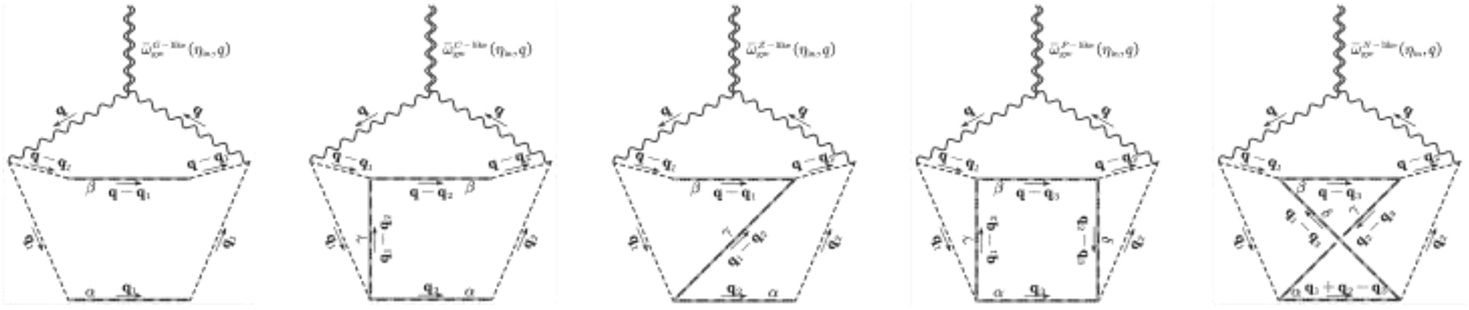
$$V_{\mathrm{R},2}^{[\xi,\mu]} = \sum_{i=0}^{\lfloor(o-\xi-\mu)/2\rfloor}\frac{(2i)!}{2^i i!}\binom{2i+\xi+\mu}{\mu}\binom{2i+\xi}{\xi}A_S^i V_0^{[2i+\xi+\mu]}\ ,$$

$$V_{\mathrm{R},3}^{[\xi,\mu,\nu]} = \sum_{i=0}^{\lfloor(o-\xi-\mu-\nu)/2\rfloor}\frac{(2i)!}{2^i i!}\binom{2i+\xi+\mu+\nu}{\nu}\binom{2i+\xi+\mu}{\mu}\binom{2i+\xi}{\xi}A_S^i V_0^{[2i+\xi+\mu+\nu]}$$

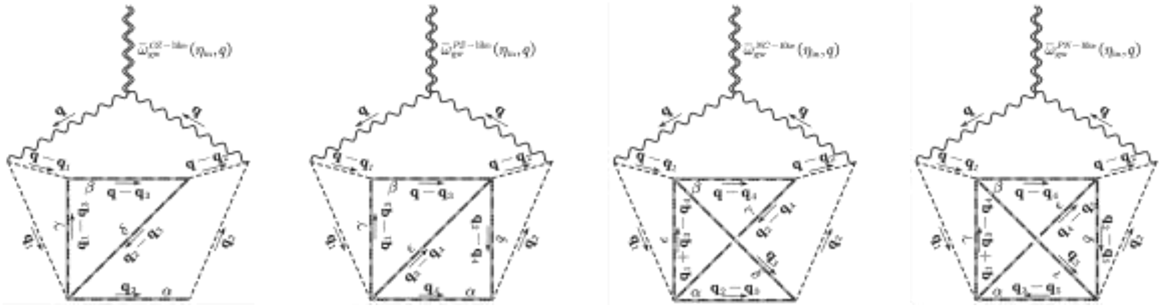
Diagrammatic Approach: "Renormalized" Diagrams

$$\bar{\omega}_{\mathrm{gw}}^{X-\mathrm{like}} = \frac{q^5}{3(2\pi)^3\mathcal{H}^2}\left\{\prod_{i=1}^{n}\int\frac{\mathrm{d}^3\mathbf{q}_i}{(2\pi)^3}\right\}\left\{\sum_{\lambda=+,\times} Q_\lambda(\mathbf{q},\mathbf{q}_1)Q_\lambda(\mathbf{q},\mathbf{q}_j)\overline{\hat{I}(|\mathbf{q}-\mathbf{q}_1|,q_1,\eta_{\mathrm{in}})\hat{I}(|\mathbf{q}-\mathbf{q}_j|,q_j,\eta_{\mathrm{in}})}\right\}\mathcal{P}^{X-\mathrm{like}}$$





$$\mathcal{P}^{G-\mathrm{like}} = 2\left(\sum_{\alpha=1}^{o} V_{\mathrm{R},1}^{[\alpha]}P_{\mathrm{R},a}^{[\alpha]}V_{\mathrm{R},1}^{[\alpha]}\right)\left(\sum_{\beta=1}^{o} V_{\mathrm{R},1}^{[\beta]}P_{\mathrm{R},a}^{[\beta]}V_{\mathrm{R},1}^{[\beta]}\right)$$

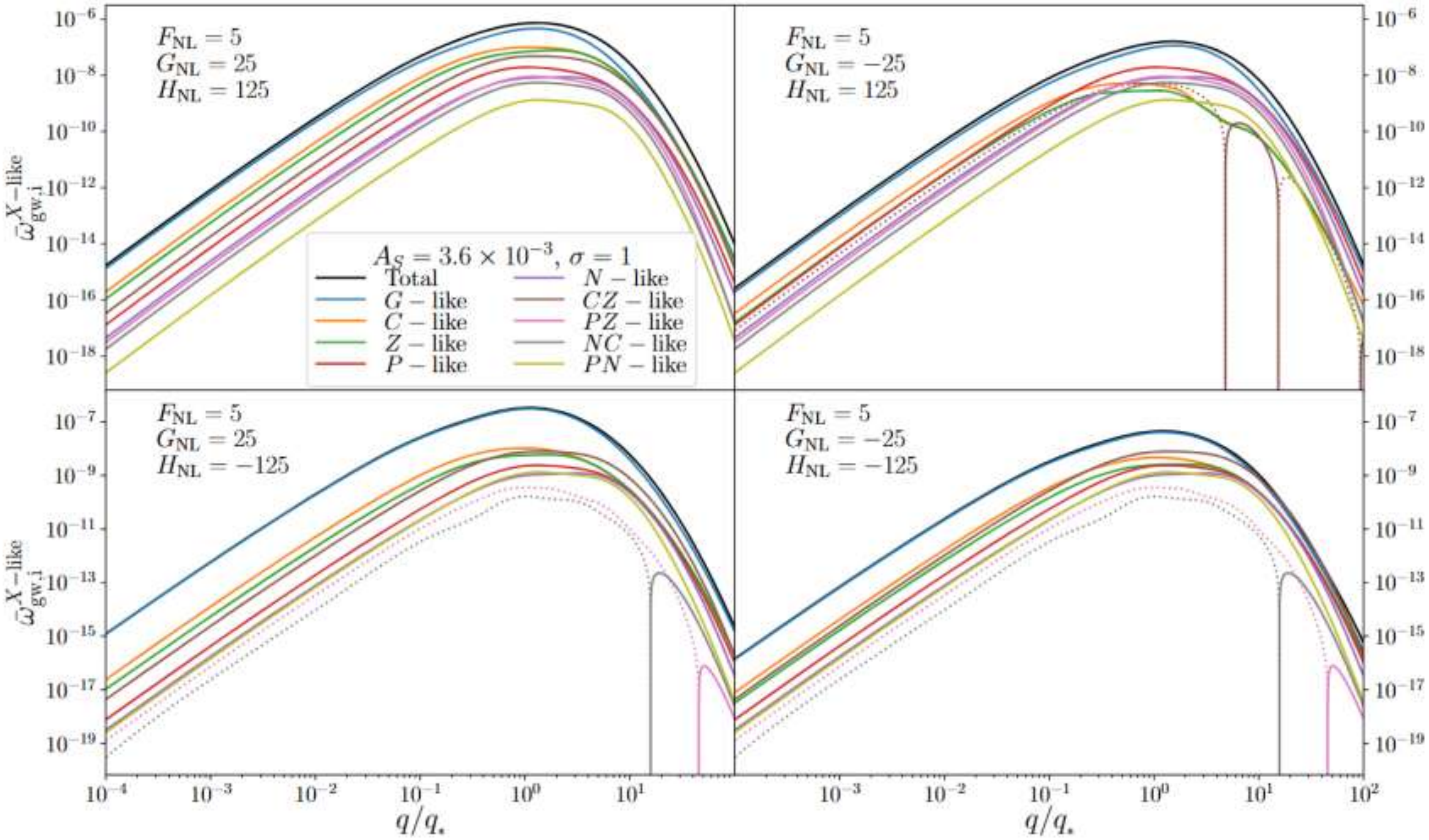


$$\mathcal{P}^{C-\mathrm{like}} = 4\sum_{\gamma=1}^{o-1}\sum_{\beta=1}^{o-\gamma}\sum_{\alpha=1}^{o-\gamma} V_{\mathrm{R},1}^{[\alpha]}P_{\mathrm{R},a}^{[\alpha]}V_{\mathrm{R},2}^{[\alpha,\gamma]}P_{\mathrm{R},b}^{[\gamma]}V_{\mathrm{R},2}^{[\gamma,\beta]}P_{\mathrm{R},a}^{[\beta]}V_{\mathrm{R},1}^{[\beta]}$$

......

$$\mathcal{P}^{PN-\mathrm{like}} = \sum_{\iota=1}^{o-2}\sum_{\epsilon=1}^{o-2}\sum_{\delta=1}^{\min(o-\epsilon,o-\iota)-1}\sum_{\gamma=1}^{\min(o-\epsilon,o-\iota)-1}\sum_{\beta=1}^{\min(o-\gamma-\epsilon,o-\delta-\iota)}\sum_{\alpha=1}^{\min(o-\gamma-\iota,o-\delta-\epsilon)} P_{\mathrm{R},a}^{[\alpha]}V_{\mathrm{R},3}^{[\alpha,\gamma,\epsilon]}P_{\mathrm{R},b}^{[\gamma]}V_{\mathrm{R},3}^{[\beta,\gamma,\iota]}P_{\mathrm{R},a}^{[\beta]}$$
$$\times V_{\mathrm{R},3}^{[\beta,\delta,\epsilon]}P_{\mathrm{R},b}^{[\delta]}V_{\mathrm{R},3}^{[\alpha,\delta,\iota]}P_{\mathrm{R},c}^{[\epsilon]}P_{\mathrm{R},c}^{[\iota]}$$

Isotropic Background: Energy-Density Fraction Spectrum

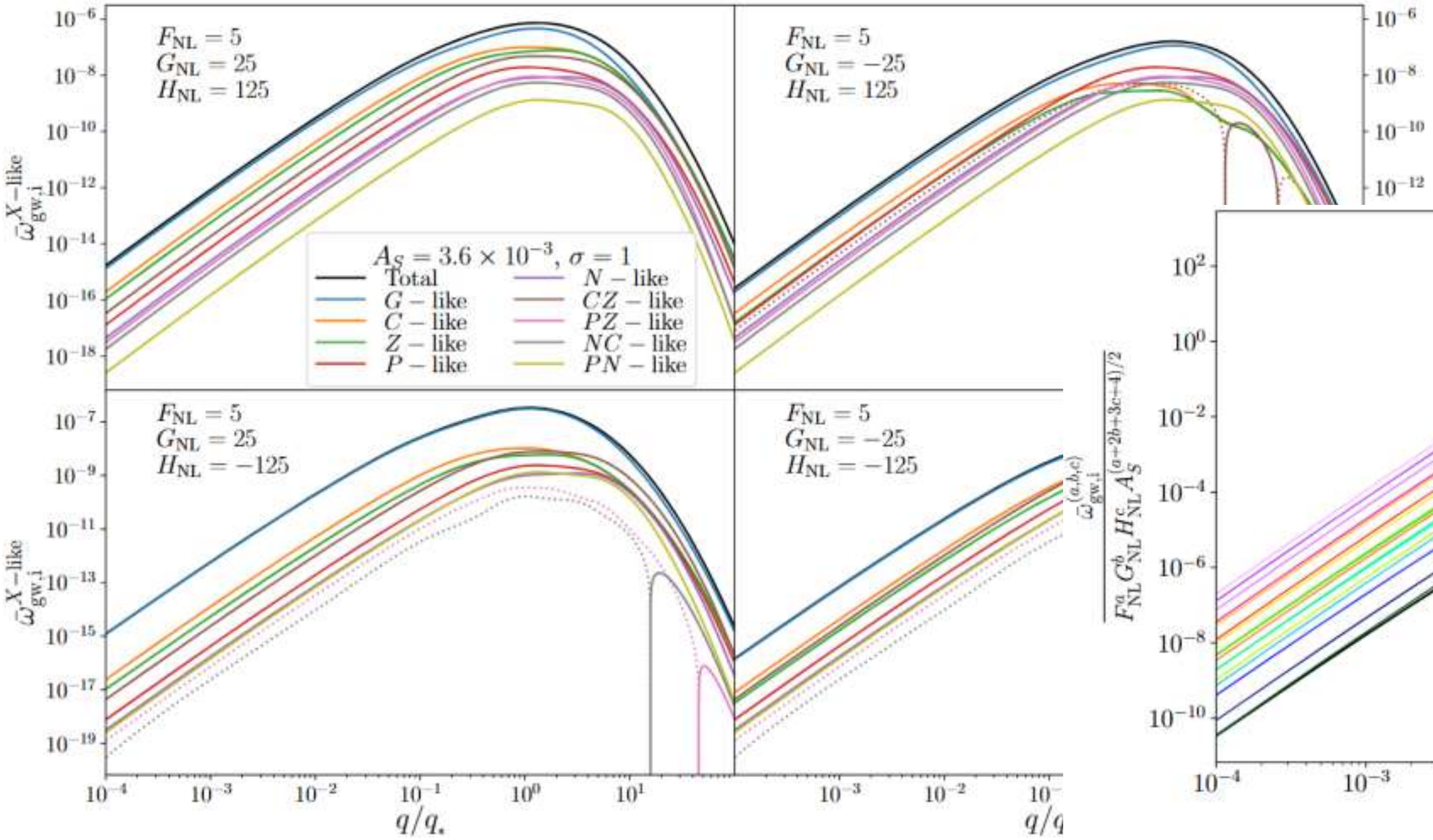


$$\Delta_{\rm S}^2(q)=\frac{A_{\rm S}}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

$$\sigma=1$$

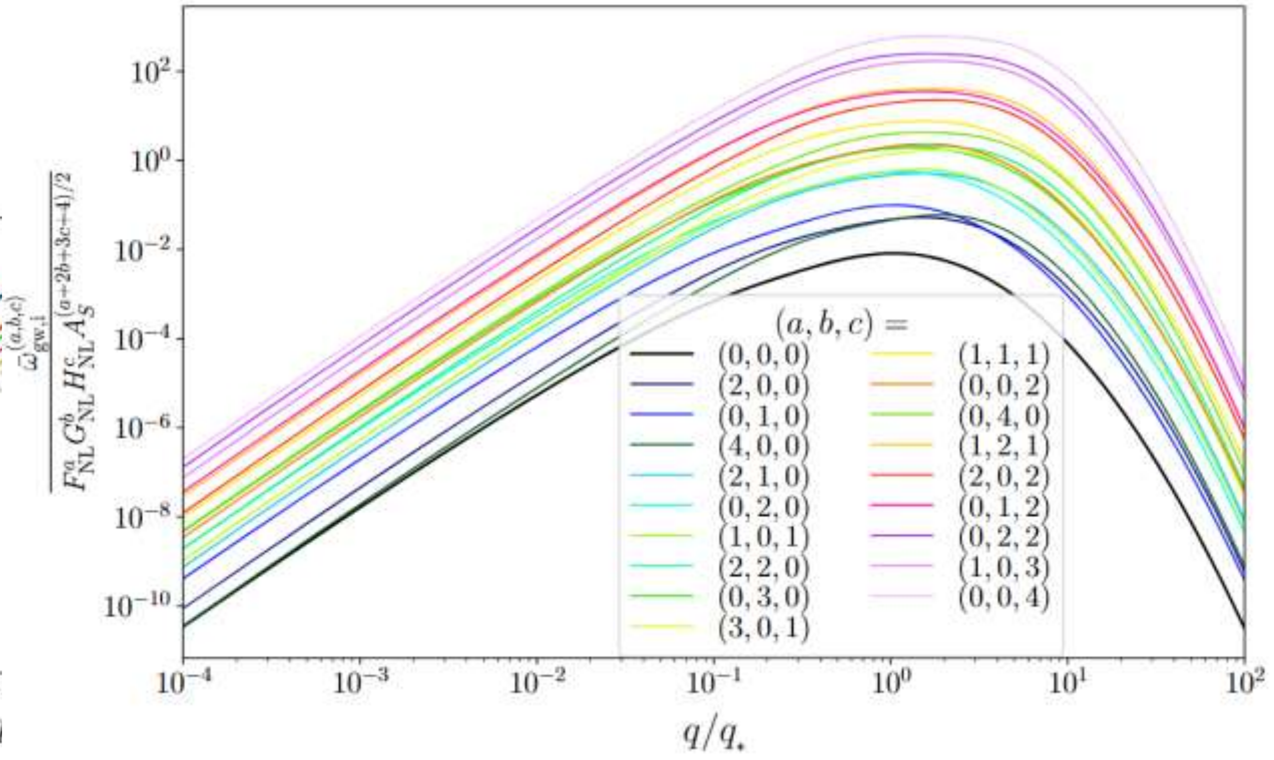
$$\bar{\omega}_{\rm gw,in}(q)=\sum_X \bar{\omega}_{\rm gw,in}^{X-\rm like}(\eta,q)$$

Isotropic Background: Energy-Density Fraction Spectrum



$$\Delta_S^2(q)=\frac{A_S}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

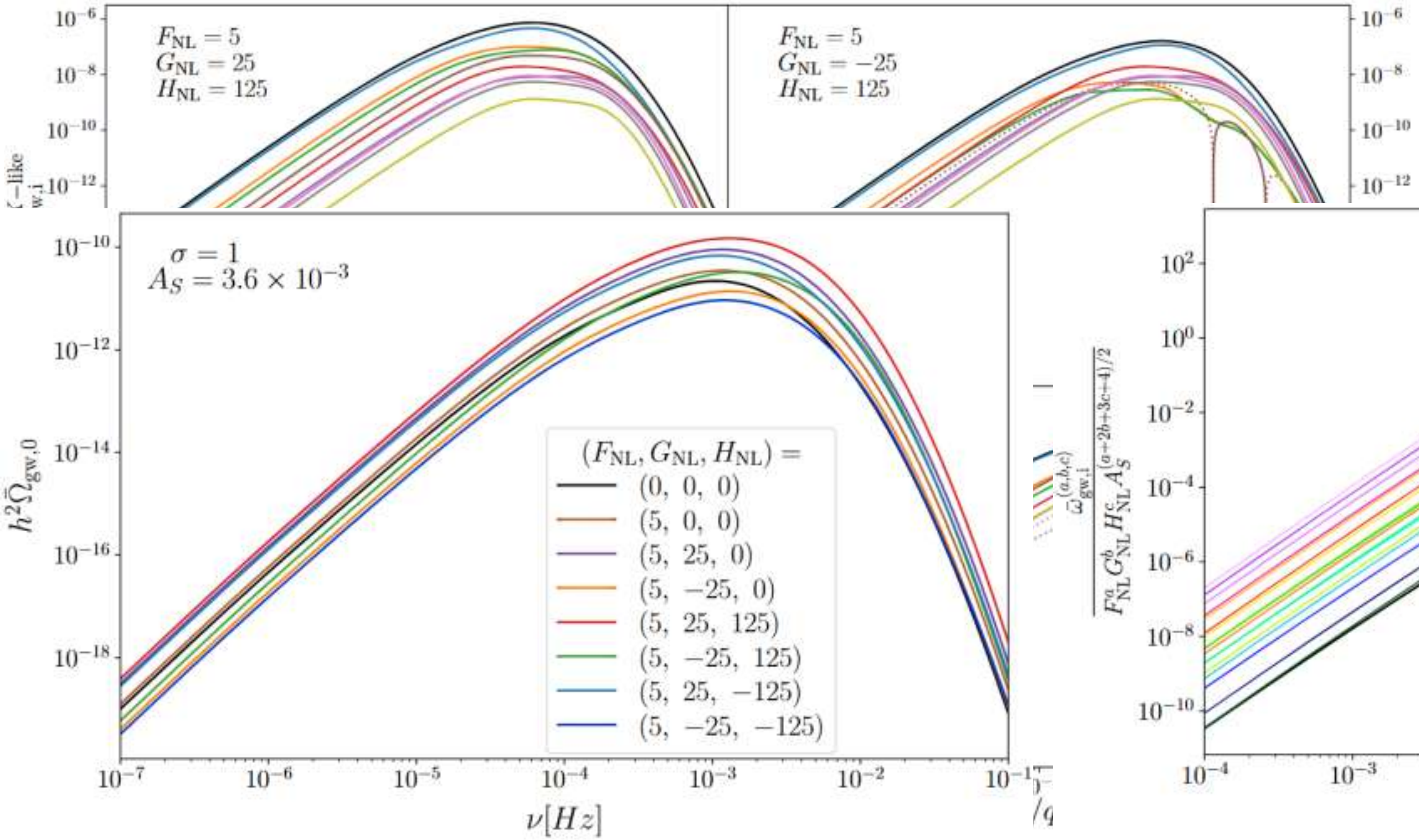
$$\sigma=1$$

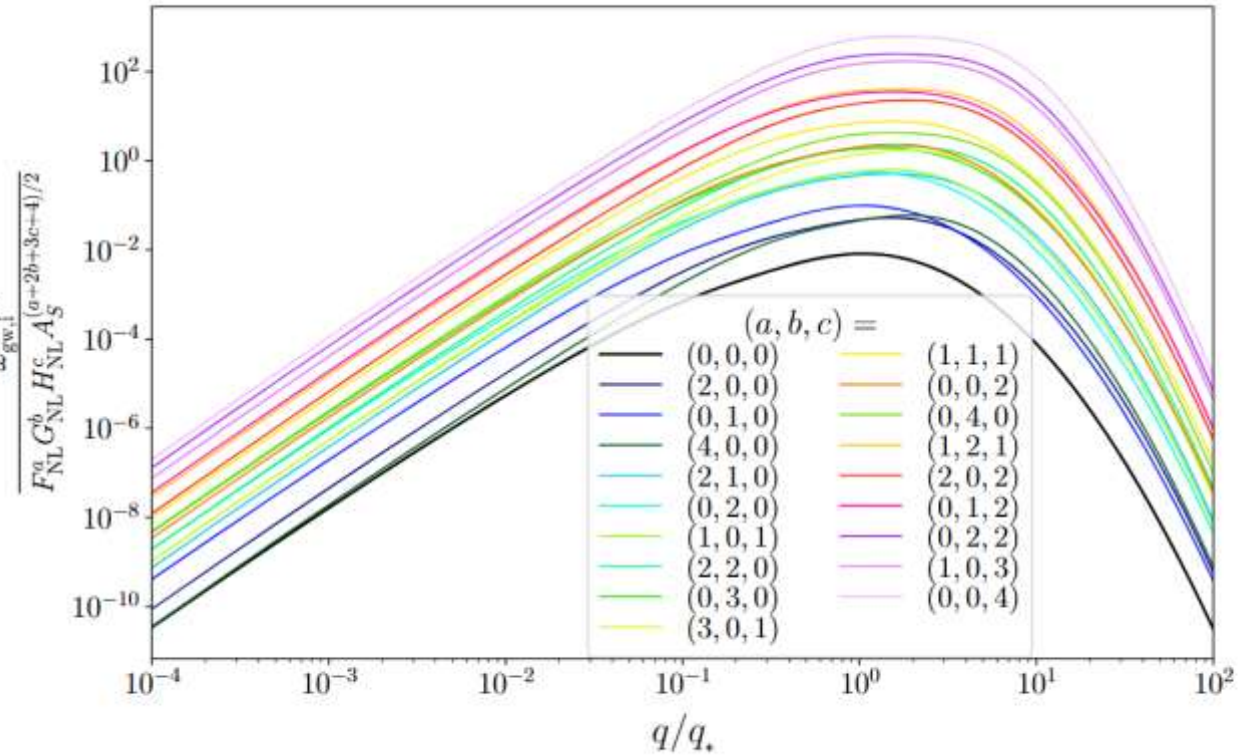


$$\bar{\omega}_{\rm gw,in}(q)=\sum_X \bar{\omega}^{X-\rm like}_{\rm gw,in}(\eta,q)=\sum_{(a,b,c)}\bar{\omega}^{(a,b,c)}_{\rm gw,in}(q)$$

$$\bar{\omega}^{(a,b,c)}_{\rm gw,in}\propto F^a_{\rm NL}G^b_{\rm NL}H^c_{\rm NL}A^{(a+2b+3c+4)/2}$$

Isotropic Background: Energy-Density Fraction Spectrum





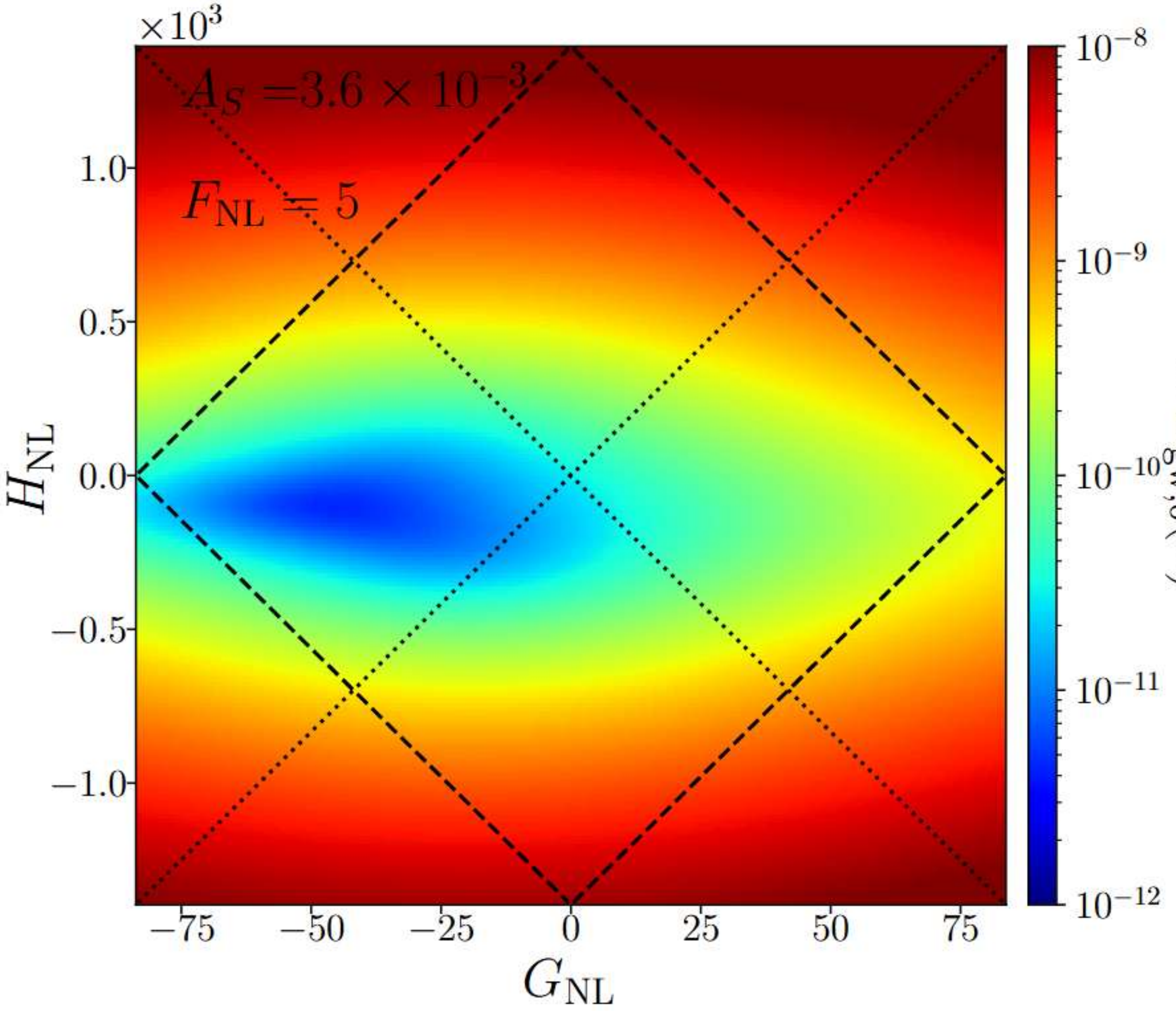
$$\Delta^2_S(q) = \frac{A_S}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

$$\sigma = 1$$

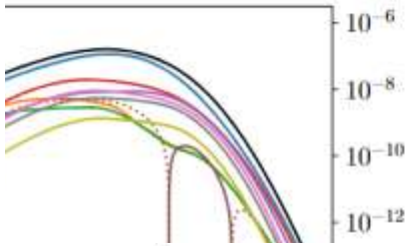
$$\bar{\omega}_{gw,in}(q) = \sum_X \bar{\omega}^{X-\mathrm{like}}_{gw,in}(\eta, q) = \sum_{(a,b,c)} \bar{\omega}^{(a,b,c)}_{gw,in}(q)$$

$$\bar{\omega}^{(a,b,c)}_{gw,in} \propto F^a_{NL} G^b_{NL} H^c_{NL} A^{(a+2b+3c+4)/2}$$

Isotropic Background: Energy-Density Fraction Spectrum

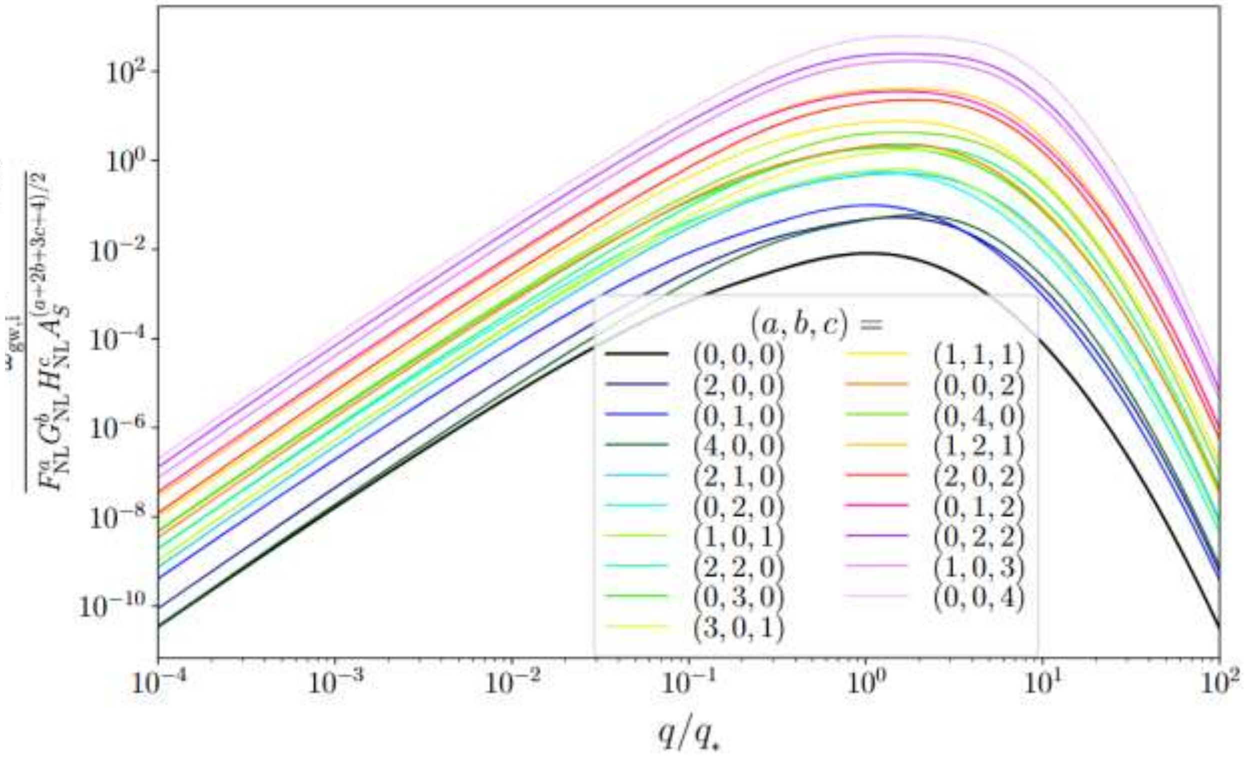


$$\bar{\omega}_{\rm gw,in}(q) = \sum_X \bar{\omega}^{X-\rm like}_{\rm gw,in}(\eta, q)$$



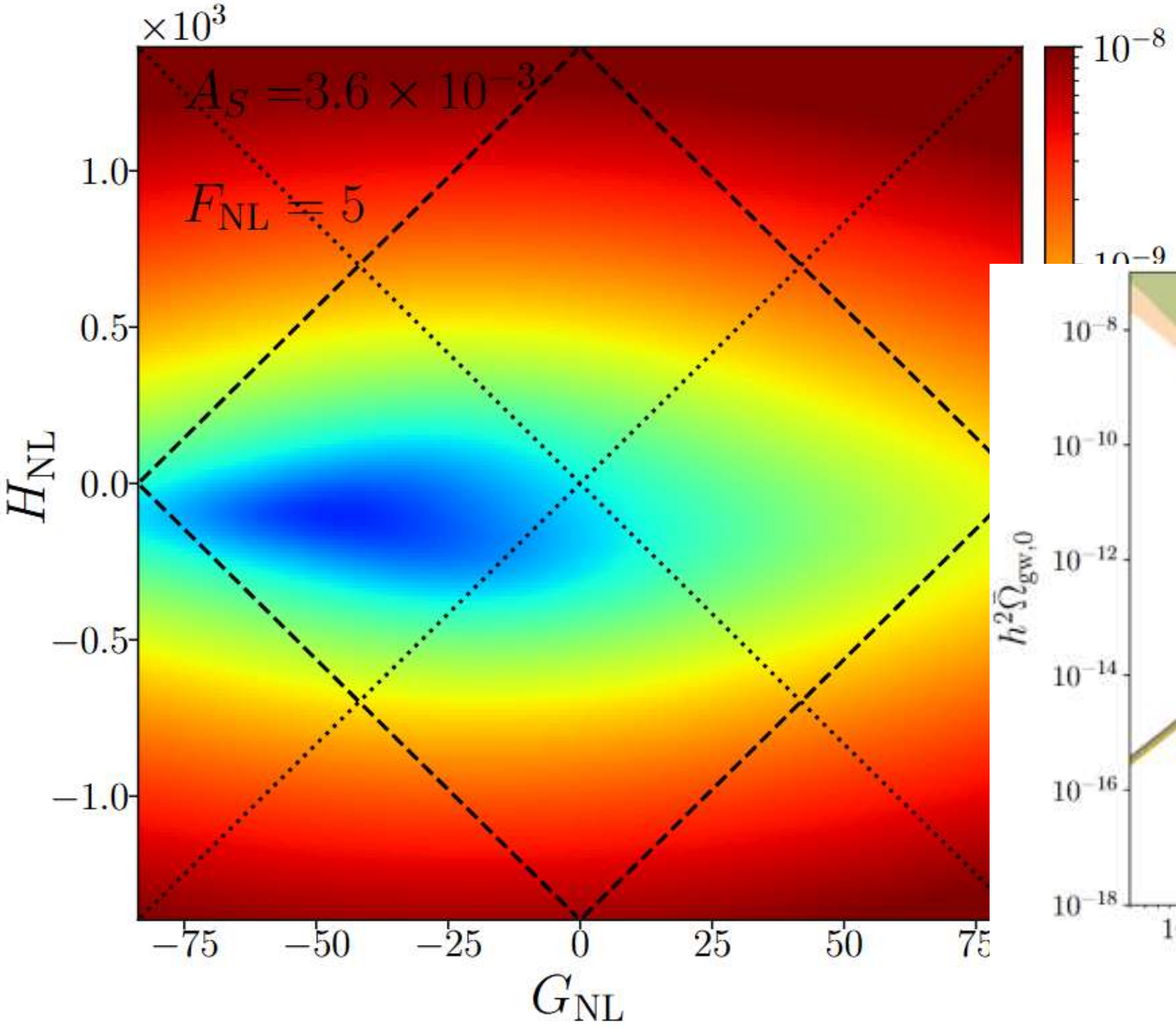
$$\Delta^2_{\rm S}(q) = \frac{A_{\rm S}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

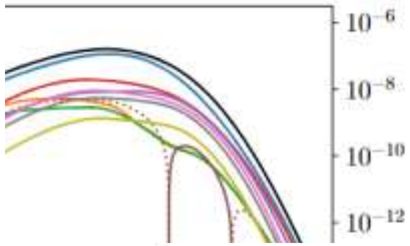
$$\sigma = 1$$



$$\bar{\omega}^{(a,b,c)}_{\rm gw,in} \propto F^a_{\rm NL} G^b_{\rm NL} H^c_{\rm NL} A^{(a+2b+3c+4)/2}$$

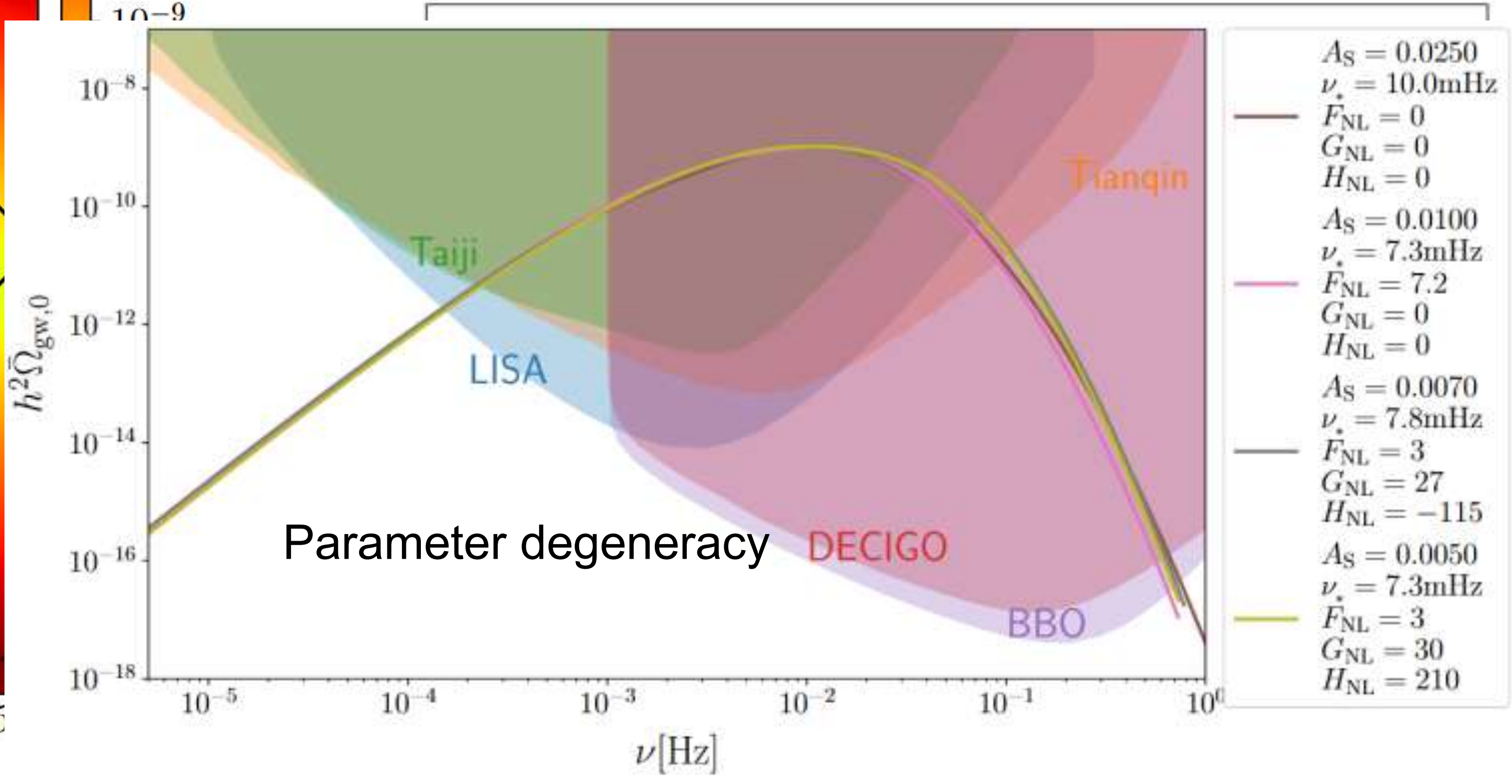
Isotropic Background: Energy-Density Fraction Spectrum





$$\Delta_S^2(q) = \frac{A_S}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

$\sigma = 1$

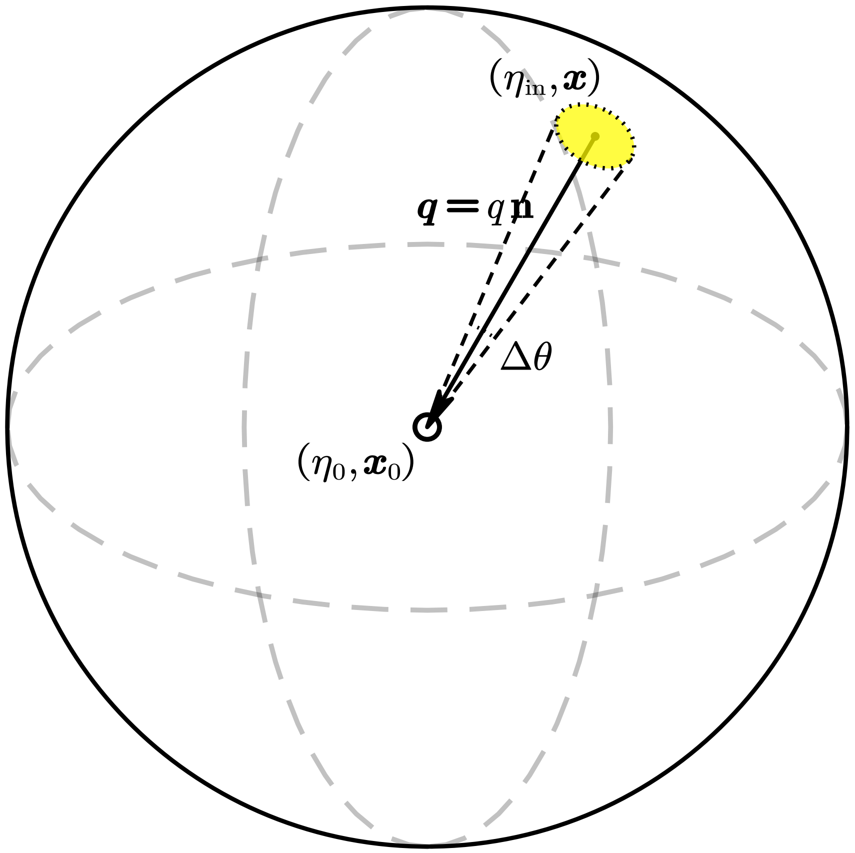


$$\bar{\omega}_{\rm gw,in}(q) = \sum_X \bar{\omega}_{\rm gw,in}^{X-\rm like}(\eta, q)$$

$$\bar{\omega}_{\rm gw,in}^{(a,b,c)} \propto F_{\rm NL}^a G_{\rm NL}^b H_{\rm NL}^c A^{(a+2b+3c+4)/2}$$

Anisotropic and Non-Gaussian Background

Angular Resolution: numerous of Hubble-horizon patches ➡ ignoring initial anisotropies

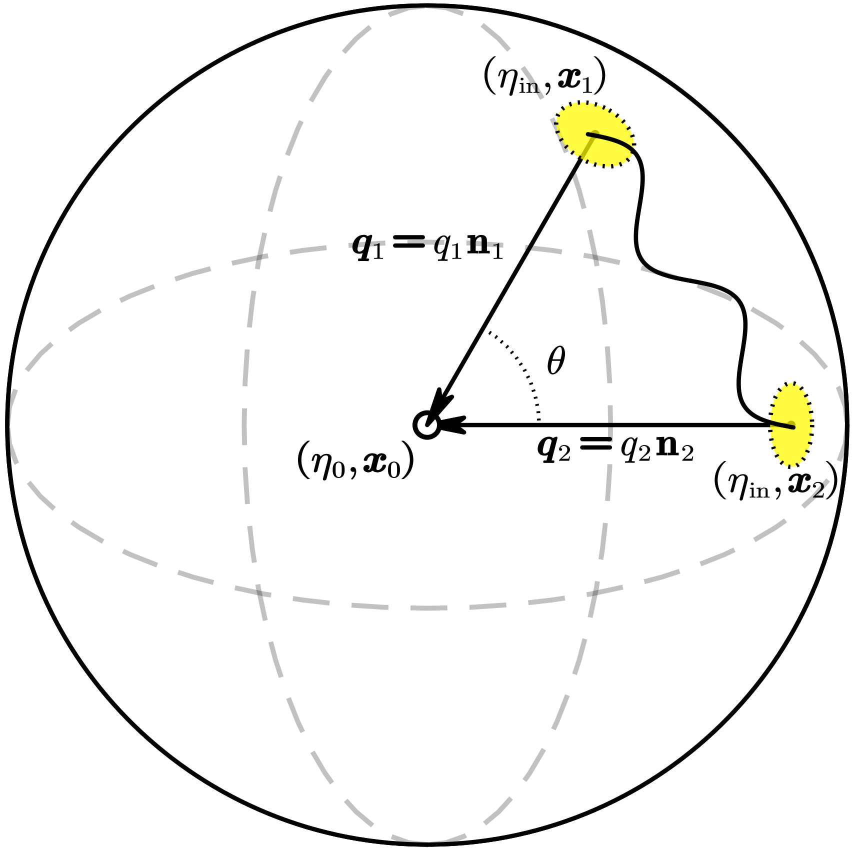


Anisotropic and Non-Gaussian Background

Angular Resolution: numerous of Hubble-horizon patches ➡ ignoring initial anisotropies

Anisotropy

$$\left\langle \prod_{i=1}^{2} \delta_{\mathrm{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \delta_{\ell_1\ell_2}\delta_{m_1 m_2}(-1)^{m_1}\tilde{C}_{\ell_1}(\nu)$$



Anisotropic and Non-Gaussian Background

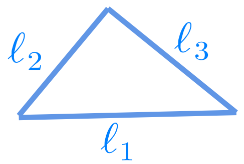
Angular Resolution: numerous of Hubble-horizon patches ➡ ignoring initial anisotropies

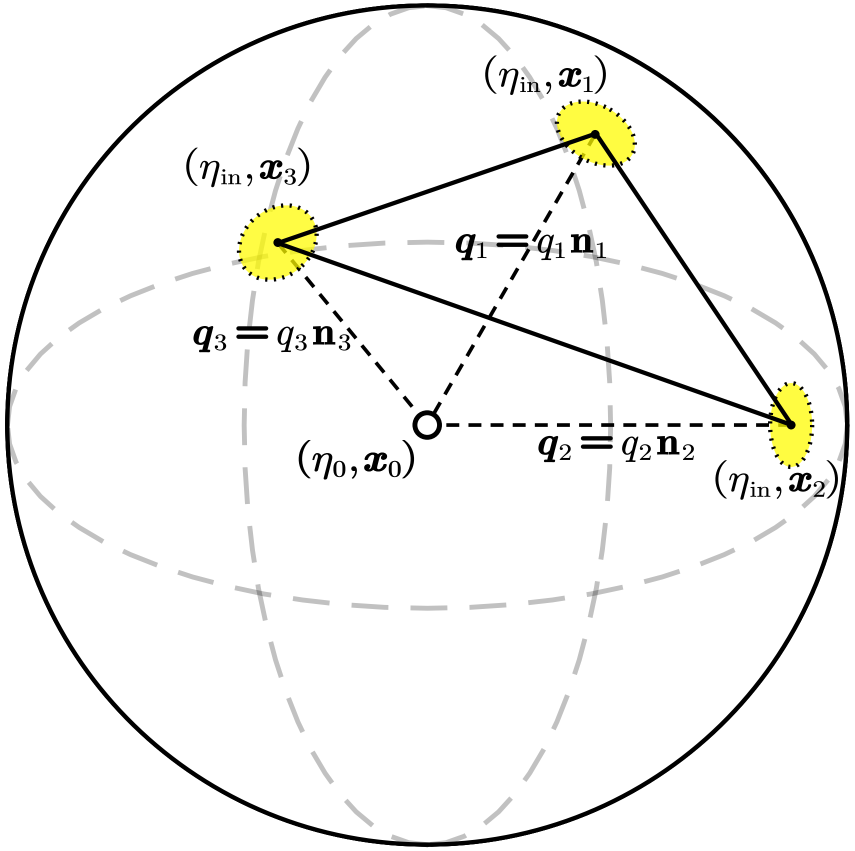
Anisotropy

$$\left\langle \prod_{i=1}^{2} \delta_{\mathrm{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \delta_{\ell_1\ell_2}\delta_{m_1m_2}(-1)^{m_1}\tilde{C}_{\ell_1}(\nu)$$

Non-Gaussianity

$$\left\langle \prod_{i=1}^{3} \delta_{\mathrm{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1\ell_2\ell_3}(\nu)$$



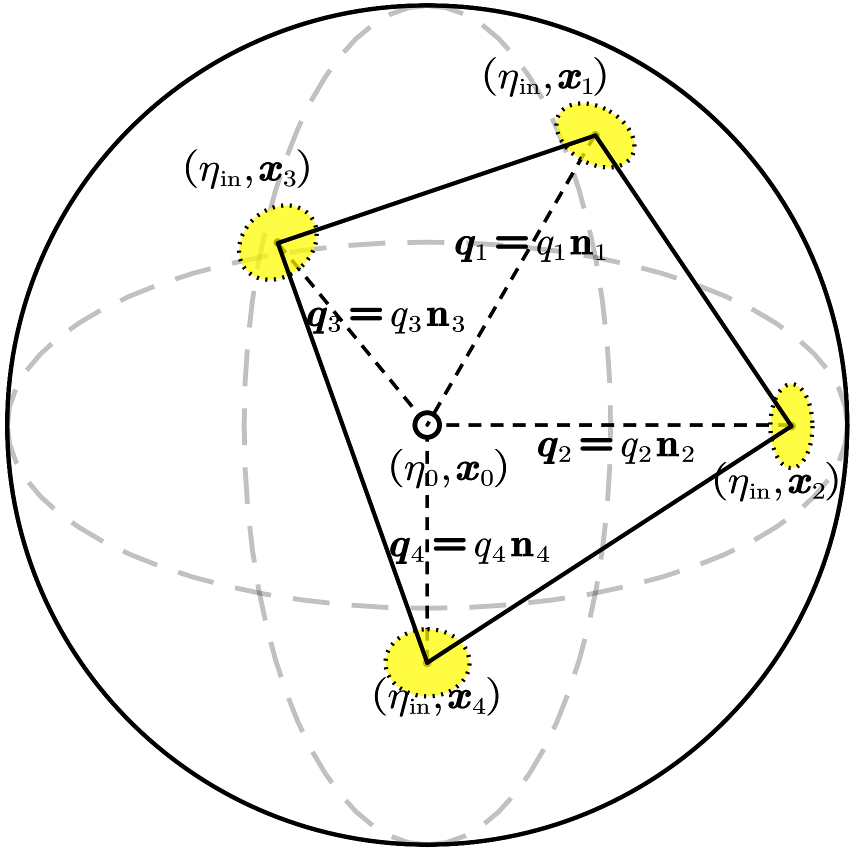


Anisotropic and Non-Gaussian Background

Angular Resolution: numerous of Hubble-horizon patches ➡ ignoring initial anisotropies

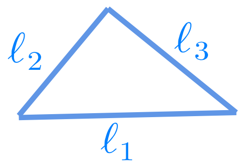
Anisotropy

$$\left\langle \prod_{i=1}^{2} \delta_{\mathrm{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \delta_{\ell_1\ell_2}\delta_{m_1m_2}(-1)^{m_1}\tilde{C}_{\ell_1}(\nu)$$



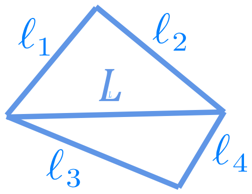
Non-Gaussianity

$$\left\langle \prod_{i=1}^{3} \delta_{\mathrm{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1\ell_2\ell_3}(\nu)$$



$$\left\langle \prod_{i=1}^{4} \delta_{\mathrm{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \sum_{LM} (-1)^M \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} \ell_3 & \ell_4 & L \\ m_3 & m_4 & M \end{pmatrix} T^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu)$$

$$T^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu) \longrightarrow T_{c}{}^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu) \longrightarrow P^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu) \longrightarrow t^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu)$$



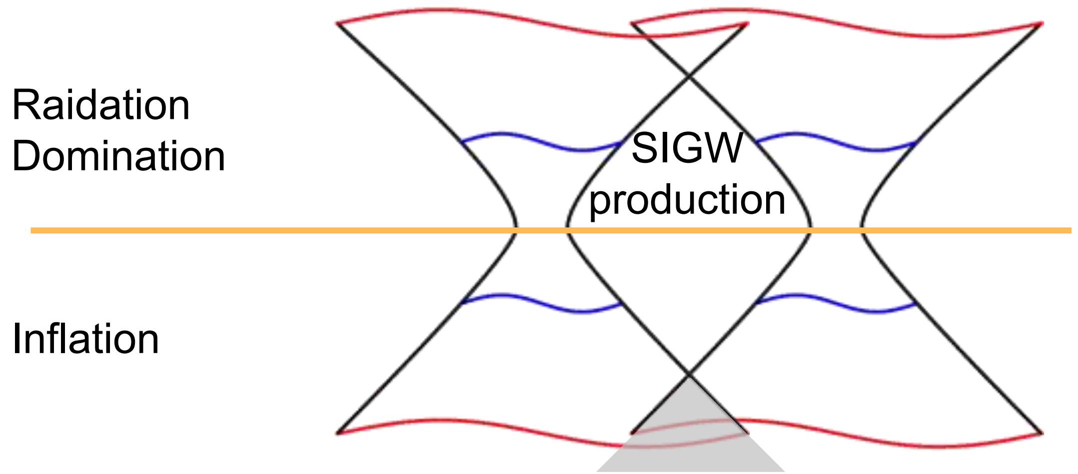
Anisotropic Background: Initial Inhomogeneities

$$\zeta = \zeta_{\mathrm{g}} + F_{\mathrm{NL}}\left(\zeta_{\mathrm{g}}^2 - \langle \zeta_{\mathrm{g}}^2 \rangle\right) + G_{\mathrm{NL}}\zeta_{\mathrm{g}}^3 + H_{\mathrm{NL}}\left(\zeta_{\mathrm{g}}^4 - 3\langle \zeta_{\mathrm{g}}^2 \rangle^2\right) + \cdots$$

$$\zeta_{\mathrm{g}} = \zeta_{\mathrm{gS}} + \zeta_{\mathrm{gL}}$$

Short modes: SIGW production

Long modes: energy-density modulation



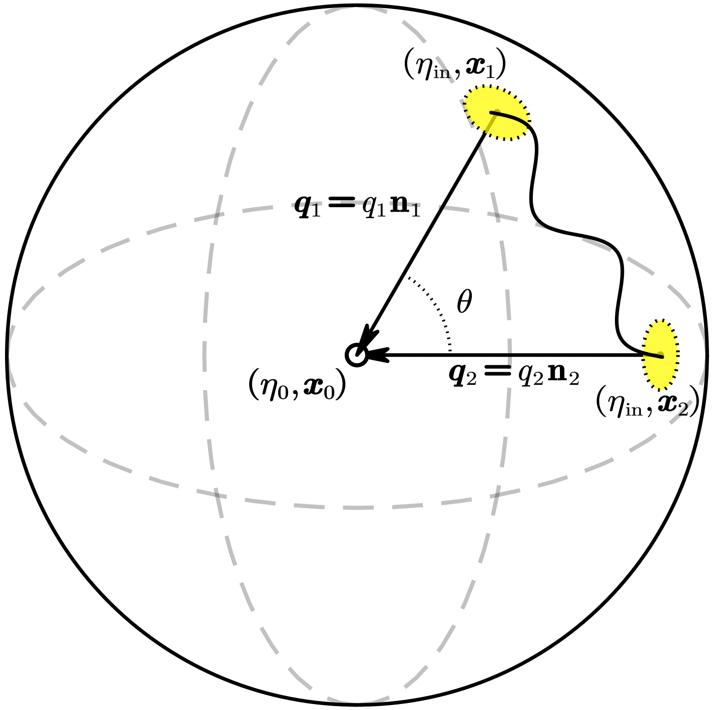
Boltzmann equation

$$\delta_{\mathrm{gw},0}(\boldsymbol{q}) = \delta_{\mathrm{gw,in}}(\boldsymbol{q}) + \left(4 - n_{\mathrm{gw}}(\nu)\right)\Phi_{\mathrm{L}}(\eta_{\mathrm{in}}, \boldsymbol{x}_{\mathrm{in}}) + 2\left(4 - n_{\mathrm{gw}}(\nu)\right)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}_0}\int_{\eta_{\mathrm{in}}}^{\eta_0} \mathrm{d}\eta\, e^{ik\mu(\eta-\eta_0)}\partial_\eta\Phi_{\mathrm{L}}(\eta, \boldsymbol{k})$$

$$\langle \delta_{\mathrm{gw,in}}(\boldsymbol{x}_1)\delta_{\mathrm{gw,in}}(\boldsymbol{x}_2)\rangle$$

$$\Downarrow$$

$$\langle \omega_{\mathrm{gw,in}}(\boldsymbol{x}_1)\omega_{\mathrm{gw,in}}(\boldsymbol{x}_2)\rangle$$



Anisotropic Background: Initial Inhomogeneities

$$\zeta = \zeta_{\mathrm{g}} + F_{\mathrm{NL}}\left(\zeta_{\mathrm{g}}^2 - \langle \zeta_{\mathrm{g}}^2 \rangle\right) + G_{\mathrm{NL}}\zeta_{\mathrm{g}}^3 + H_{\mathrm{NL}}\left(\zeta_{\mathrm{g}}^4 - 3\langle \zeta_{\mathrm{g}}^2 \rangle^2\right) + \cdots$$

$$\zeta_{\mathrm{g}} = \zeta_{\mathrm{gS}} + \zeta_{\mathrm{gL}}$$

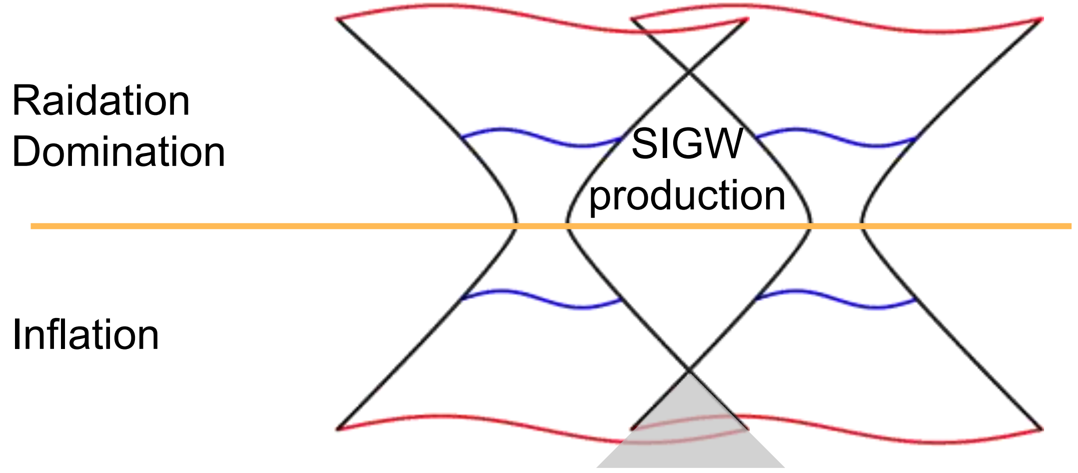
Boltzmann equation

$$\bar{\rho}_{\mathrm{gw}} = \frac{m_{\mathrm{PL}} \boxed{e^{2\Phi_{\mathrm{L}}}}}{16a^2(\eta)} \left\langle \overline{\partial_l h_{ij} \partial_l h_{ij}} \right\rangle$$

$$\delta_{\mathrm{gw},0}(\boldsymbol{q}) = \delta_{\mathrm{gw,in}}(\boldsymbol{q}) + \left(4 - n_{\mathrm{gw}}(\nu)\right) \Phi_{\mathrm{L}}(\eta_{\mathrm{in}}, \boldsymbol{x}_{\mathrm{in}})$$

Short modes: SIGW production

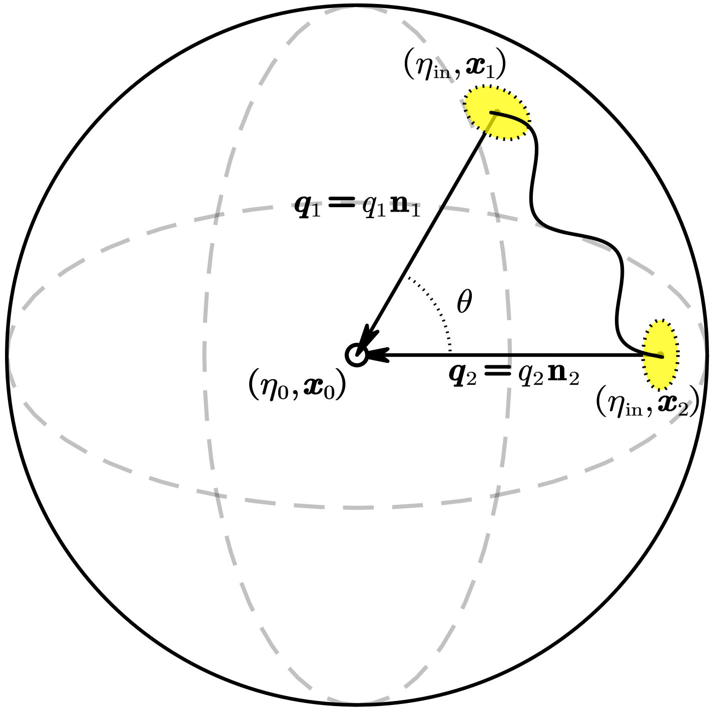
Long modes: energy-density modulation



$$\langle \delta_{\mathrm{gw,in}}(\boldsymbol{x}_1) \delta_{\mathrm{gw,in}}(\boldsymbol{x}_2) \rangle$$

$$\Downarrow$$

$$\langle \omega_{\mathrm{gw,in}}(\boldsymbol{x}_1) \omega_{\mathrm{gw,in}}(\boldsymbol{x}_2) \rangle$$



Anisotropic Background: Initial Inhomogeneities

$$\zeta = \zeta_{\rm g} + F_{\rm NL}\left(\zeta_{\rm g}^2 - \langle \zeta_{\rm g}^2 \rangle\right) + G_{\rm NL}\zeta_{\rm g}^3 + H_{\rm NL}\left(\zeta_{\rm g}^4 - 3\langle \zeta_{\rm g}^2 \rangle^2\right) + \cdots$$

$$\zeta_{\rm g} = \zeta_{\rm gS} + \zeta_{\rm gL}$$

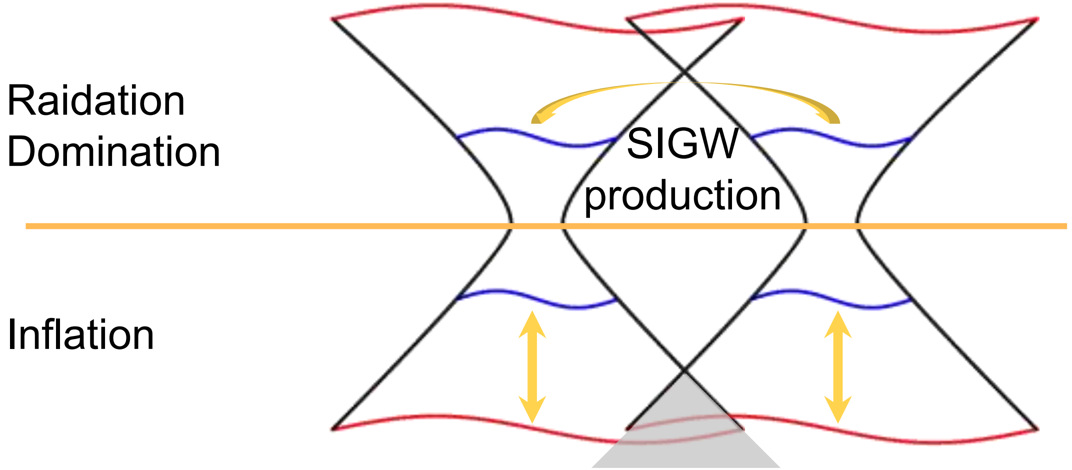
Boltzmann equation

$$\bar{\rho}_{\rm gw} = \frac{m_{\rm PL}\boxed{e^{2\Phi_{\rm L}}}}{16a^2(\eta)}\left\langle \overline{\partial_l h_{ij}\partial_l h_{ij}} \right\rangle$$

$$\delta_{\rm gw,0}(\boldsymbol{q}) = \delta_{\rm gw,in}(\boldsymbol{q}) + \left(4 - n_{\rm gw}(\nu)\right)\Phi_{\rm L}(\eta_{\rm in}, \boldsymbol{x}_{\rm in})$$

Short modes: SIGW production

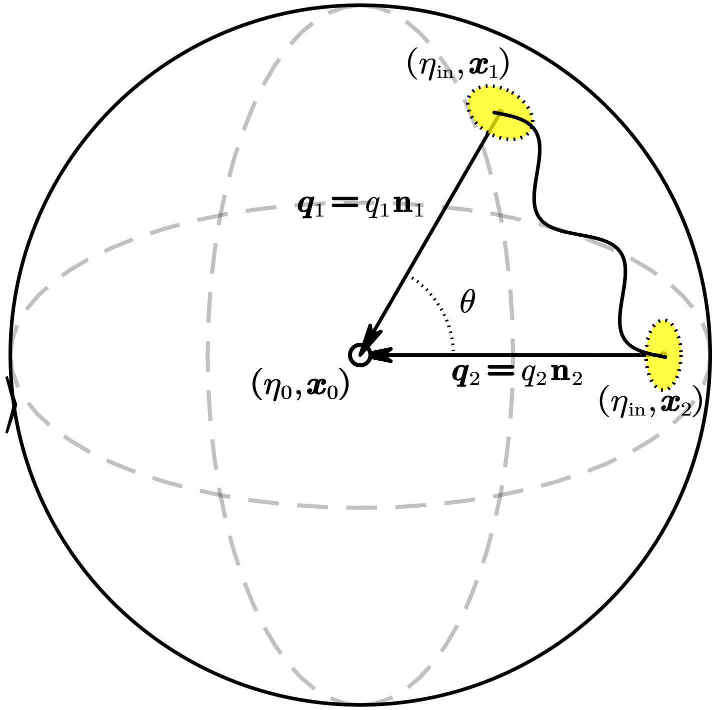
Long modes: energy-density modulation



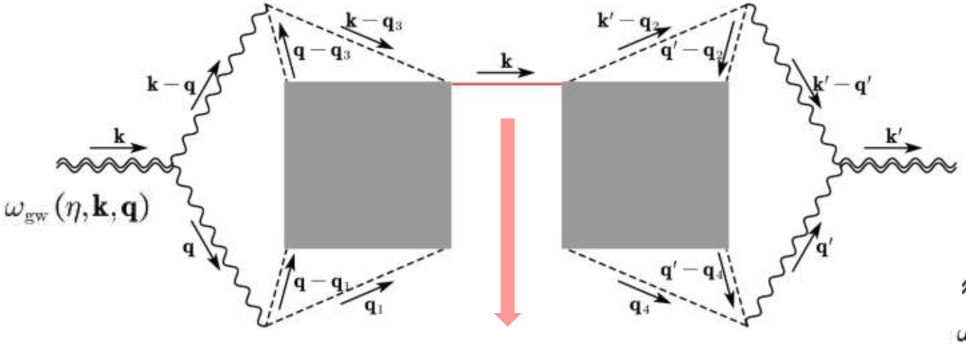
$$\left\langle \delta_{\rm gw,in}(\boldsymbol{x}_1, \boldsymbol{q}_1)\delta_{\rm gw,in}(\boldsymbol{x}_2, \boldsymbol{q}_2) \right\rangle$$

$$\Big\downarrow$$

$$\left\langle \omega_{\rm gw,in}(\boldsymbol{x}_1, \boldsymbol{q}_1)\omega_{\rm gw,in}(\boldsymbol{x}_2, \boldsymbol{q}_2) \right\rangle$$



Anisotropic Background: Two-point Angular Correlator

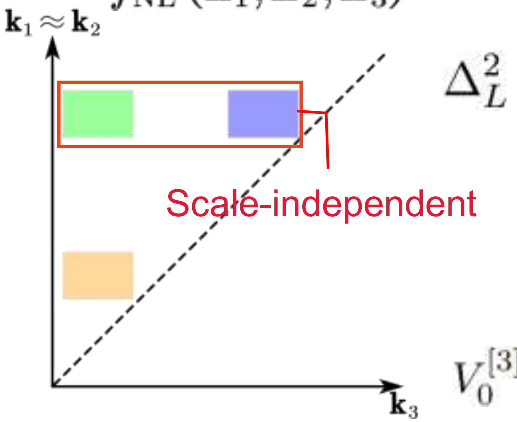


Non-Gaussian bridge

$$\langle\omega_{\mathrm{gw}}(\eta_{\mathrm{in}},\mathbf{x}_{\mathrm{in}},\mathbf{q})\omega_{\mathrm{gw}}(\eta_{\mathrm{in}},\mathbf{x}'_{\mathrm{in}},\mathbf{q}')\rangle$$

$$\sim\langle\langle\omega_{\mathrm{gw}}(\eta_{\mathrm{in}},\mathbf{k},\mathbf{q})\rangle_{\mathbf{x}_{\mathrm{in}}}\langle\omega^*_{\mathrm{gw}}(\eta_{\mathrm{in}},\mathbf{k}',\mathbf{q}')\rangle_{\mathbf{x}'_{\mathrm{in}}}\rangle^{\mathcal{O}(\Delta_L^2)}$$

$f_{\mathrm{NL}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$



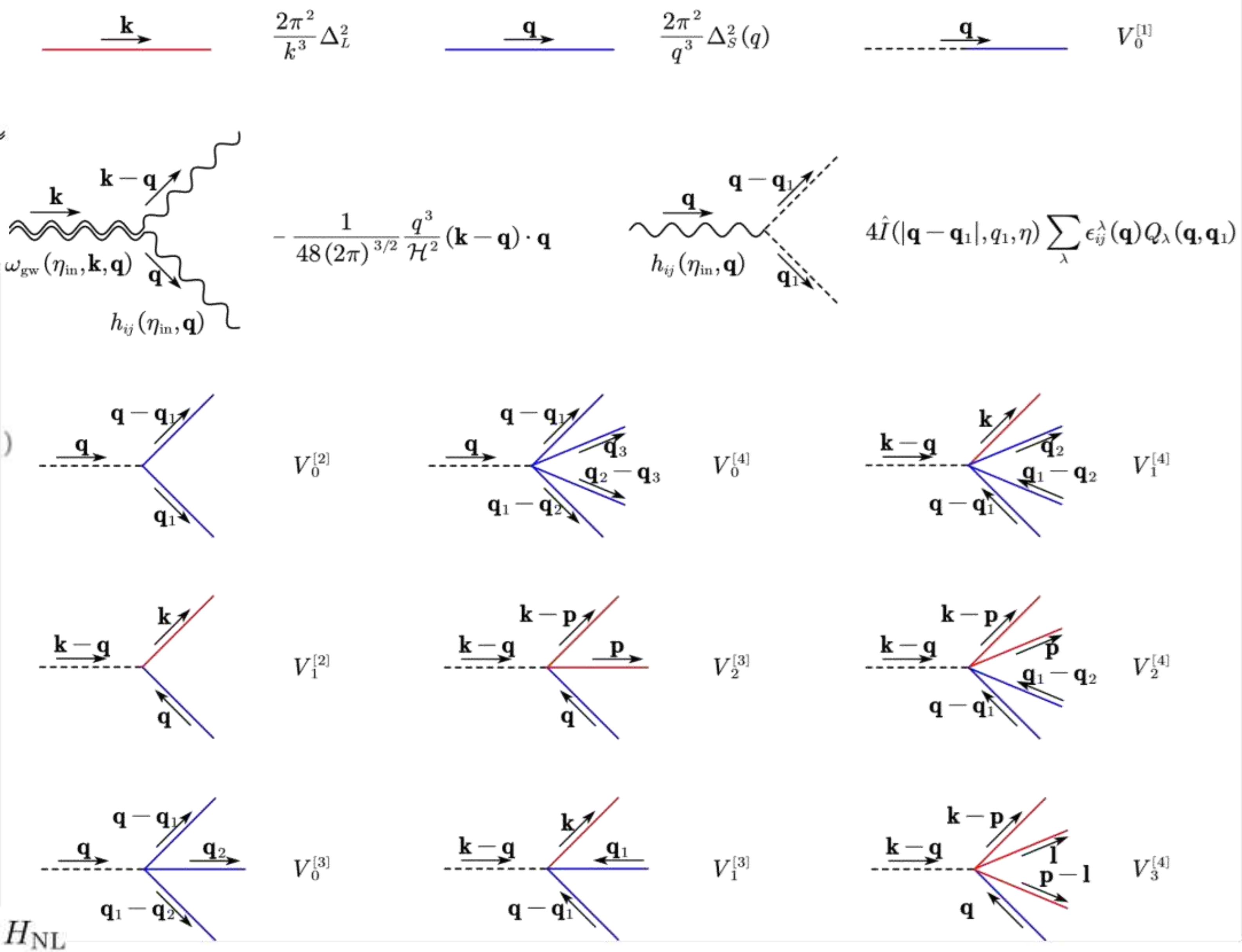
$$\Delta_L^2 = A_L \simeq 2.1\times10^{-9}$$

$$V_0^{[1]} = 1$$

$$V_0^{[2]} = V_1^{[2]} = F_{\mathrm{NL}}$$

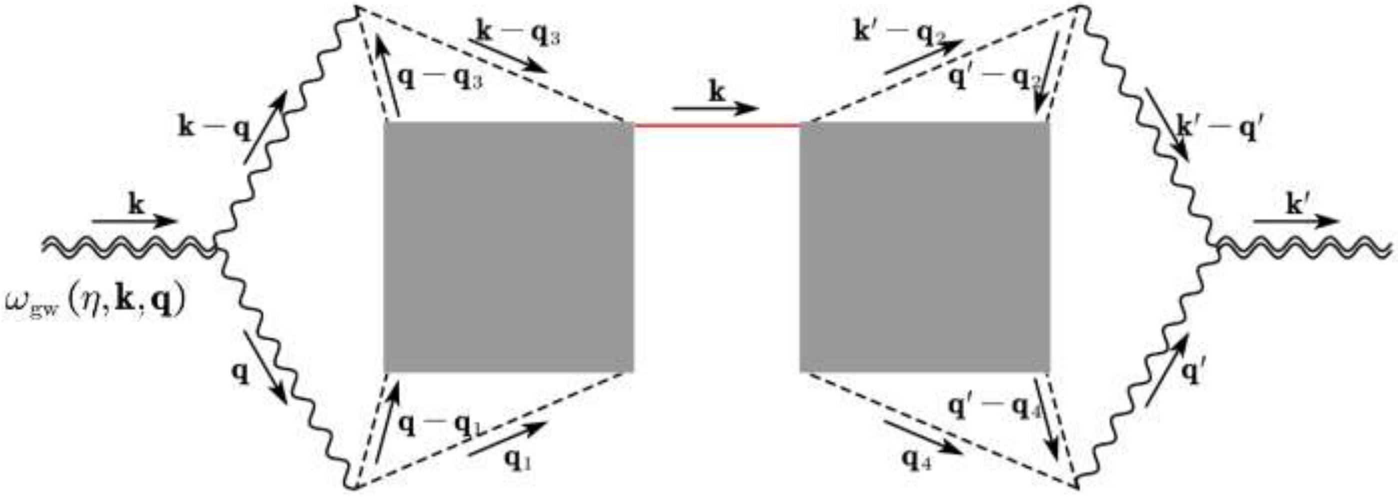
$$V_0^{[3]} = V_1^{[3]} = V_2^{[3]} = G_{\mathrm{NL}}$$

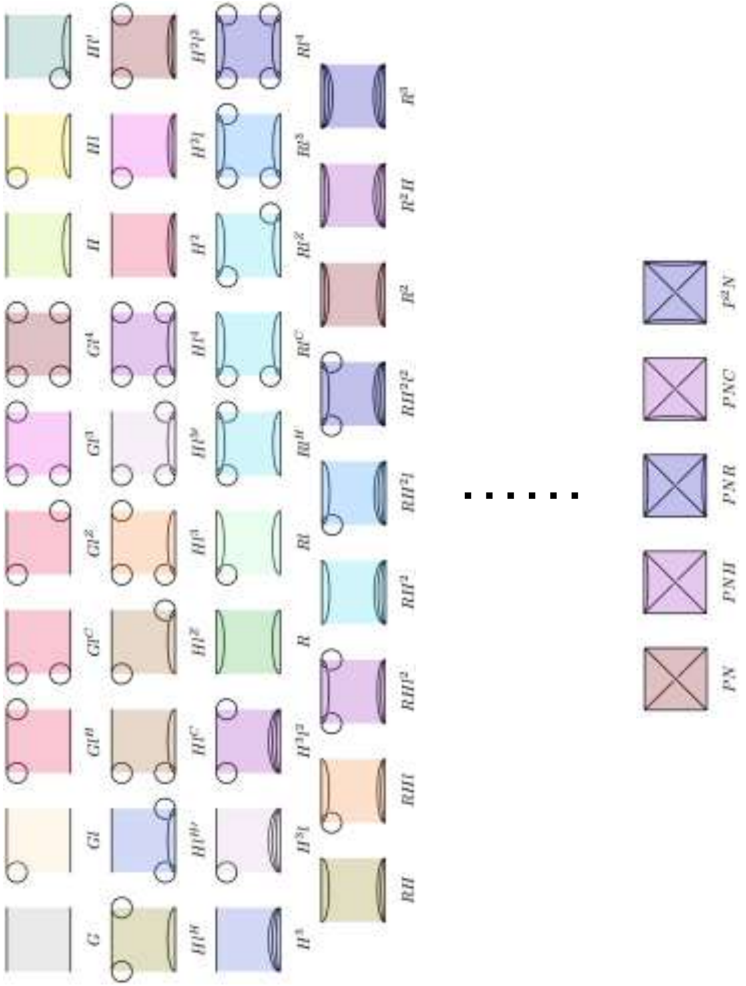
$$V_0^{[4]} = V_1^{[4]} = V_2^{[4]} = V_3^{[4]} = H_{\mathrm{NL}}$$



Anisotropic Background: Two-point Angular Correlator

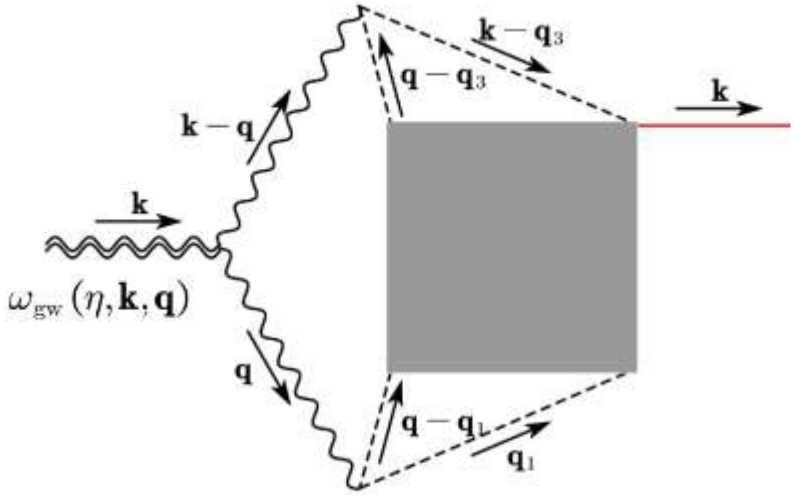
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023) 056
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 06 (2024) 039

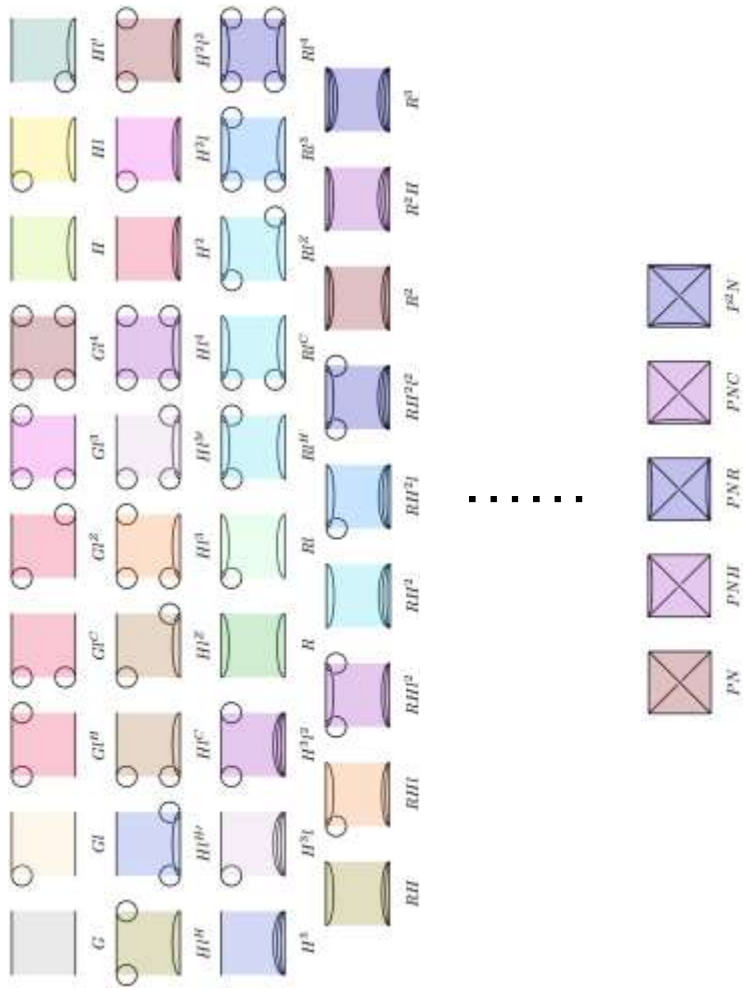




Anisotropic Background: Two-point Angular Correlator

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023) 056
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 06 (2024) 039

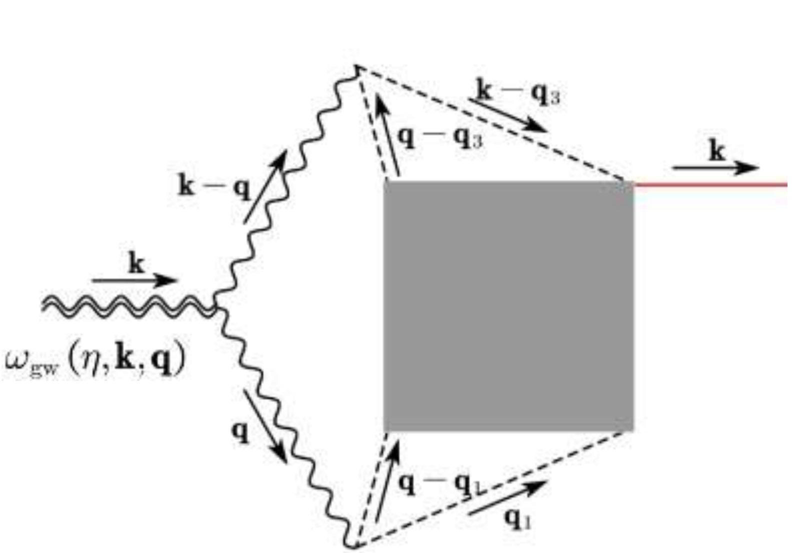


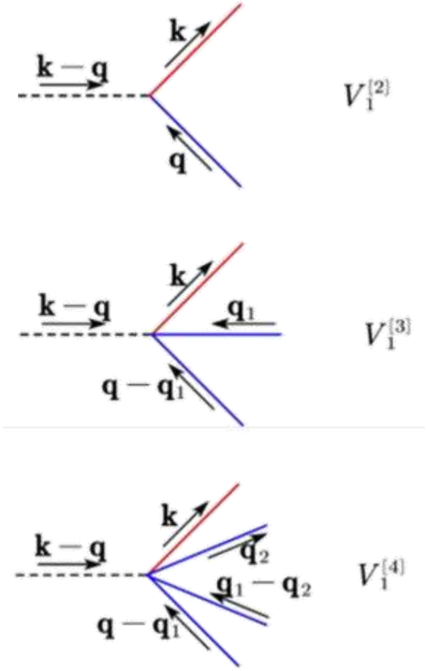


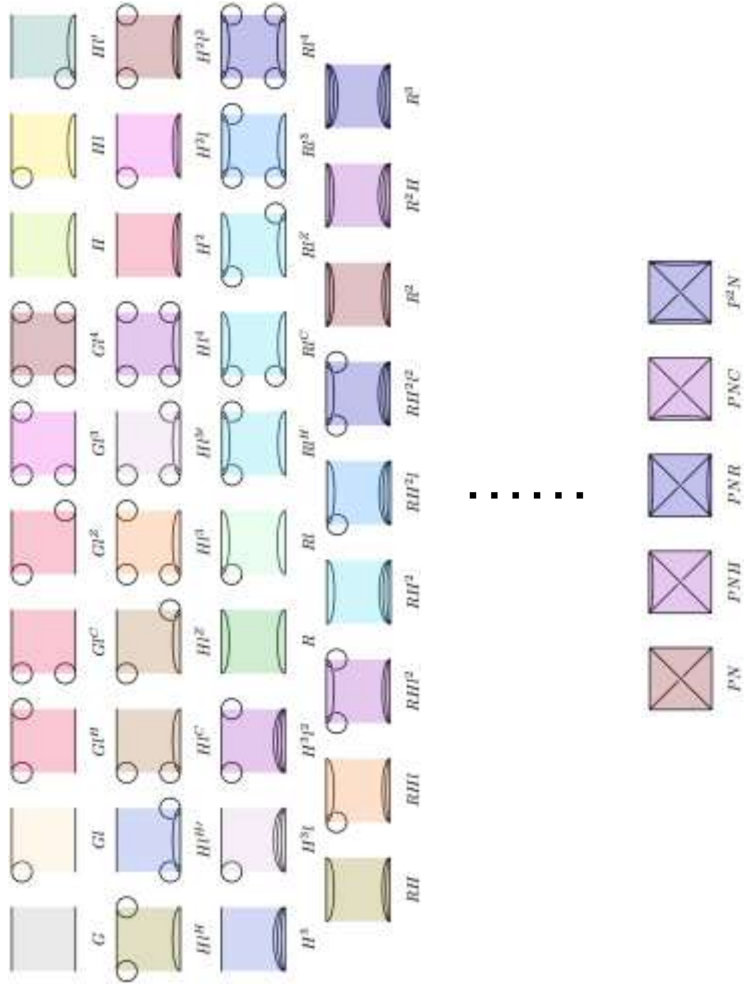
$$\omega^{(1)}_{\rm gw,in}(\boldsymbol{q}) = \omega^{(1)}_{\rm ng,in}(q) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \zeta_{\rm gL}(\boldsymbol{k})$$

Anisotropic Background: Two-point Angular Correlator

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023) 056
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 06 (2024) 039







$$\omega^{(1)}_{\rm gw,in}(\boldsymbol{q}) = \omega^{(1)}_{\rm ng,in}(q) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \zeta_{\rm gL}(\boldsymbol{k})$$

$$\omega^{(1)}_{\rm ng,in}(q) = \sum_{c=0}^{4} \sum_{b=0}^{\lfloor 4-c \rfloor} \sum_{c=0}^{\lfloor 4-b-c \rfloor} \bar{\omega}^{(a,b,c)}_{\rm gw,in}(q) \sum_{i=1}^{o-1=3} \left(\frac{(i+1)!}{i!} \frac{V_1^{[i+1]}}{V_0^{[i]}} \right)^1 \binom{N_i}{1}$$

$$\bar{\omega}^{(a,b,c)}_{\rm gw,in} \propto F^a_{\rm NL} G^b_{\rm NL} H^c_{\rm NL} A^{(a+2b+3c+4)/2}$$

Anisotropic Background: Reduced Angular Power Spectrum

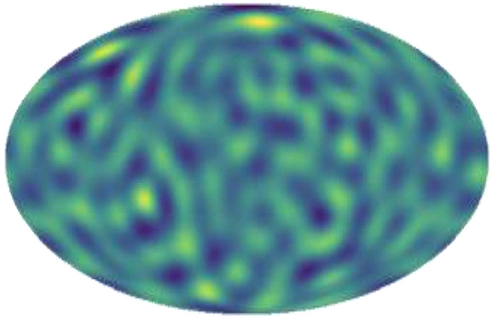
PNG-induced Inhomogeneity   SW effect & NL-effect

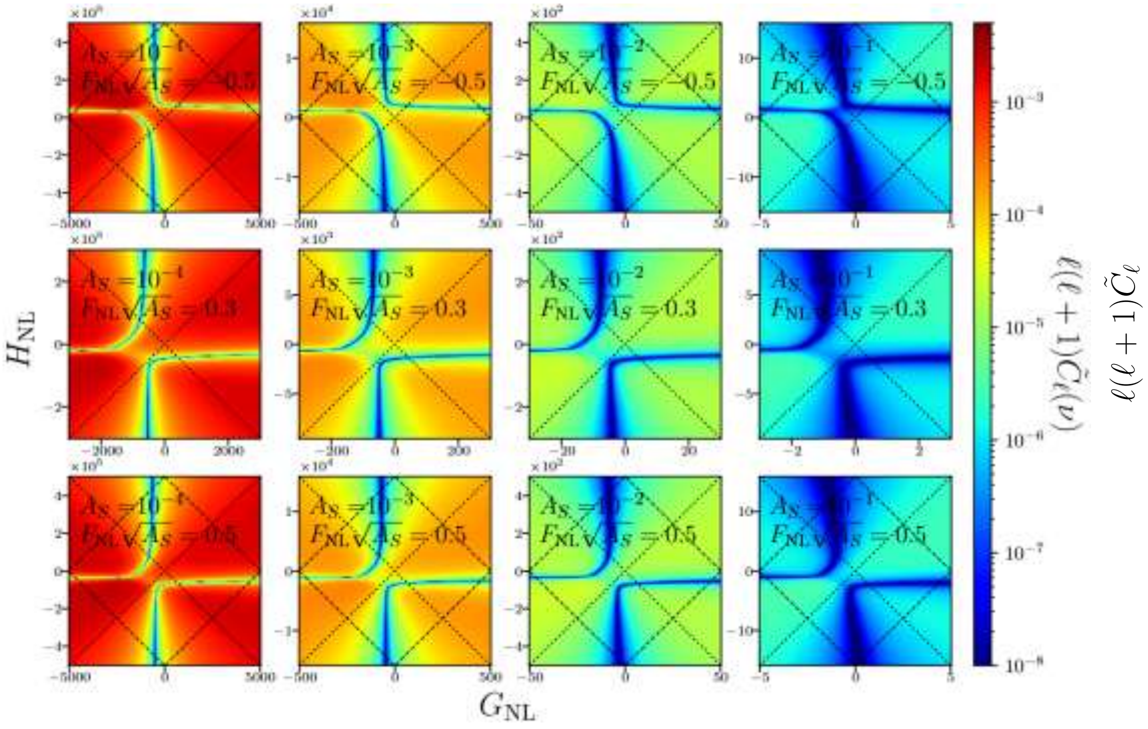
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 06 (2024) 039

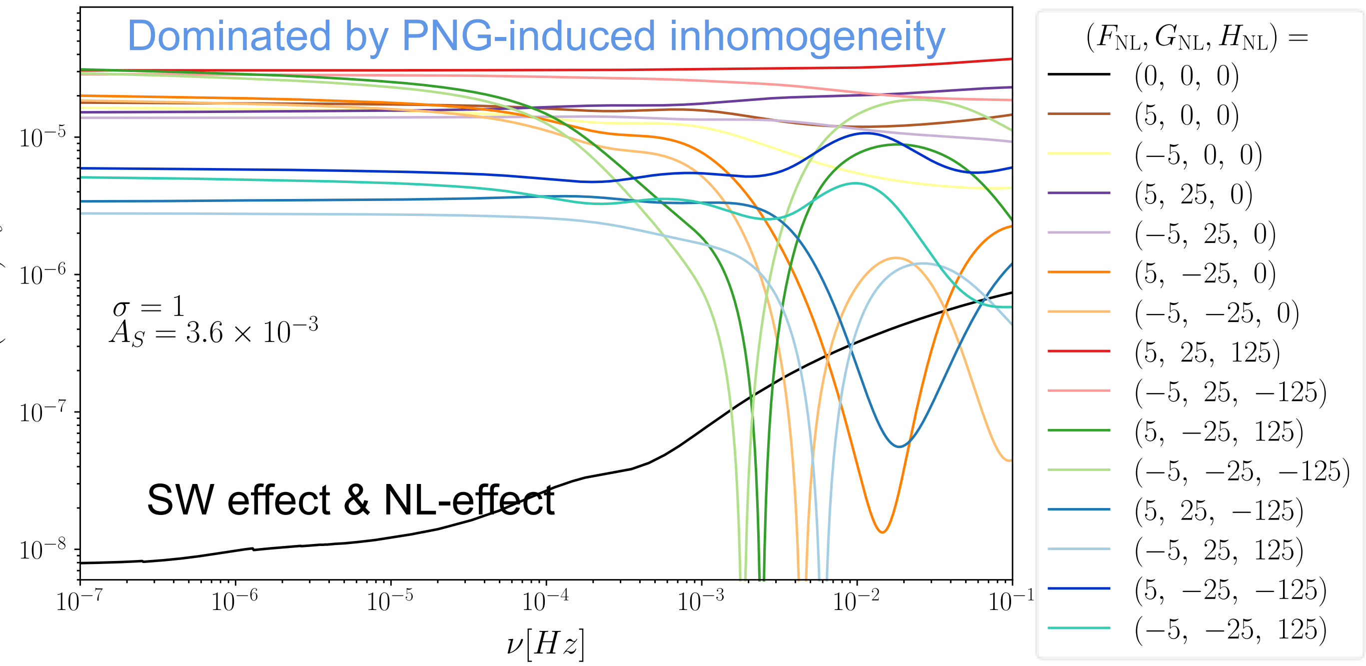
$$\delta^{(1)}_{\mathrm{gw},0}(\boldsymbol{q}) = \left[\frac{\omega^{(1)}_{\mathrm{ng,in}}(q)}{\bar{\omega}_{\mathrm{gw,in}}(q)} + \frac{3}{5}\left(2 - n_{\mathrm{gw}}(\nu)\right)\right]\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\zeta_{\mathrm{gL}}(\boldsymbol{k})$$

$$\left\langle \prod_{i=1}^{2} \delta_{\mathrm{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \delta_{\ell_1\ell_2}\delta_{m_1m_2}(-1)^{m_1}\tilde{C}_{\ell_1}(\nu)$$

$$\boxed{\tilde{C}_\ell(\nu) = \frac{2\pi A_{\mathrm{L}}}{\ell(\ell+1)}\left[\frac{\omega^{(1)}_{\mathrm{ng,in}}(2\pi\nu)}{\bar{\omega}_{\mathrm{gw,in}}(2\pi\nu)} + \frac{3}{5}\left(2 - n_{\mathrm{gw}}(\nu)\right)\right]^2}$$



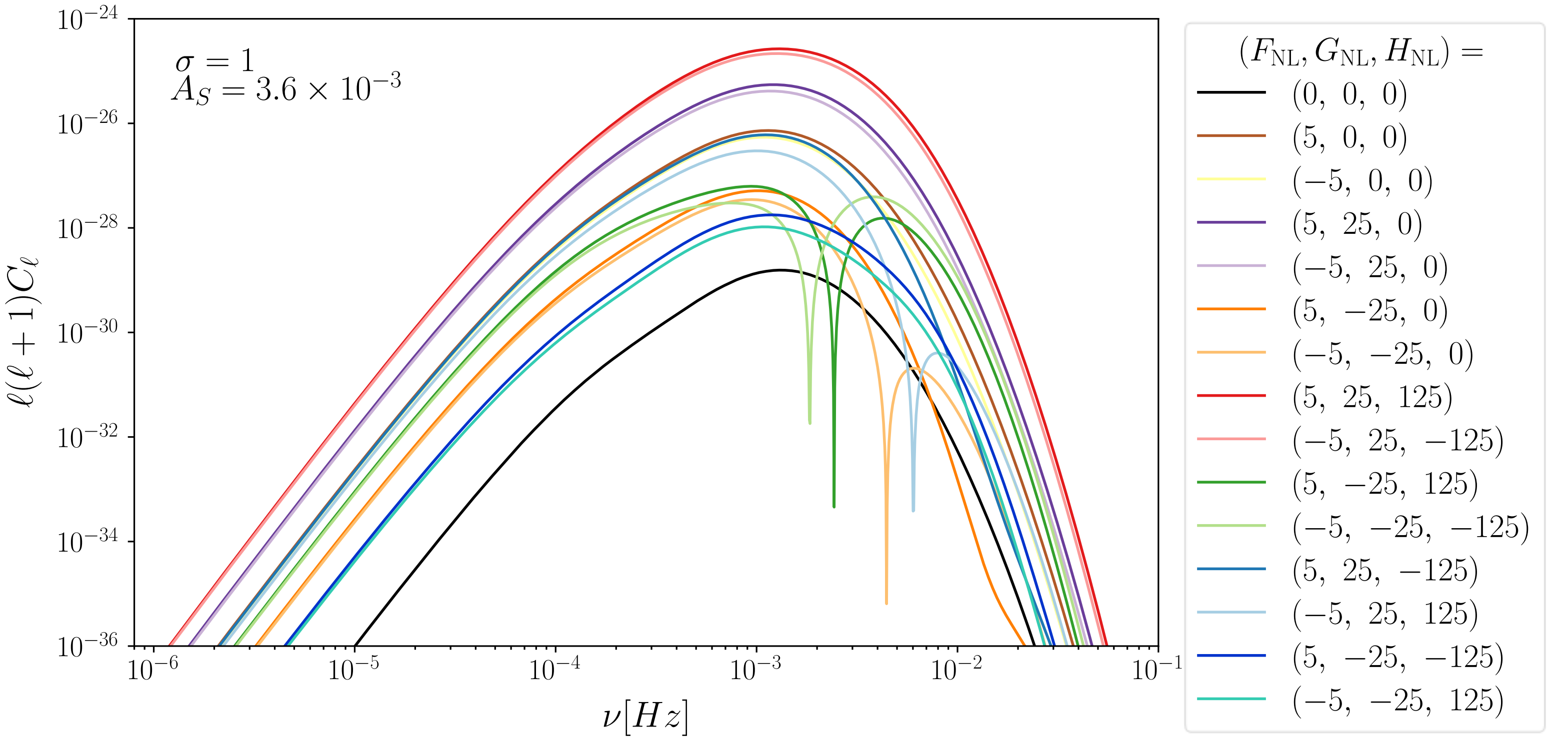




Anisotropic Background: Angular Power Spectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, arXiv:2309.07792

$$C_\ell(\nu) = \left[\frac{\bar{\Omega}_{\mathrm{gw},0}(\nu)}{4\pi}\right]^2 \tilde{C}_\ell(\nu)$$



SIGWs: $C_\ell \propto [\ell(\ell+1)]^{-1}$

Astrophysical GWs: $C_\ell \propto (\ell+1/2)^{-1}$

Anisotropic Background: Angular Power Spectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, arXiv:2309.07792

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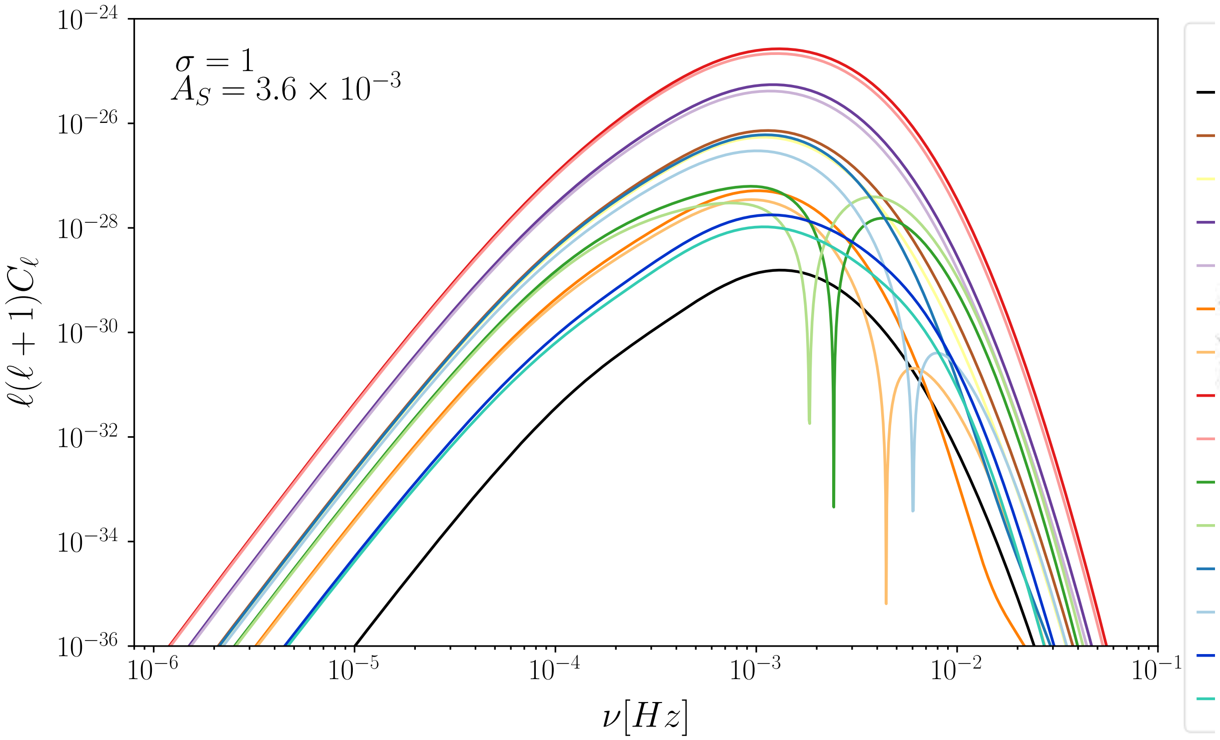

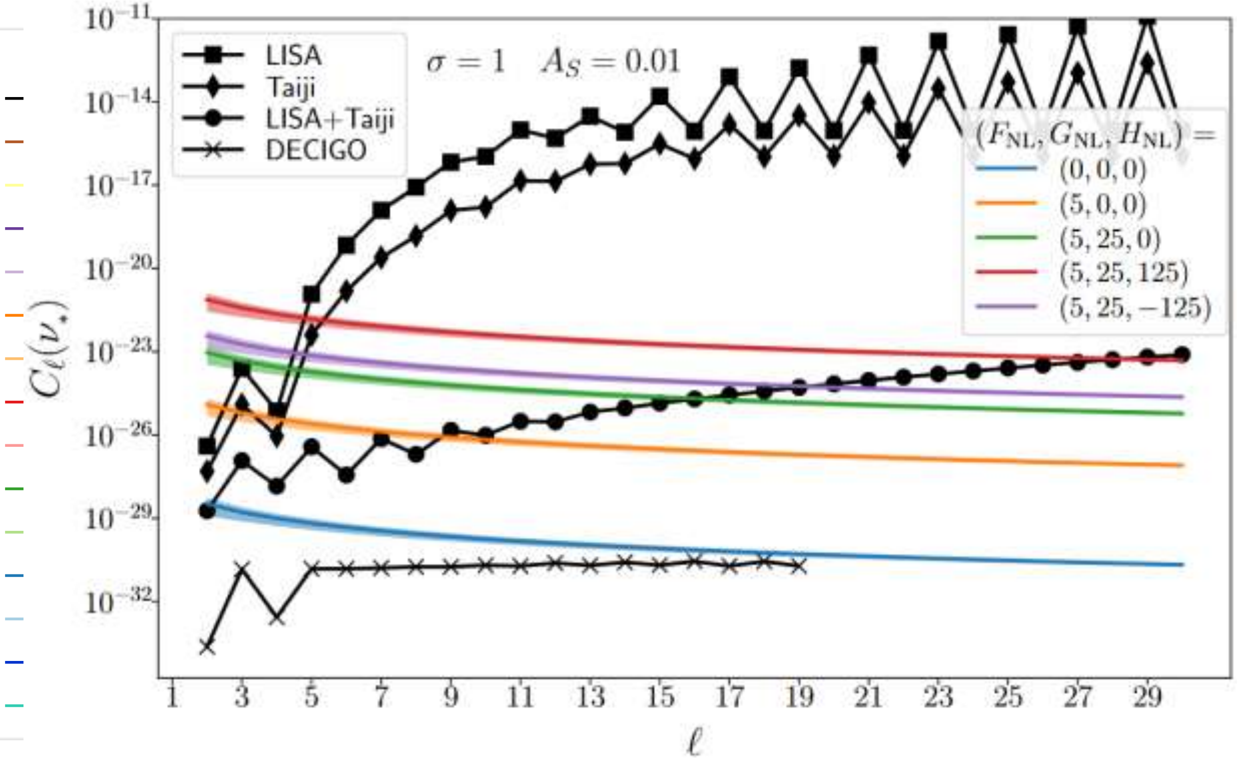

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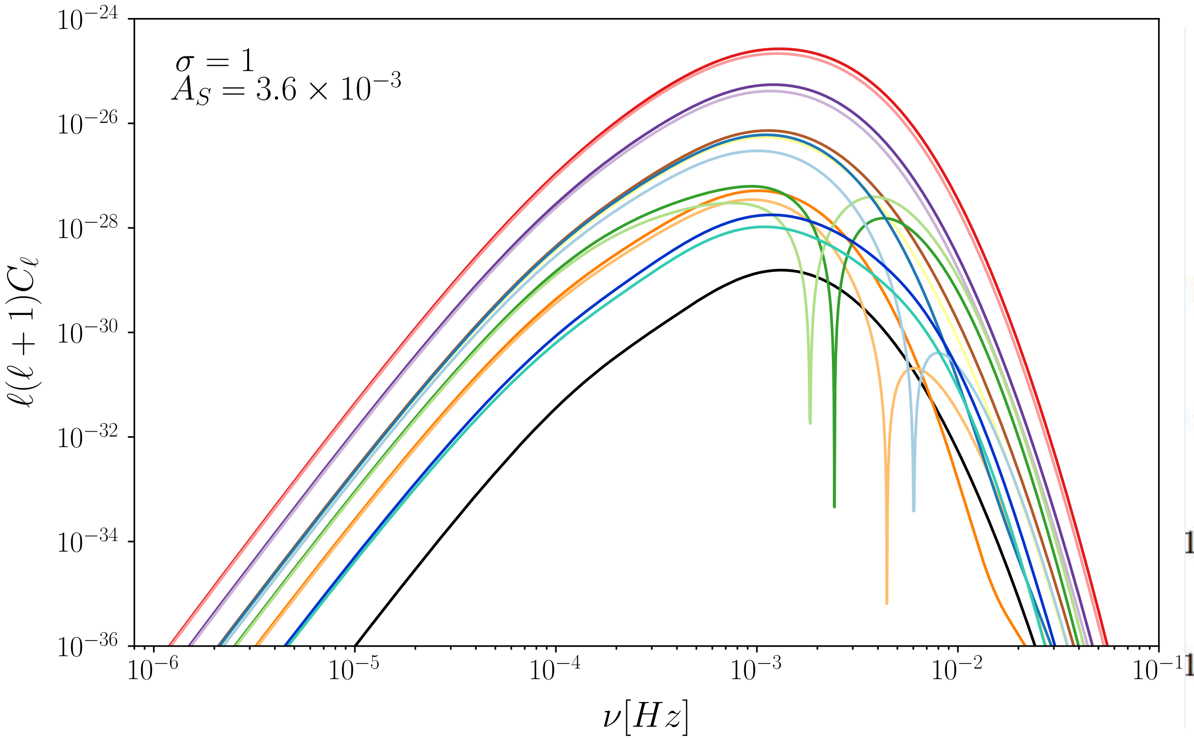
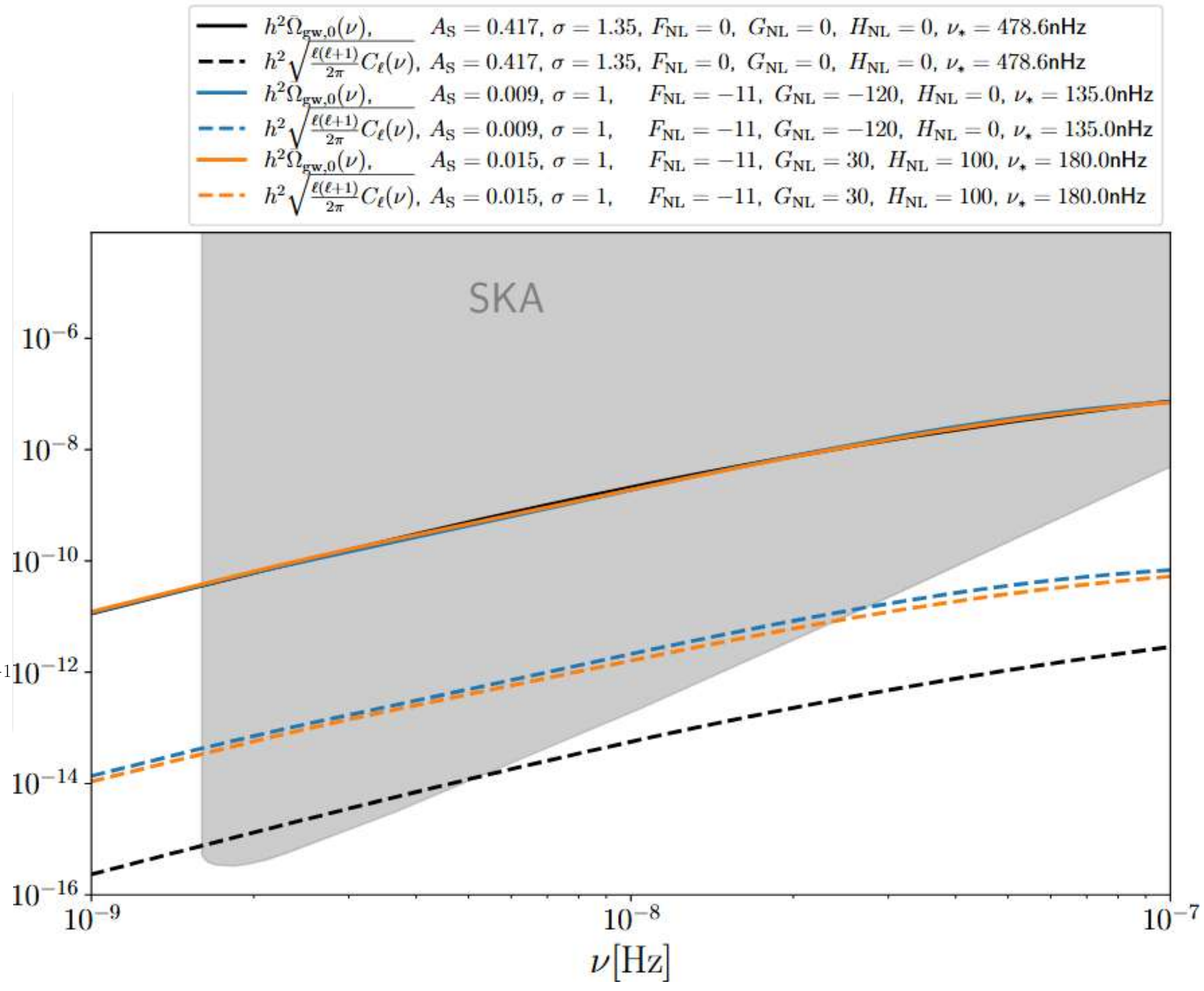




Parameter degeneracy breaking

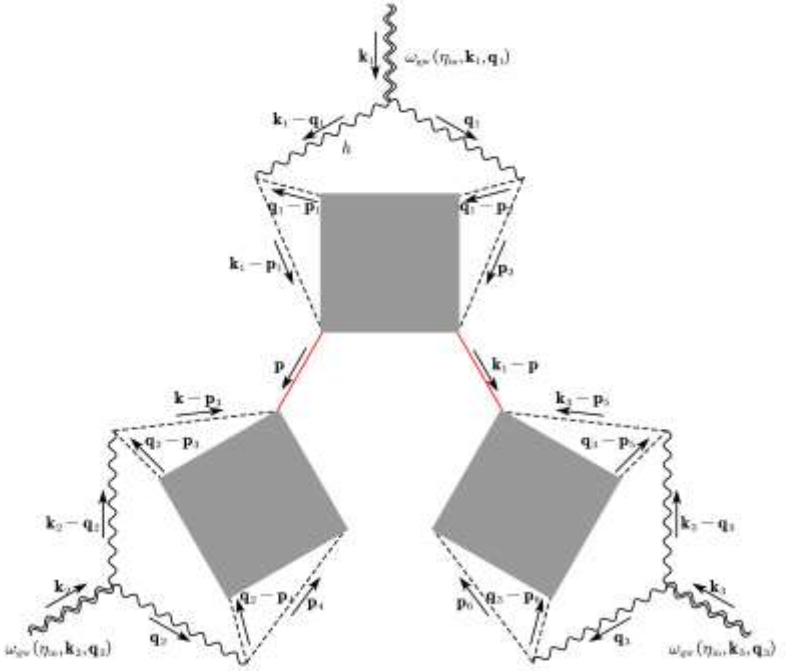
SIGWs: $C_\ell \propto [\ell(\ell+1)]^{-1}$

Astrophysical GWs: $C_\ell \propto (\ell+1/2)^{-1}$

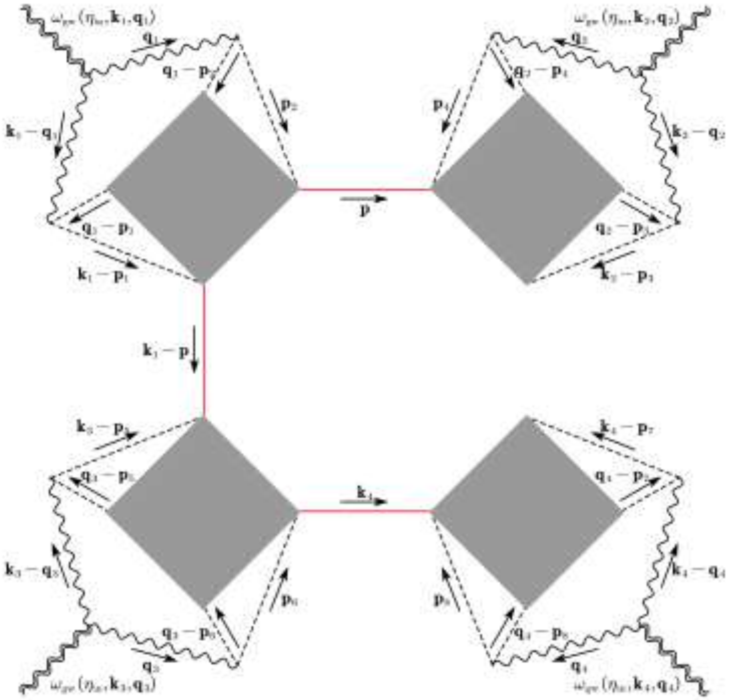
Non-Gaussian Background: Angular Bispectrum and Trispectrum

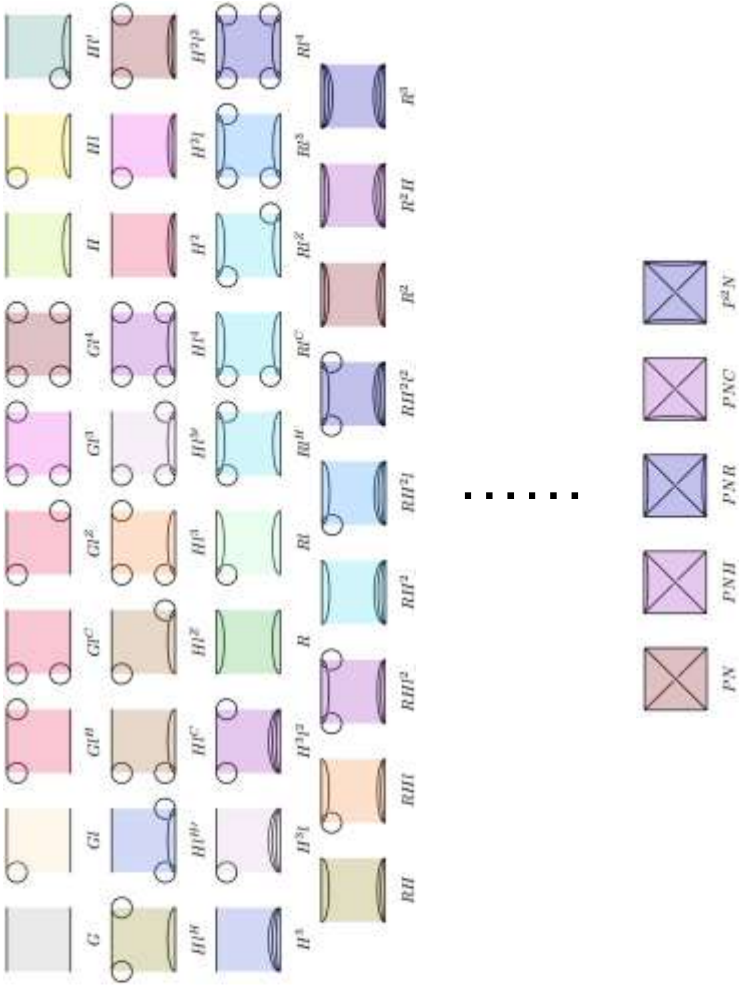
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

$$\langle \prod_{i=1}^{3} \omega_{\mathrm{gw}}(\eta_{\mathrm{in}}, \mathbf{k}_i, \mathbf{q}_i) \rangle$$



$$\langle \prod_{i=1}^{4} \omega_{\mathrm{gw}}(\eta_{\mathrm{in}}, \mathbf{k}_i, \mathbf{q}_i) \rangle$$





$$\omega_{\mathrm{gw,in}}(\boldsymbol{q}) = \bar{\omega}_{\mathrm{gw,in}}(q) + \omega^{(1)}_{\mathrm{gw,in}}(\boldsymbol{q}) + \omega^{(2)}_{\mathrm{gw,in}}(\boldsymbol{q})$$

$$\omega^{(2)}_{\mathrm{gw,in}}(\boldsymbol{q}) = \omega^{(2)}_{\mathrm{ng,in}}(q) \int \frac{\mathrm{d}^3\boldsymbol{k}\, \mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \zeta_{\mathrm{gL}}(\boldsymbol{p}) \zeta_{\mathrm{gL}}(\boldsymbol{k}-\boldsymbol{p})$$

Non-Gaussian Background: Angular Bispectrum and Trispectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

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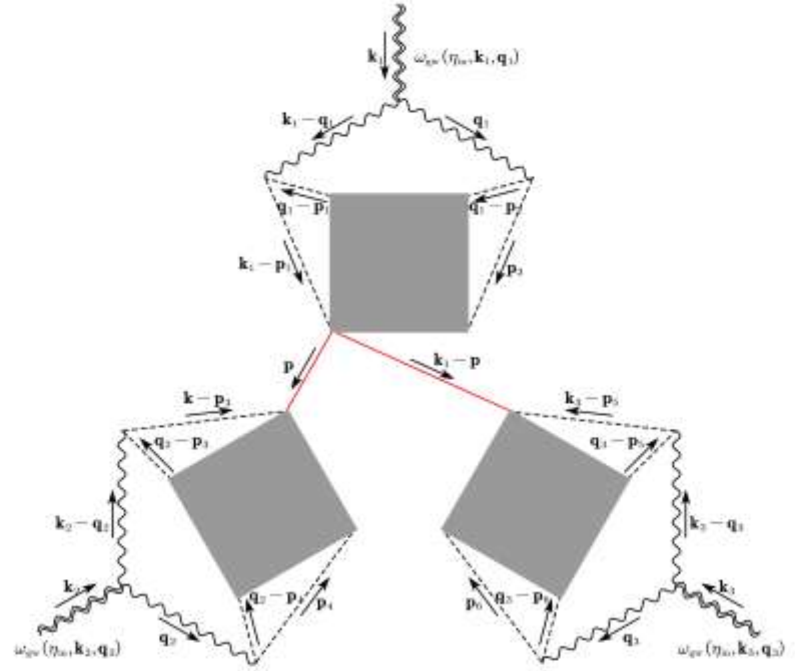

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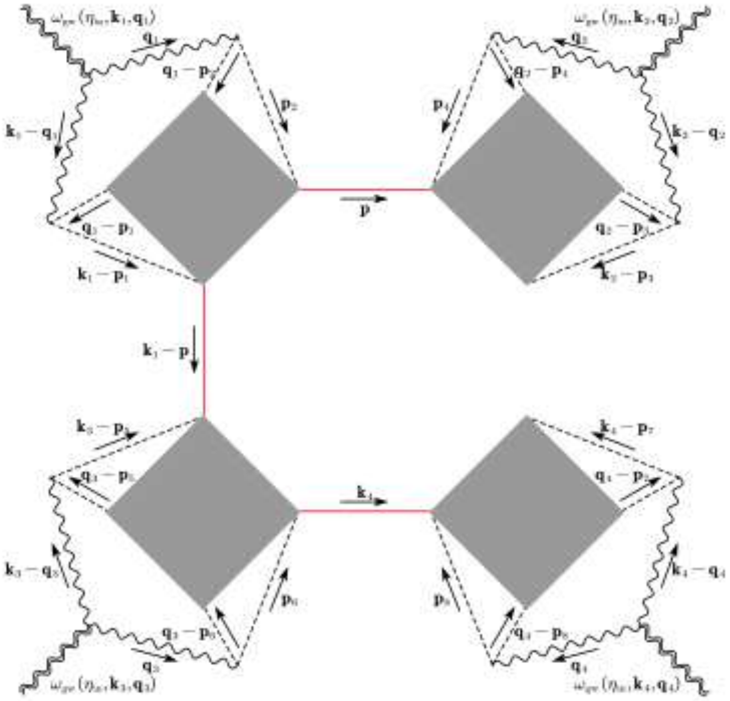

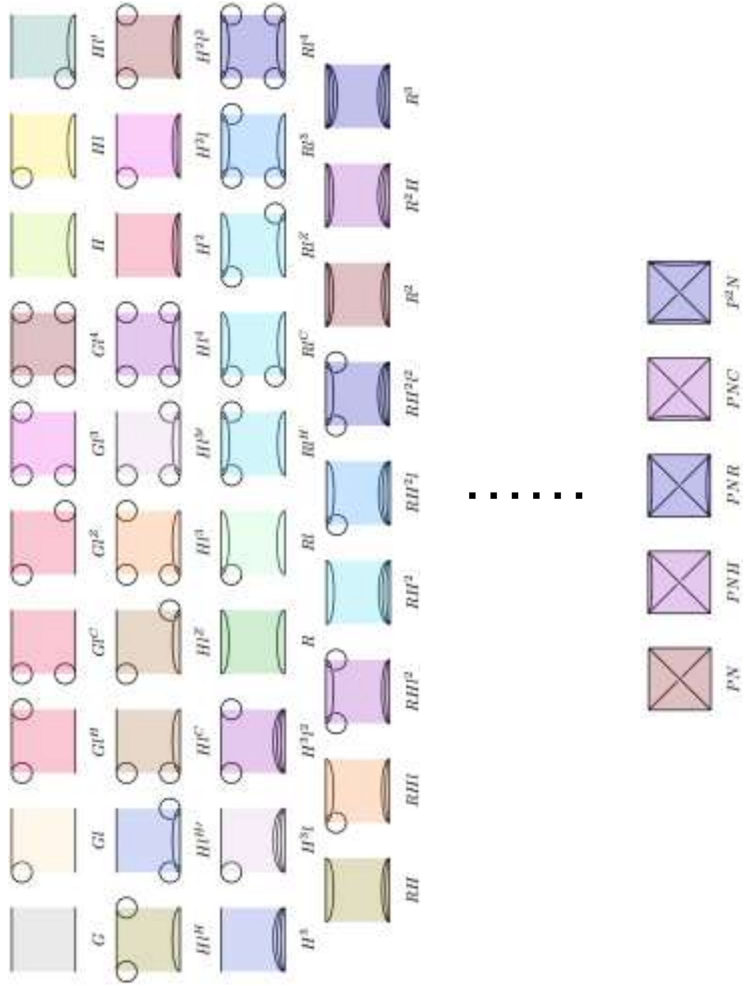

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Non-Gaussian Background: Angular Bispectrum and Trispectrum

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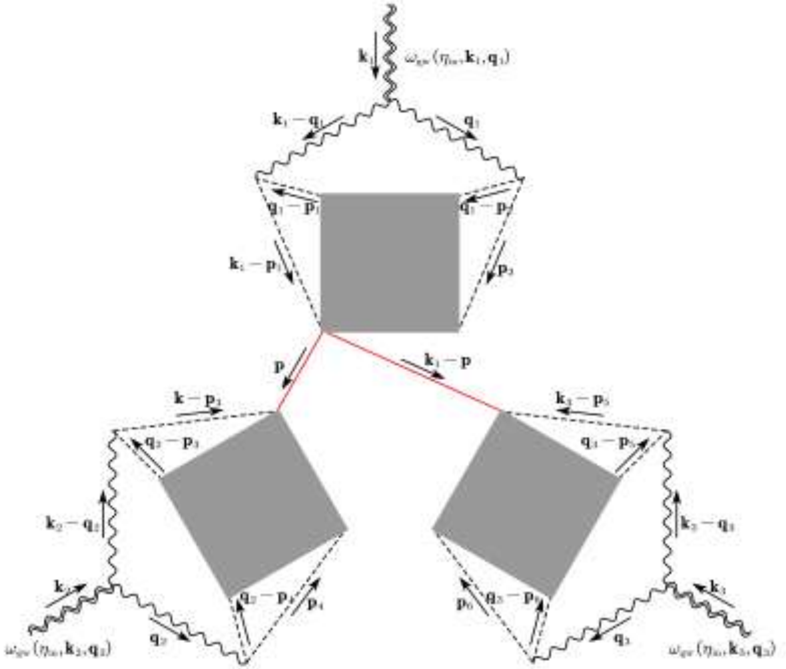

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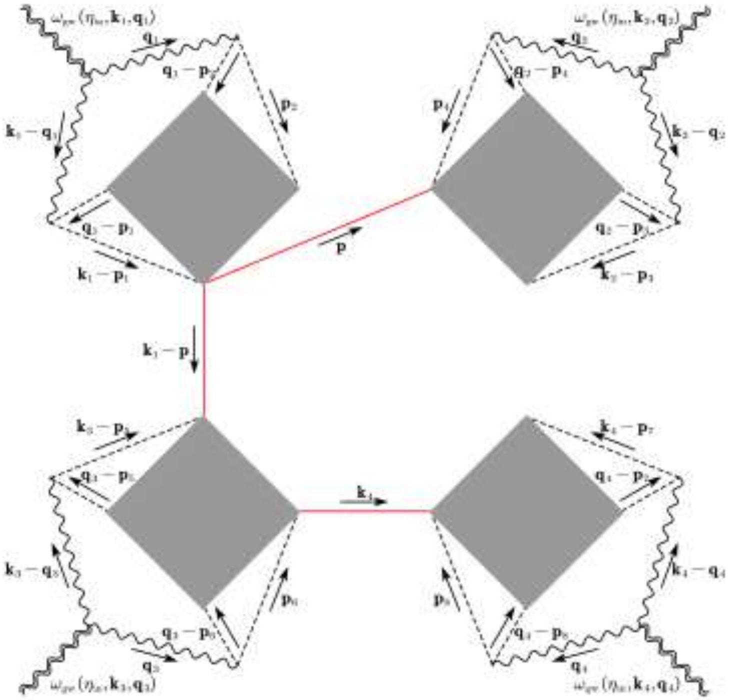

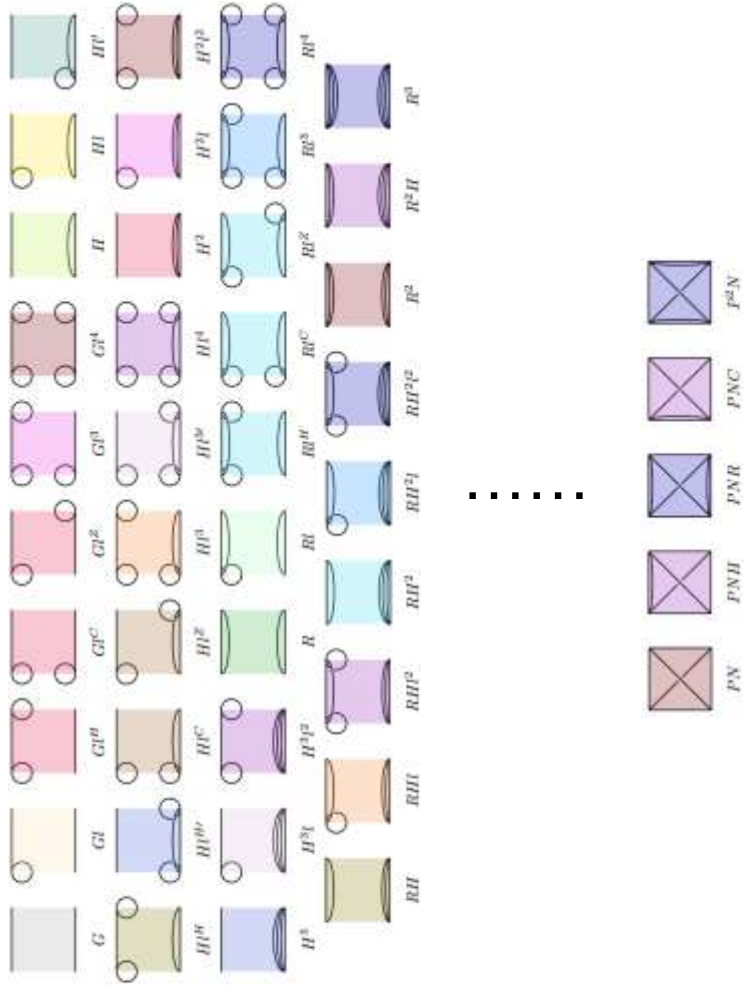

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JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

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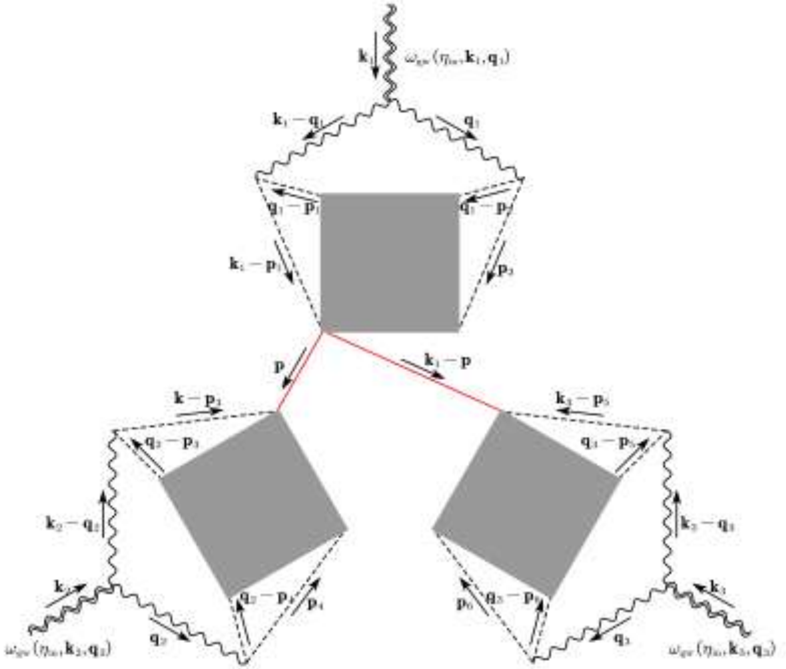

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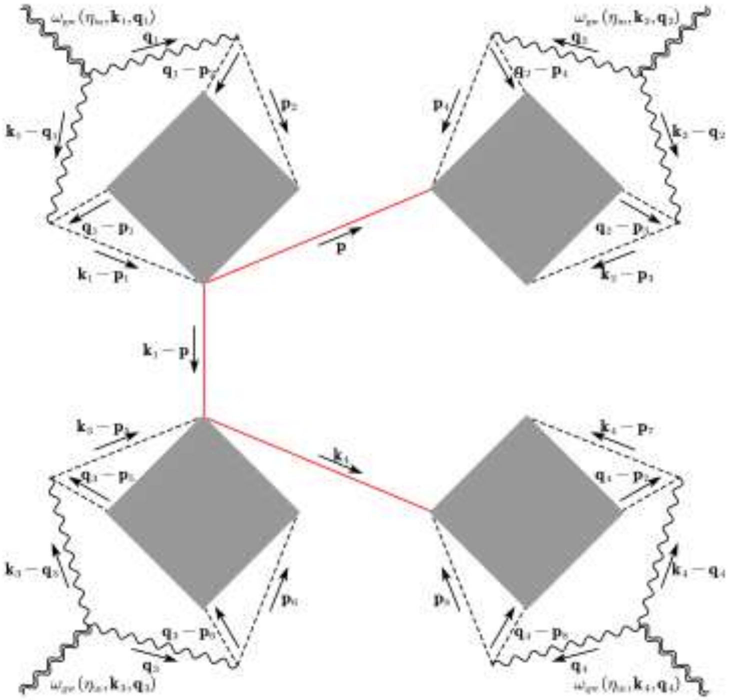

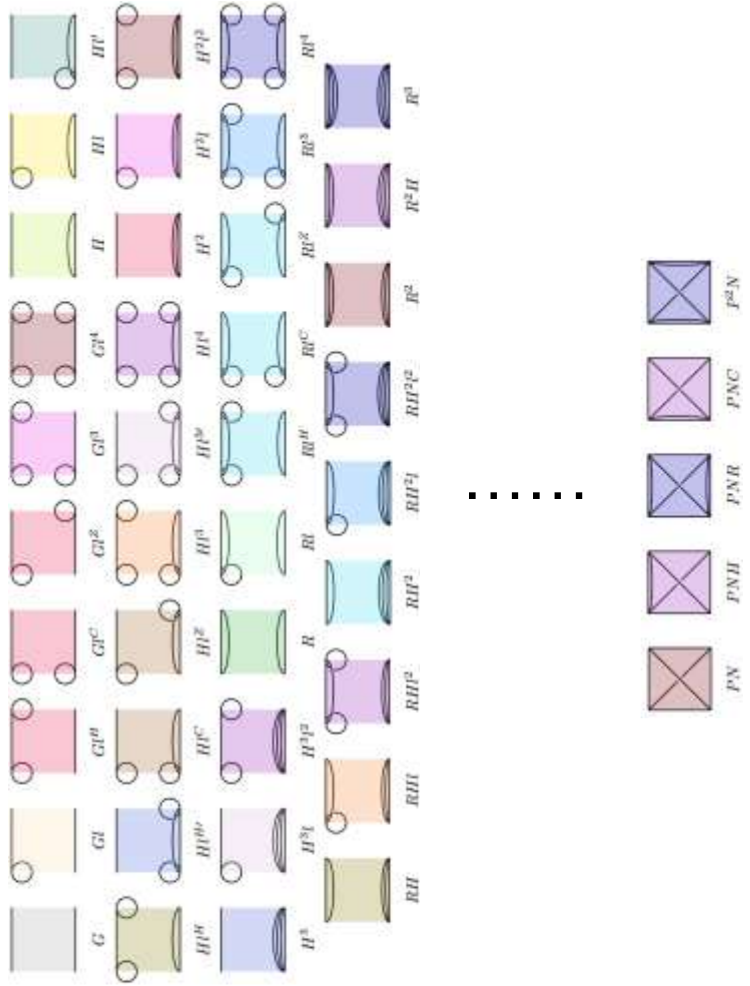

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Non-Gaussian Background: Angular Bispectrum and Trispectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

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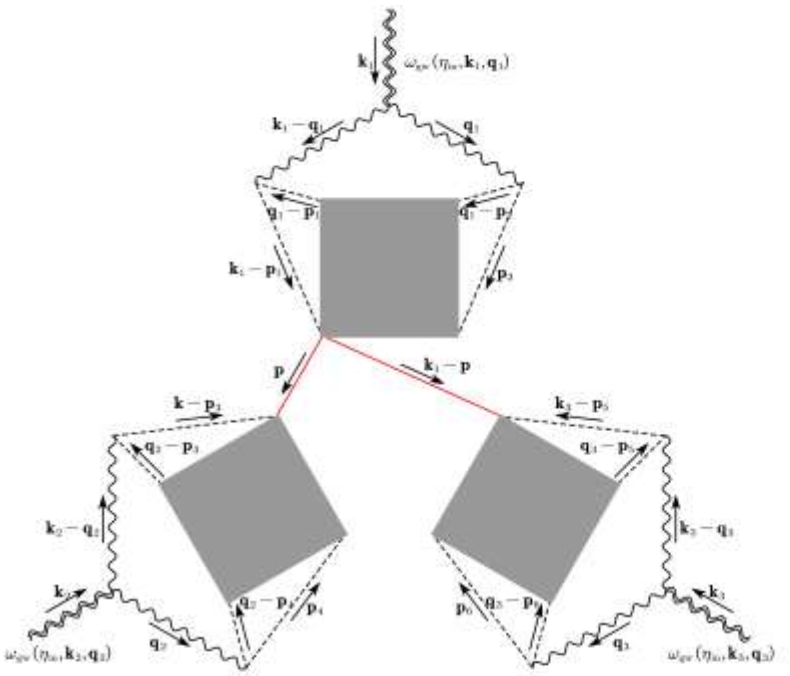

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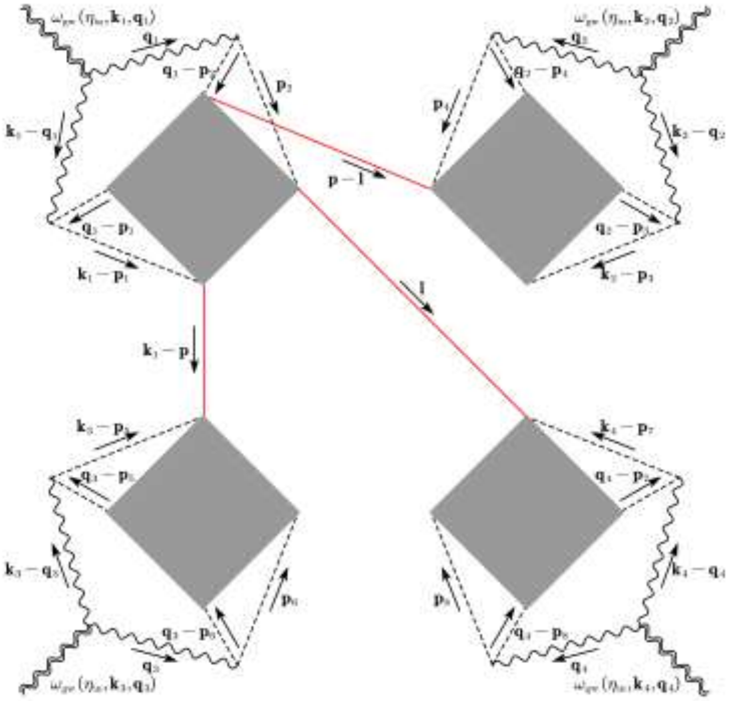

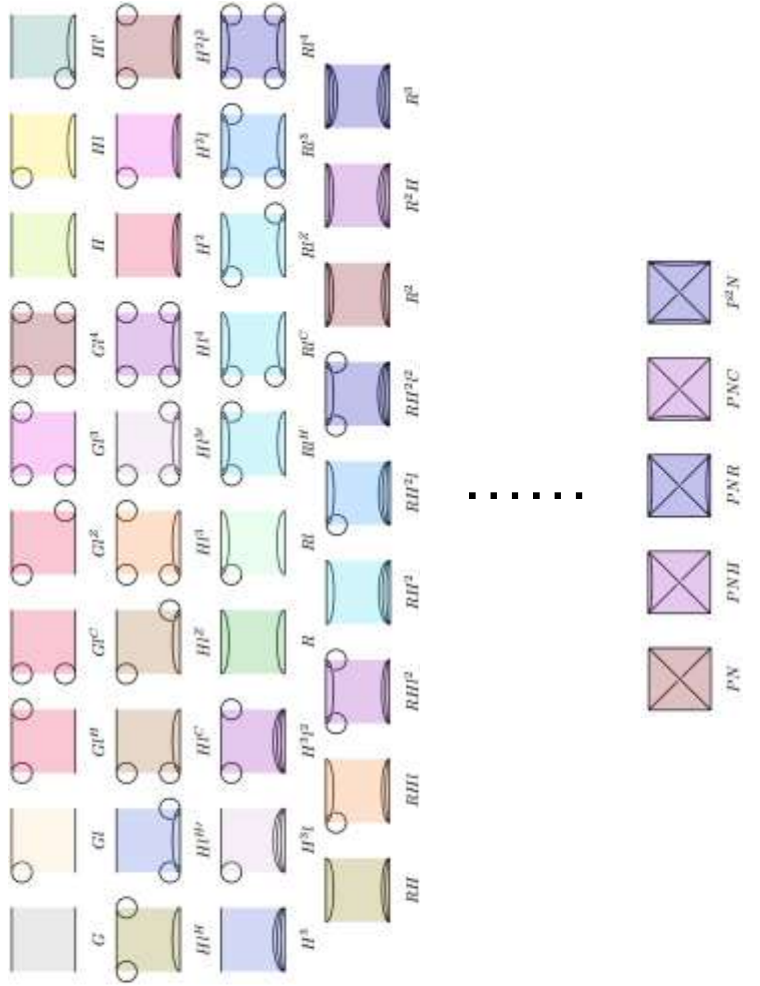

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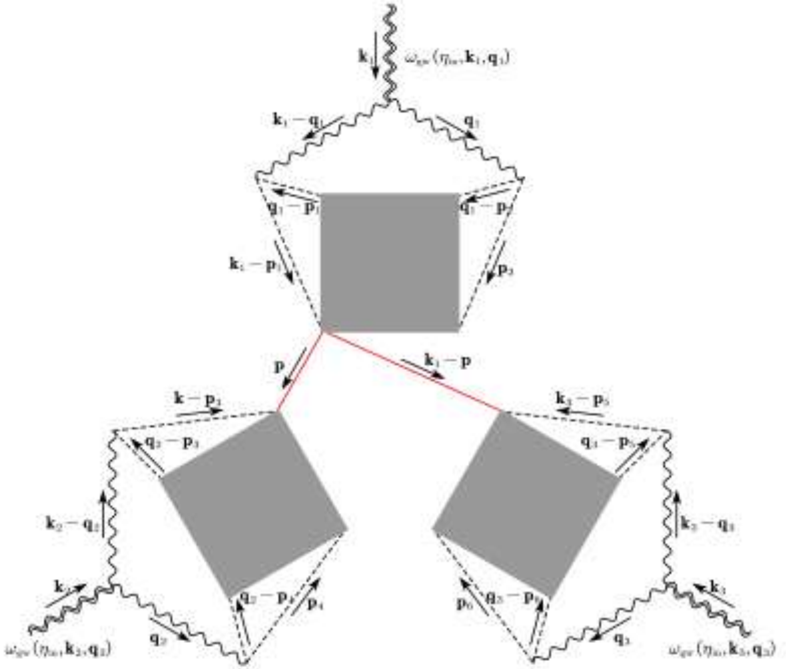
$$\omega^{(3)}_{\rm gw,in}(\boldsymbol{q}) = \omega^{(3)}_{\rm ng,in}(q) \int \frac{{\rm d}^3\boldsymbol{k}\,{\rm d}^3\boldsymbol{p}\,{\rm d}^3\boldsymbol{l}}{(2\pi)^{9/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \zeta_{\rm gL}(\boldsymbol{l})\zeta_{\rm gL}(\boldsymbol{p}-\boldsymbol{l})\zeta_{\rm gL}(\boldsymbol{k}-\boldsymbol{p})$$

Non-Gaussian Background: Angular Bispectrum and Trispectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

$$\langle \prod_{i=1}^{3} \omega_{\mathrm{gw}}(\eta_{\mathrm{in}}, \mathbf{k}_i, \mathbf{q}_i) \rangle$$



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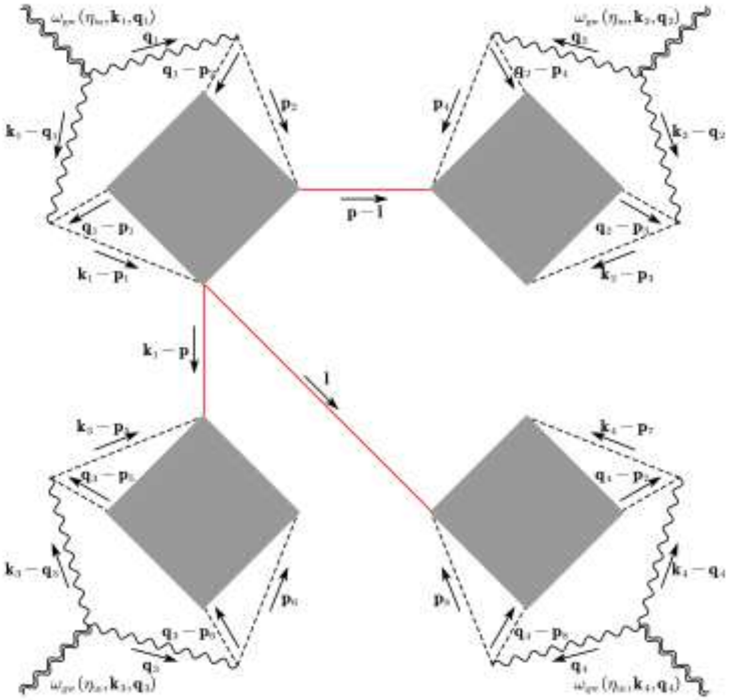

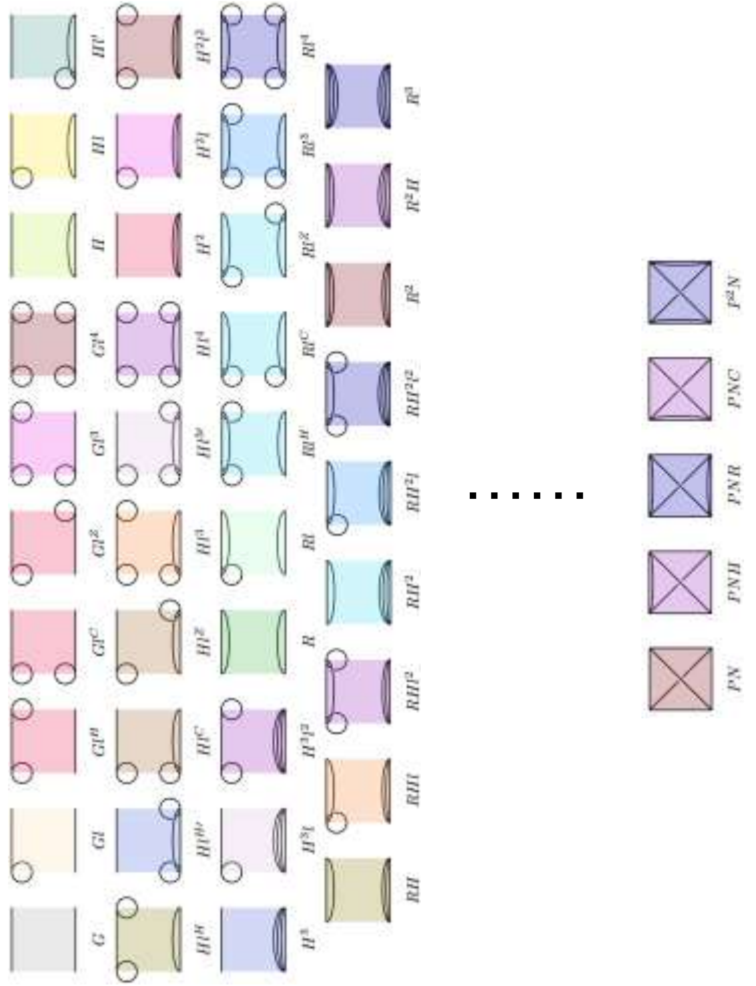

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Non-Gaussian Background: Angular Bispectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

$$B_{\ell_1\ell_2\ell_3}(\nu) = b(\nu) h_{\ell_1\ell_2\ell_3}\left[\frac{1}{\ell_1\ell_2(\ell_1+1)(\ell_2+1)} + (\ell_1 \leftrightarrow \ell_3) + (\ell_2 \leftrightarrow \ell_3)\right]$$

$$b(\nu) = 2(2\pi A_{\rm L})^2\left[\frac{\omega^{(1)}_{\rm ng,in}(2\pi\nu)}{\bar{\omega}_{\rm gw,in}(2\pi\nu)} + \frac{3}{5}\left(2-n_{\rm gw}(\nu)\right)\right]^2\left[\frac{\omega^{(2)}_{\rm ng,in}(2\pi\nu)}{\bar{\omega}_{\rm gw,in}(2\pi\nu)} + \frac{3}{5}F_{\rm NL}\left(2-n_{\rm gw}(\nu)\right)\right]$$

PNG-induced Inhomogeneity SW effect & NL effect

Gaussian ζ ➡ Gaussian SIGW

$$\omega^{(2)}_{\rm ng,in}(q) = \sum_{c=0}^{4}\sum_{b=0}^{\lfloor 4-c\rfloor}\sum_{a=0}^{\lfloor 4-b-c\rfloor} \bar{\omega}^{(a,b,c)}_{\rm gw,in}(q)\left(\mathcal{T}^{(2)}_1 + \mathcal{T}^{(2)}_2\right),$$

$$\mathcal{T}^{(2)}_1 = \sum_{l=1}^{2}\sum_{n=1}^{2}\sum_{i=1}^{o-l}\left(\frac{(i+l)!}{i!}\frac{V_l^{[i+l]}}{V_0^{[i]}}\right)^n \binom{N_i}{n}\delta_{ln,2},$$

$$\mathcal{T}^{(2)}_2 = \sum_{i=1}^{4-1}\sum_{j=1}^{\min(i-1,o-1)}\left(\frac{(i+1)!}{i!}\frac{V_1^{[i+1]}}{V_0^{[i]}}\right)^1\binom{N_i}{1}\left(\frac{(j+1)!}{j!}\frac{V_1^{[j+1]}}{V_0^{[j]}}\right)^1\binom{N_j}{1}$$

Non-Gaussian Background: Angular Bispectrum

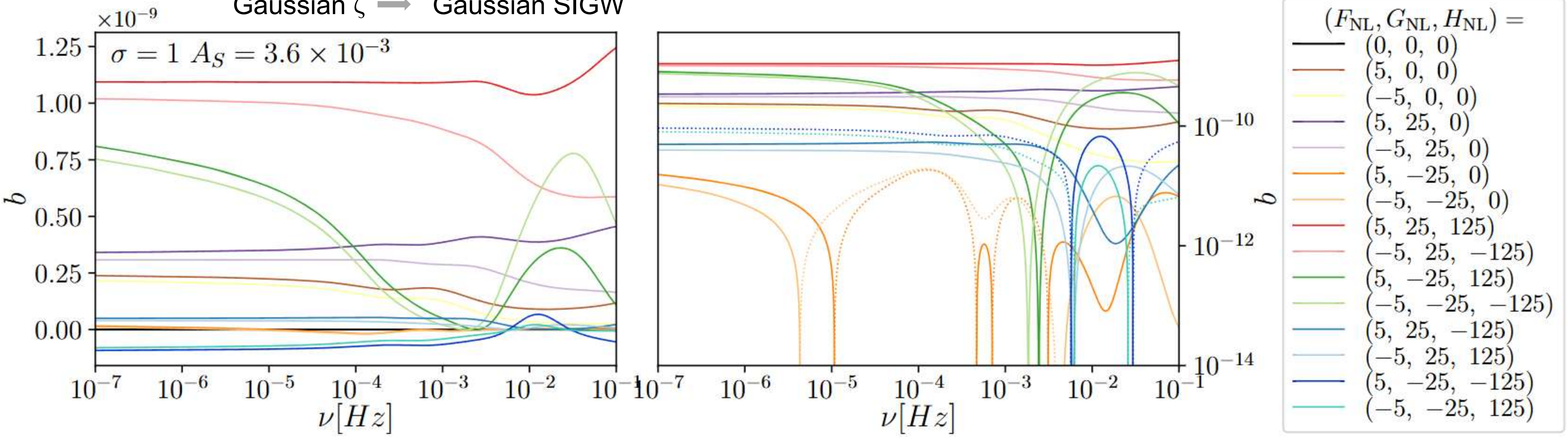
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

$$B_{\ell_1\ell_2\ell_3}(\nu) = b(\nu) h_{\ell_1\ell_2\ell_3}\left[\frac{1}{\ell_1\ell_2(\ell_1+1)(\ell_2+1)} + (\ell_1 \leftrightarrow \ell_3) + (\ell_2 \leftrightarrow \ell_3)\right]$$

$$b(\nu) = 2(2\pi A_L)^2\left[\frac{\omega^{(1)}_{\rm ng,in}(2\pi\nu)}{\bar{\omega}_{\rm gw,in}(2\pi\nu)} + \frac{3}{5}(2-n_{\rm gw}(\nu))\right]^2\left[\frac{\omega^{(2)}_{\rm ng,in}(2\pi\nu)}{\bar{\omega}_{\rm gw,in}(2\pi\nu)} + \frac{3}{5}F_{\rm NL}(2-n_{\rm gw}(\nu))\right]$$

PNG-induced Inhomogeneity SW effect & NL effect

Gaussian ζ ➡ Gaussian SIGW



Non-Gaussian Background: Angular Bispectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

$$B_{\ell_1\ell_2\ell_3}(\nu) = b(\nu) h_{\ell_1\ell_2\ell_3}\left[\frac{1}{\ell_1\ell_2(\ell_1+1)(\ell_2+1)} + (\ell_1 \leftrightarrow \ell_3) + (\ell_2 \leftrightarrow \ell_3)\right]$$

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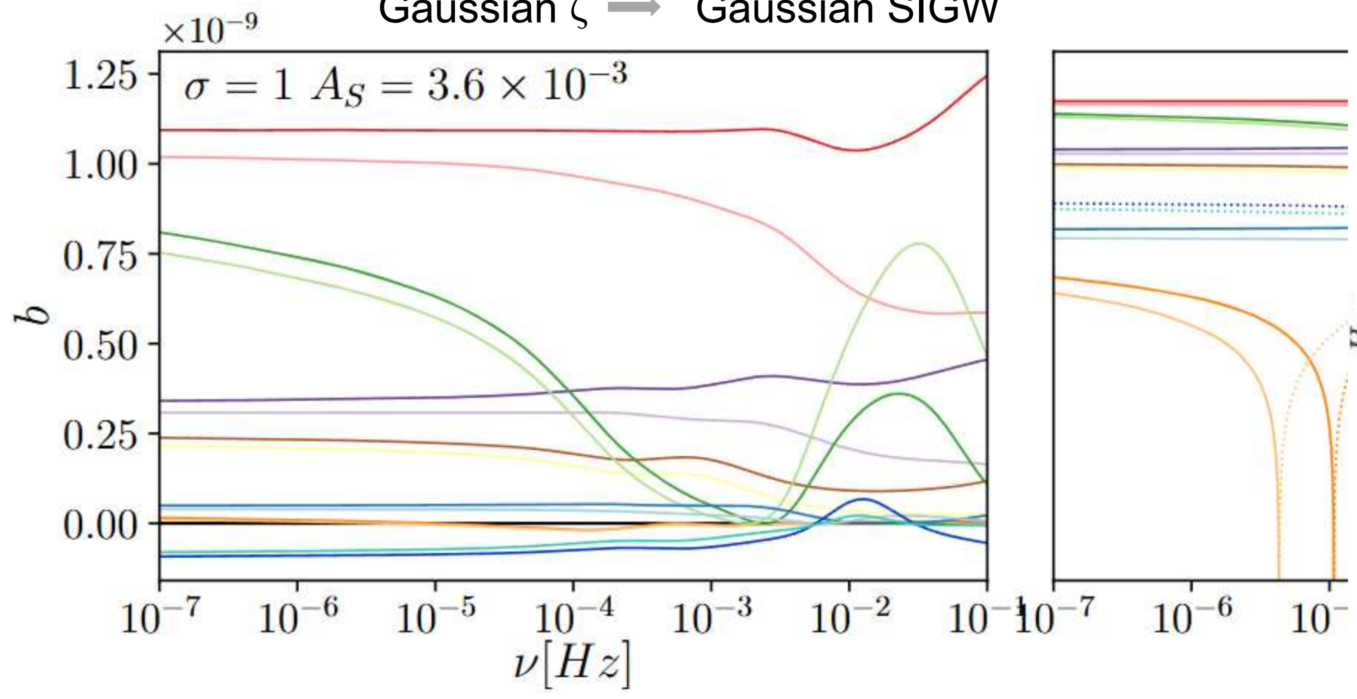
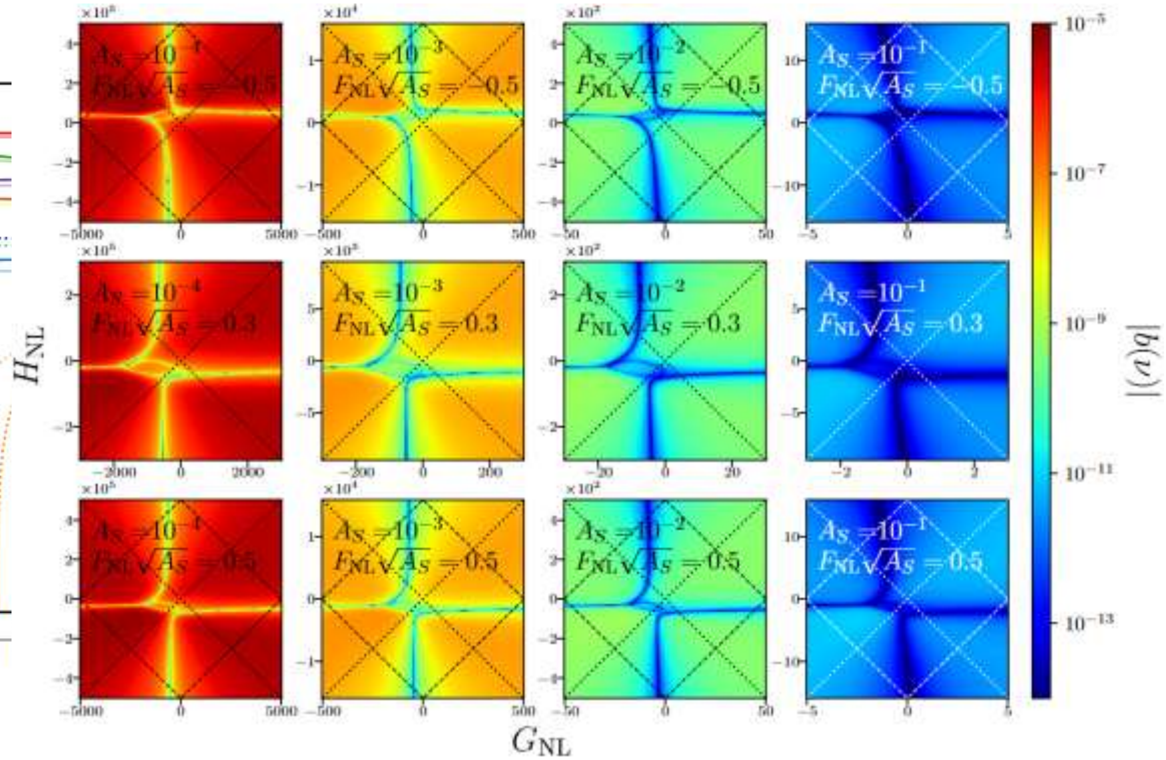
PNG-induced Inhomogeneity   SW effect & NL effect

Gaussian ζ ⟶ Gaussian SIGW





Non-Gaussian Background: Angular Trispectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

$$t^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu) \simeq \left[\frac{t_1(\nu)}{L(L+1)} + \frac{t_2(\nu)}{\ell_1(\ell_1+1)} + \frac{t_2(\nu)}{\ell_3(\ell_3+1)}\right] \frac{h_{\ell_1\ell_2 L}}{\ell_2(\ell_2+1)} \frac{h_{\ell_3\ell_4 L}}{\ell_4(\ell_4+1)}$$

$$t_1(\nu) = 4\cdot(2\pi A_{\rm L})^3 \left[\frac{\omega^{(1)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}(2-n_{\rm gw}(\nu))\right]^2 \left[\frac{\omega^{(2)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}F_{\rm NL}(2-n_{\rm gw}(\nu))\right]^2$$

$$t_2(\nu) = \frac{(2\pi A_{\rm L})^3}{2\pi} \left[\frac{\omega^{(1)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}(2-n_{\rm gw}(\nu))\right]^3 \left[\frac{\omega^{(3)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}G_{\rm NL}(2-n_{\rm gw}(\nu))\right]$$

PNG-induced Inhomogeneity

SW effect & NL effect

$$\omega^{(3)}_{\rm ng,in}(q) = \sum_{c=0}^{4}\sum_{b=0}^{\lfloor 4-c\rfloor}\sum_{c=0}^{\lfloor 4-b-c\rfloor} \bar\omega^{(a,b,c)}_{\rm gw,in}(q)\left(\mathscr{T}^{(3)}_1 + \mathscr{T}^{(3)}_2 + \mathscr{T}^{(3)}_3 + \mathscr{T}^{(3)}_4\right) ,$$

$$\mathscr{T}^{(3)}_1 = \sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{i=1}^{o-l}\left(\frac{(i+l)!}{i!}\frac{V_l^{[i+l]}}{V_0^{[i]}}\right)^n \binom{N_i}{n}\delta_{ln,3} ,$$

$$\mathscr{T}^{(3)}_2 = \sum_{i=1}^{o-2}\left(\frac{(i+2)!}{i!}\frac{V_2^{[i+2]}}{V_0^{[i]}}\right)^1\binom{N_i}{1}\left(\frac{(i+1)!}{i!}\frac{V_1^{[i+1]}}{V_0^{[i]}}\right)^1\binom{N_i-1}{1} ,$$

$$\mathscr{T}^{(3)}_3 = \sum_{l_1=1}^{2}\sum_{l_2=1}^{2}\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\sum_{i=2}^{o-l_1}\sum_{j=1}^{\min(i-1,o-l_2)}\left(\frac{(i+l_1)!}{i!}\frac{V_{l_1}^{[i+l_1]}}{V_0^{[i]}}\right)^{n_1}\binom{N_i}{n_1}\left(\frac{(j+l_2)!}{j!}\frac{V_{l_2}^{[j+l_2]}}{V_0^{[j]}}\right)^{n_2}\binom{N_j}{n_2}\delta_{l_1n_1+l_2n_2,3} ,$$

$$\mathscr{T}^{(3)}_4 = \sum_{i=3}^{o-1}\sum_{j=2}^{\min(i-1,o-1)}\sum_{k=1}^{\min(j-1,o-1)}\left(\frac{(i+1)!}{i!}\frac{V_1^{[i+1]}}{V_0^{[i]}}\right)^1\binom{N_i}{1}\left(\frac{(j+1)!}{j!}\frac{V_1^{[j+1]}}{V_0^{[j]}}\right)^1\binom{N_j}{1}\left(\frac{(k+1)!}{k!}\frac{V_1^{[k+1]}}{V_0^{[k]}}\right)^1\binom{N_k}{1}$$

Non-Gaussian Background: Angular Trispectrum

$$t^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu) \simeq \left[\frac{t_1(\nu)}{L(L+1)} + \frac{t_2(\nu)}{\ell_1(\ell_1+1)} + \frac{t_2(\nu)}{\ell_3(\ell_3+1)}\right]\frac{h_{\ell_1\ell_2L}}{\ell_2(\ell_2+1)}\frac{h_{\ell_3\ell_4L}}{\ell_4(\ell_4+1)}$$

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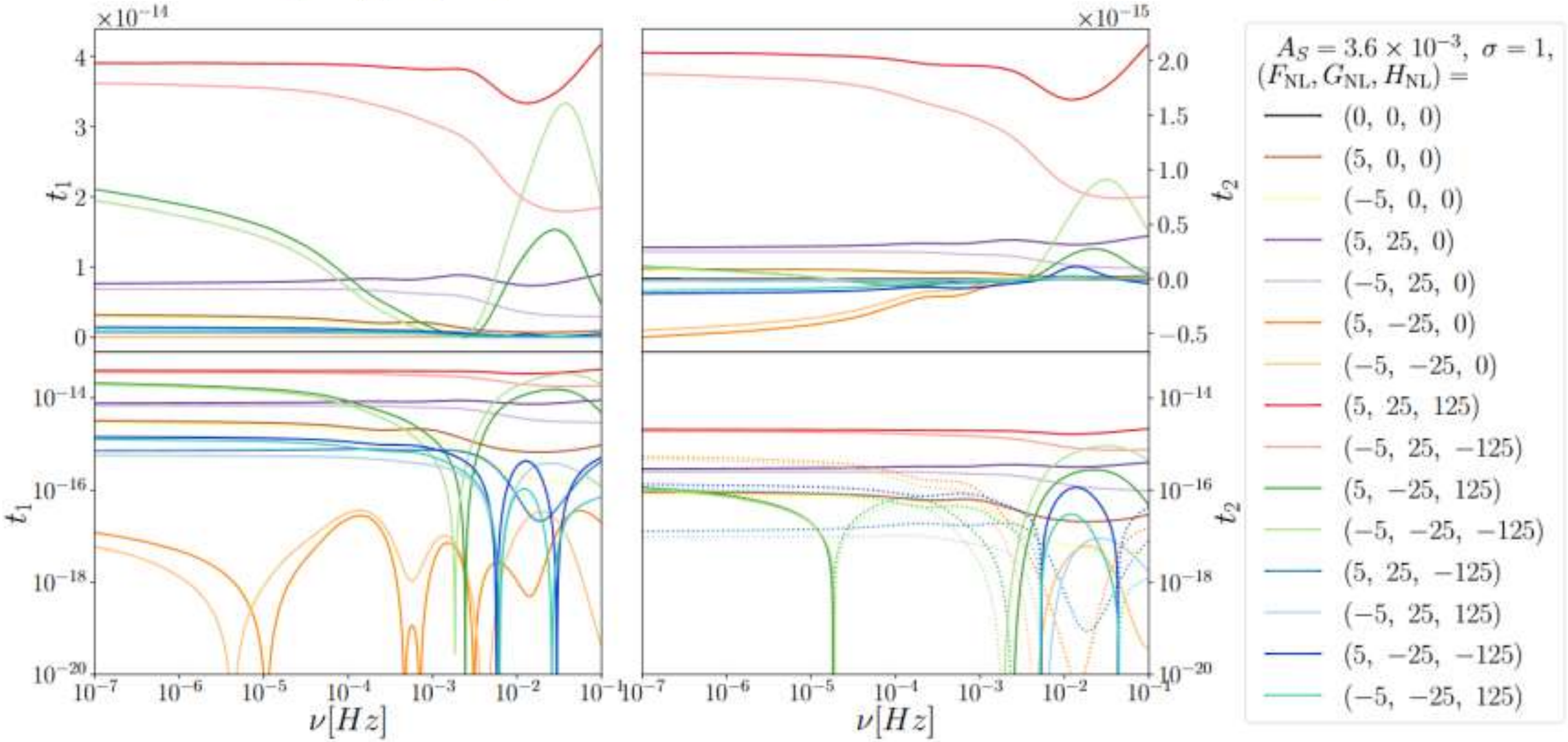
$$t_1(\nu) = 4\cdot(2\pi A_{\rm L})^3\left[\frac{\omega^{(1)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}\left(2-n_{\rm gw}(\nu)\right)\right]^2\left[\frac{\omega^{(2)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}F_{\rm NL}\left(2-n_{\rm gw}(\nu)\right)\right]^2$$

$$t_2(\nu) = \frac{(2\pi A_{\rm L})^3}{2\pi}\left[\frac{\omega^{(1)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}\left(2-n_{\rm gw}(\nu)\right)\right]^3\left[\frac{\omega^{(3)}_{\rm ng,in}(q)}{\bar\omega_{\rm gw,in}(q)} + \frac{3}{5}G_{\rm NL}\left(2-n_{\rm gw}(\nu)\right)\right]$$

PNG-induced Inhomogeneity

SW effect & NL effect



$$\Big)^{n_2}\binom{N_j}{n_2}\delta_{l_1n_1+l_2n_2,3}\ ,$$

$$\left(\frac{(k+1)!}{k!}\frac{V_1^{[k+1]}}{V_0^{[k]}}\right)^1\binom{N_k}{1}$$

Non-Gaussian Background: Angular Trispectrum

$$t^{\ell_1\ell_2}_{\ell_3\ell_4}(L,\nu) \simeq \left[\frac{t_1(\nu)}{L(L+1)} + \frac{t_2(\nu)}{\ell_1(\ell_1+1)} + \frac{t_2(\nu)}{\ell_3(\ell_3+1)}\right]\frac{h_{\ell_1\ell_2L}}{\ell_2(\ell_2+1)}\frac{h_{\ell_3\ell_4L}}{\ell_4(\ell_4+1)}$$

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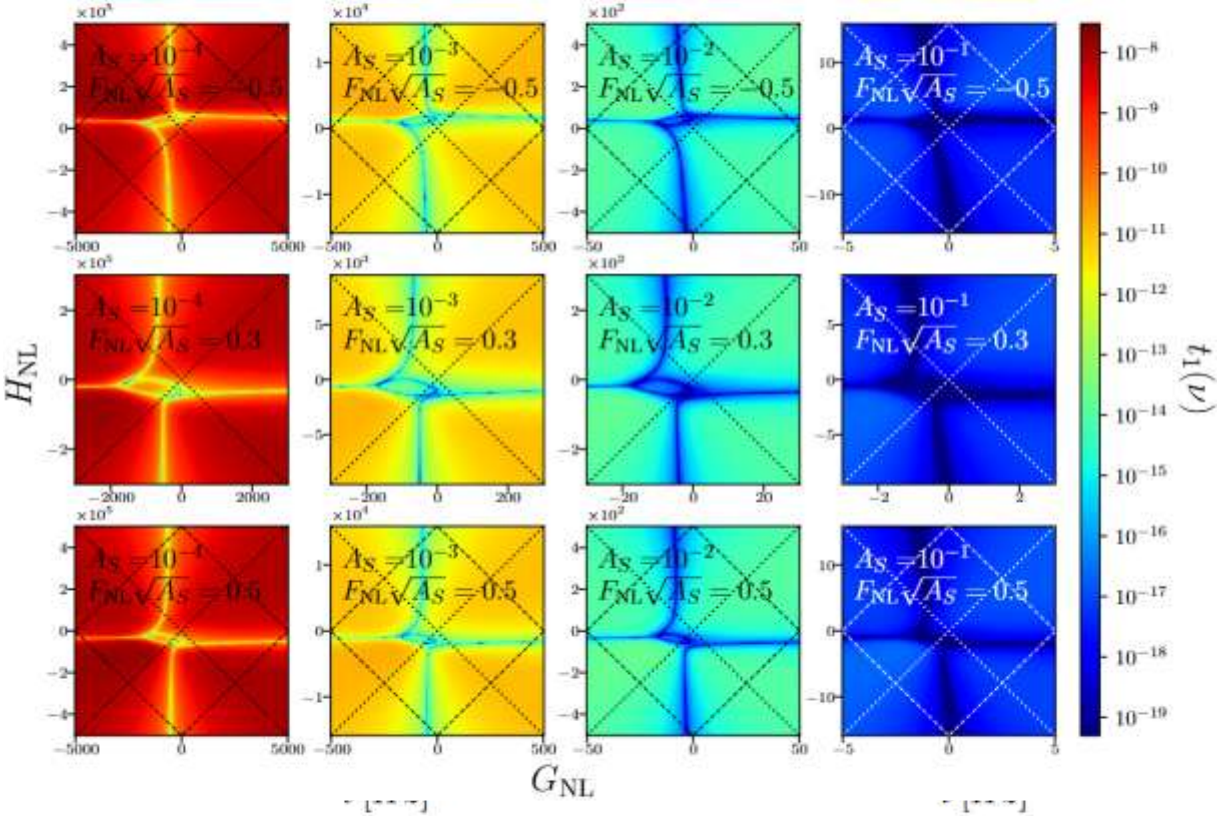
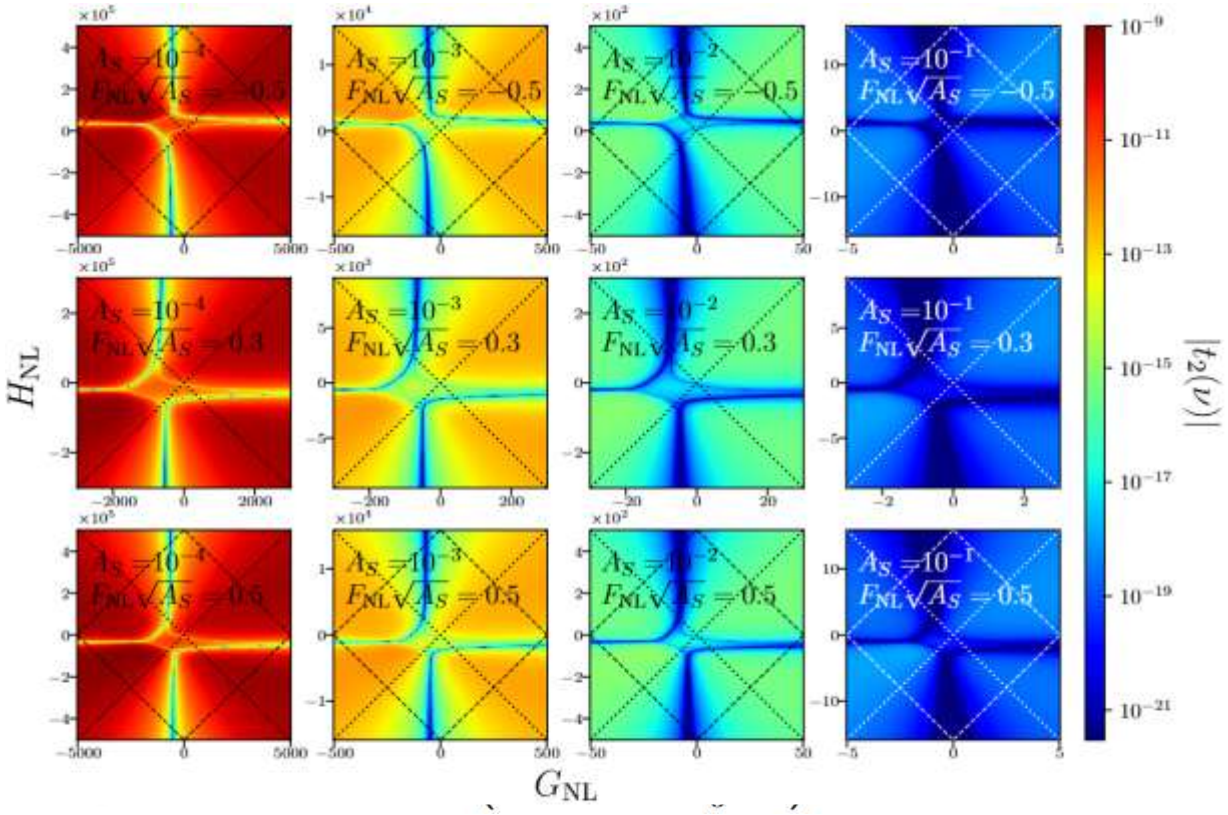
$$t_1(\nu) = 4\cdot(2\pi A_{\rm L})^3\left[\frac{\omega^{(1)}_{\rm ng,in}(q)}{\bar{\omega}_{\rm gw,in}(q)} + \frac{3}{5}(2-n_{\rm gw}(\nu))\right]^2\left[\frac{\omega^{(2)}_{\rm ng,in}(q)}{\bar{\omega}_{\rm gw,in}(q)} + \frac{3}{5}F_{\rm NL}(2-n_{\rm gw}(\nu))\right]^2$$

$$t_2(\nu) = \frac{(2\pi A_{\rm L})^3}{2\pi}\left[\frac{\omega^{(1)}_{\rm ng,in}(q)}{\bar{\omega}_{\rm gw,in}(q)} + \frac{3}{5}(2-n_{\rm gw}(\nu))\right]^3\left[\frac{\omega^{(3)}_{\rm ng,in}(q)}{\bar{\omega}_{\rm gw,in}(q)} + \frac{3}{5}G_{\rm NL}(2-n_{\rm gw}(\nu))\right]$$

PNG-induced Inhomogeneity

SW effect & NL effect



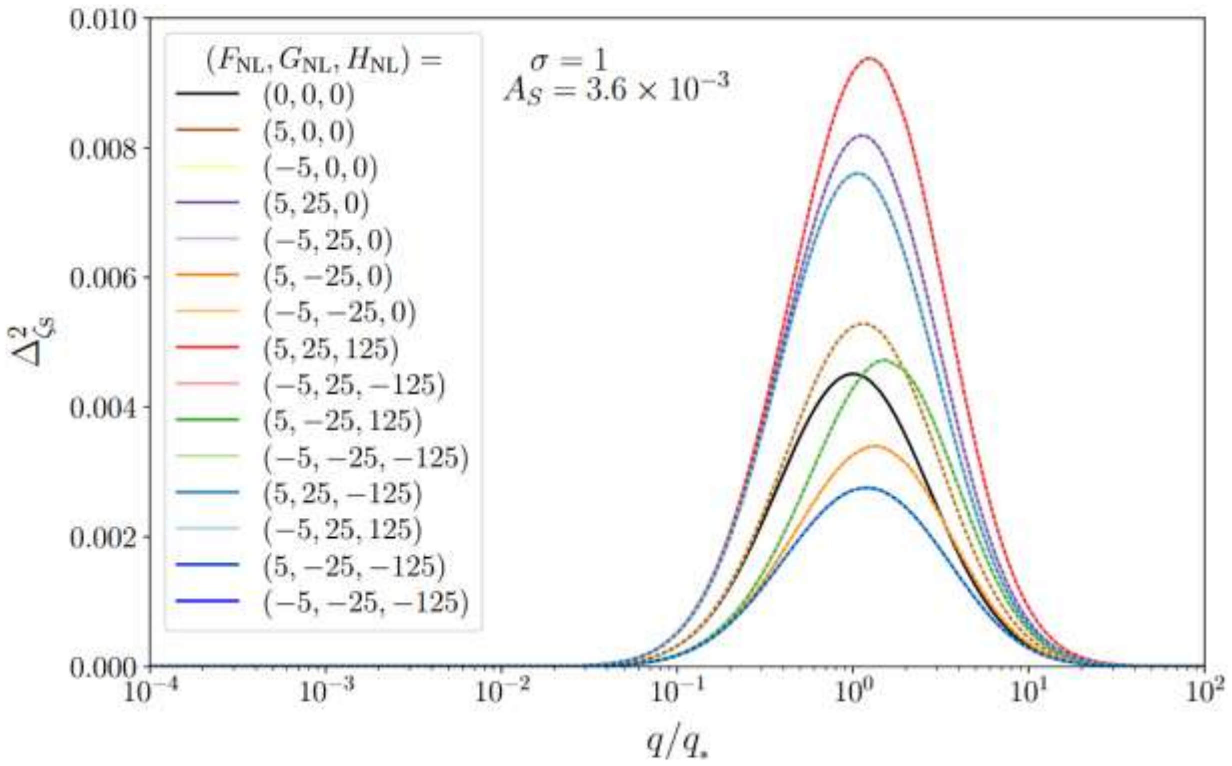


Entended Study: Primordial Trispectrum

$$T_\zeta(k_1,k_2,k_3,k_4) = \frac{54}{25} g_{\mathrm{NL}}[P(k_1)P(k_2)P(k_3) + 3\mathrm{perms}]$$
$$+\tau_{\mathrm{NL}}[P(k_1)P(k_2)P(k_{13}) + 11\mathrm{perms}]$$

$$\tau_{\mathrm{NL}} \geq \left(\frac{6f_{\mathrm{NL}}}{5}\right)^2$$

$$P_{\zeta_S}(q) = (1+3A_S G_{\mathrm{NL}})^2 P^{[1]}(q) + 2(F_{\mathrm{NL}} + 6A_S H_{\mathrm{NL}})^2 P^{[2]}(q)$$
$$+6G_{\mathrm{NL}}^2 P^{[3]}(q) + 24H_{\mathrm{NL}}^2 P^{[4]}(q)$$

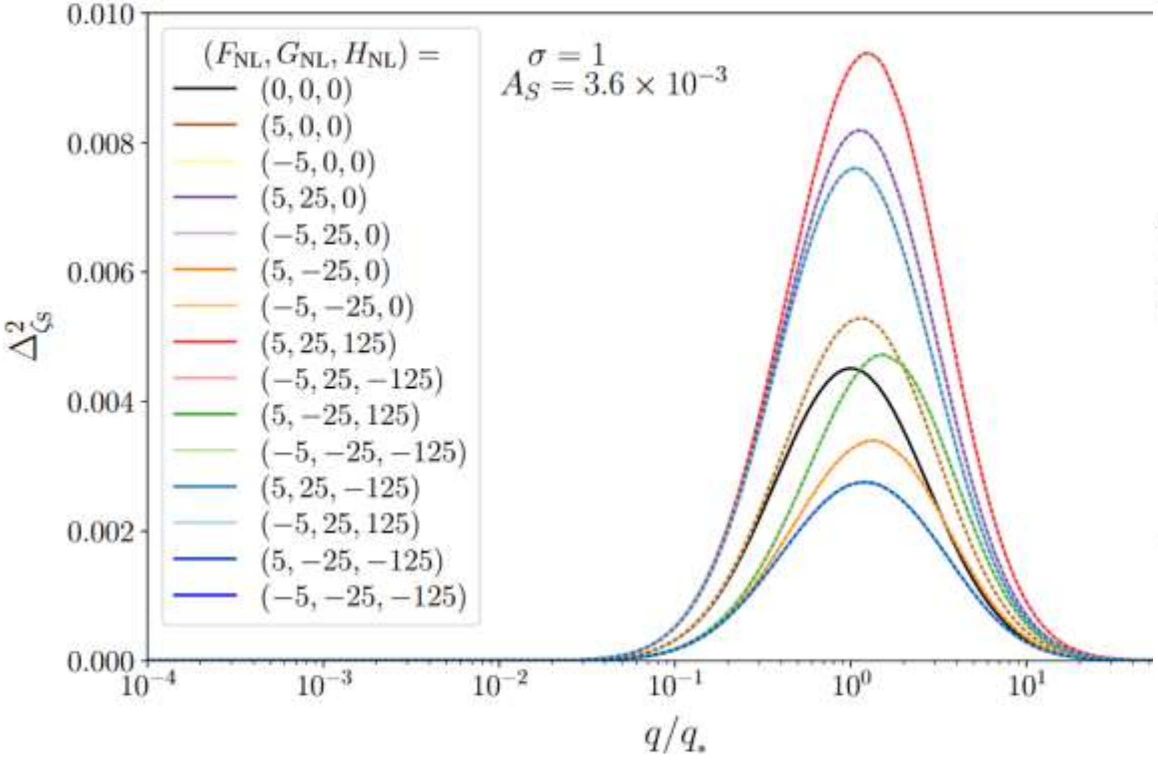


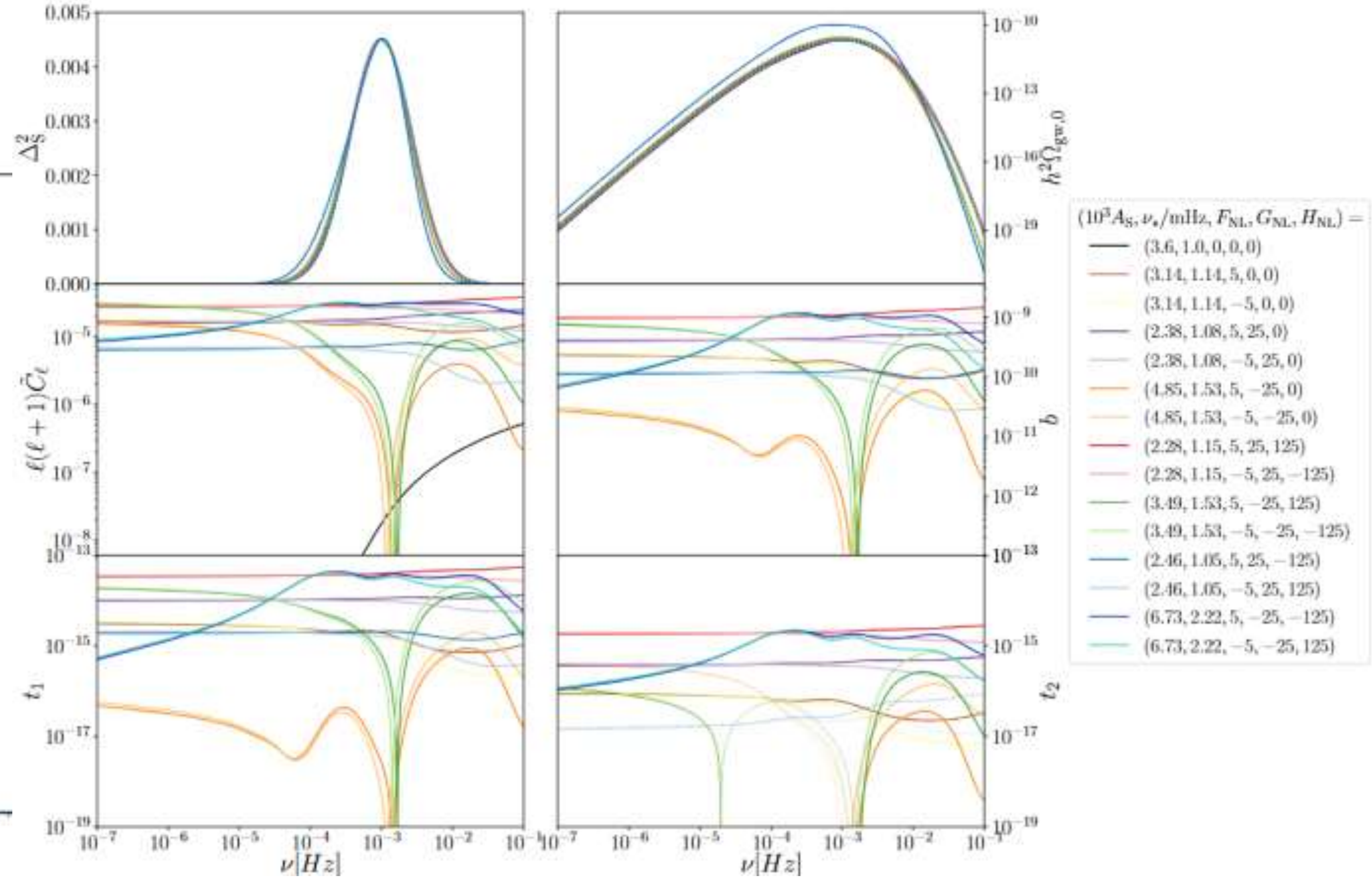
Entended Study: Primordial Trispectrum

$$T_\zeta(k_1,k_2,k_3,k_4) = \frac{54}{25} g_{\rm NL}[P(k_1)P(k_2)P(k_3) + 3\text{perms}] + \tau_{\rm NL}[P(k_1)P(k_2)P(k_{13}) + 11\text{perms}]$$

$$\tau_{\rm NL} \geq \left(\frac{6 f_{\rm NL}}{5}\right)^2$$

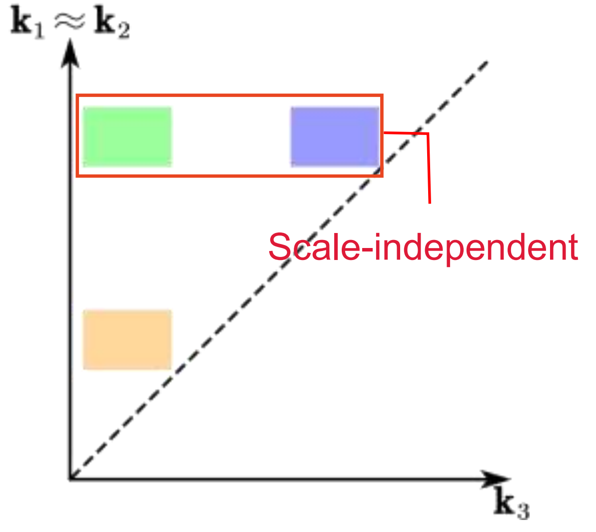
$$P_{\zeta_S}(q) = (1 + 3A_S G_{\rm NL})^2 P^{[1]}(q) + 2(F_{\rm NL} + 6A_S H_{\rm NL})^2 P^{[2]}(q) + 6G_{\rm NL}^2 P^{[3]}(q) + 24 H_{\rm NL}^2 P^{[4]}(q)$$





Entended Study: Scale-Dependent PNG

$$f_{\rm NL}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right)$$

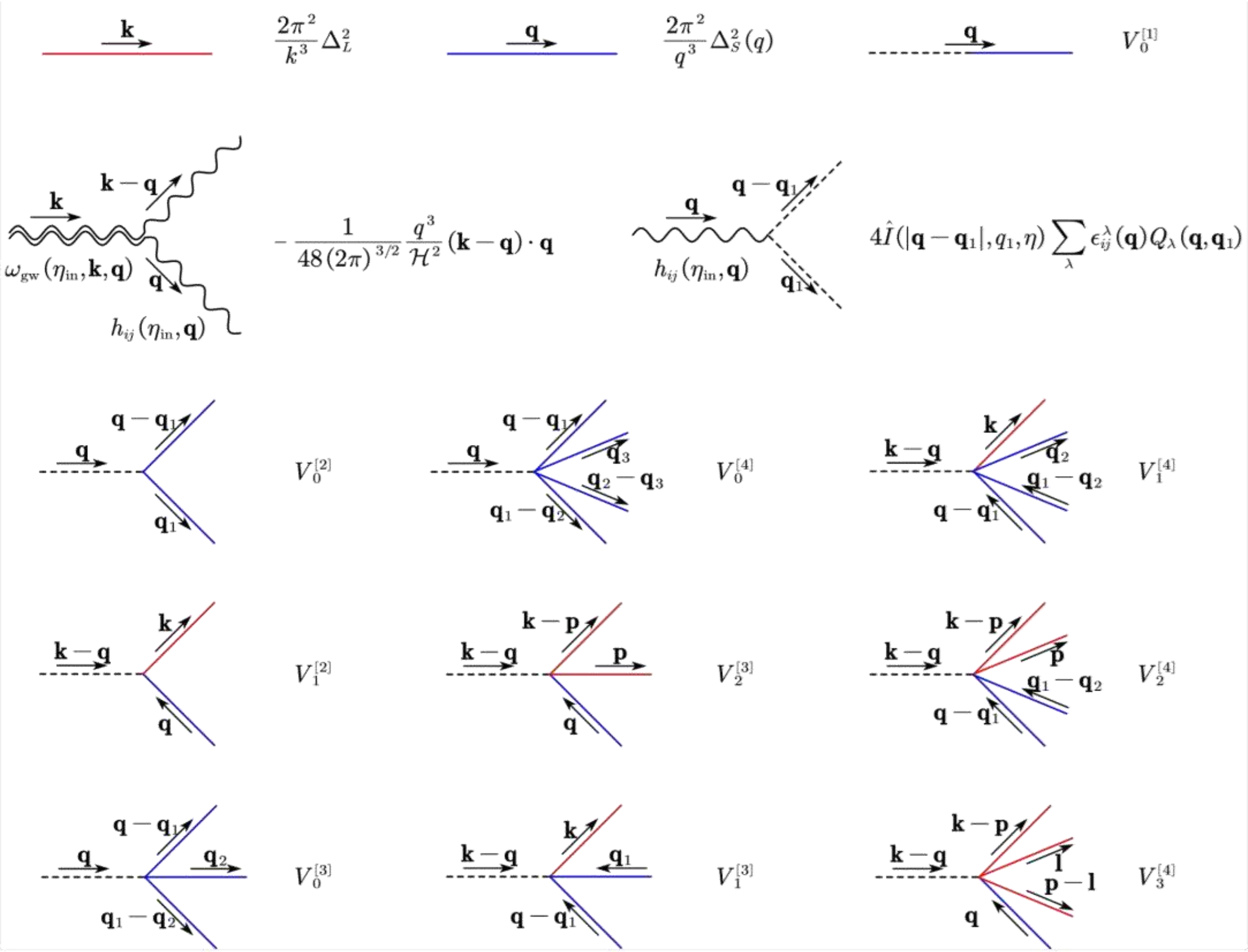


$$V_0^{[1]} = 1$$

$$V_0^{[2]} \neq V_1^{[2]}$$

$$V_0^{[3]} \neq V_1^{[3]} \neq V_2^{[3]}$$

$$V_0^{[4]} \neq V_1^{[4]} \neq V_2^{[4]} \neq V_3^{[4]}$$



Summary

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What Have We Done?

- Developed a "renormalized" diagrammatic approach to compute the energy-density spectrum, angular power spectrum, bispectrum, and trispectrum of SIGWs for primordial non-Gaussianity up to arbitrary order, which enables the direct, automated generation of all necessary integrals for these spectra with only simple Python/Mathematica codes;
- Computed the numerical results for these observables for PNG up to h_{NL} order.

Scientific Implications & Future Prospects

- A supplementary probe for physics in the early universe that surpasses the reach of measurements from CMB and LSS;
- A promising tool for the search for PBHs;
- Extraction of SIGW signals from the stochastic background and inference of model parameters;
- …

Thanks!