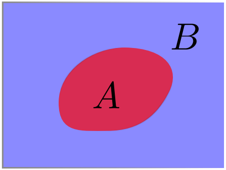
Symmetry-protected topological phases and quantum entanglement:

Shinsei Ryu
Univ. of Illinois, Urbana-Champaign

- Introduction
- Non-interacting SPT phases
- (1+1)d SPT phases
- "Twist" and entanglement spectrum

Entanglement and entropy of entanglement

(i) Bipartition the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



(ii) Partial trace $\rho_{\rm tot} = |\Psi\rangle\langle\Psi|$

$$\rho_A = {\rm tr}_B |\Psi\rangle\langle\Psi| = \sum_j p_j |\psi_j\rangle_A \langle\psi_j|_A \quad \left(\sum_j p_j = 1\right)$$

(iii) Entanglement entropy (EE)

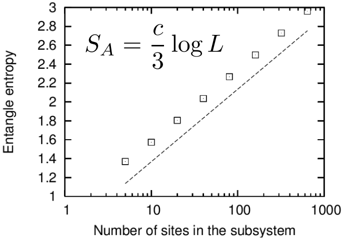
$$S_A = -{\rm tr}_A\left[\rho_A \ln \rho_A\right] = -\sum_j p_j \ln p_j$$

(iv) Entanglment spectrum (ES) $\rho_A \propto \exp(-H_e)/Z$

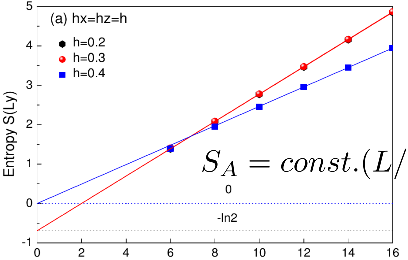
$$\{\xi_i\}_i \quad \text{where} \quad p_i =: \exp(-\xi_i)/Z$$

Entanglement measure as a numerical tool

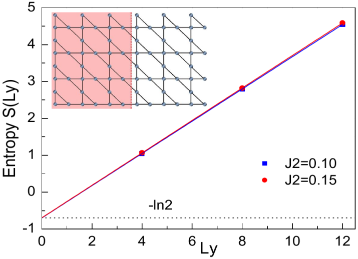
- (1+1)d CFTs



- (2+1)d Topologically ordered states



$$S_A = const.(L/\epsilon) - \gamma$$

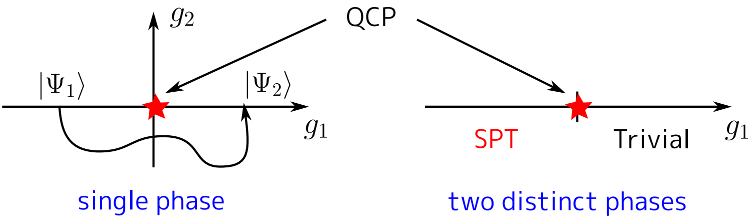


[Jiang-Wang-Balents (12)]

Symmetry protected topological phases (SPTs)

- Question: structure of phase diagram in the presence of symmetries

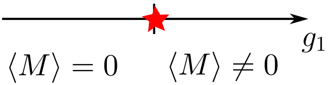
- Two gapped states of matter may be adiabatically connected,
  but sharply distinct once symmetries are enforced:

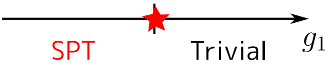


- SPT is not a topologically-ordered phase without symmetry
  "deformable" to a trivial phase (state w/o entanglement)

- E.g. quantum spin Hall effect, topological insulators

- Landau's symmetry-breaking framework does not apply.





- Can SPT be distinguished by using concept of quantum entanglement

Key concept: Symmetry-protected degenearcy in ES

- Non-interacting SPT phases

- (1+1)d SPT phases

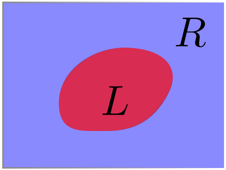
- "Symmetry twist" and entanglement spectrum

ES in non-interacting systems

- ES in non-interacting systems can be computed from 2pt functions

[Peschel(03)]

$$C_{IJ} = \langle \Psi_G | \psi_I^\dagger \psi_J | \Psi_G \rangle \qquad I = \mathbf{r}, \sigma, i$$



$$C = \begin{pmatrix} C_L & C_{LR} \\ C_{RL} & C_R \end{pmatrix}, \quad C_{RL} = C_{LR}^\dagger$$

$$Q_{IJ} := 1 - 2C_{IJ}$$

- Correlation matrix is a projector:

$$C^2 = C \qquad Q^2 = 1$$

- Entanglement Hamiltonian:

$$\rho_L = \exp(-H_e) \qquad H_e = \sum_{IJ \in L} \psi_I^\dagger K_{IJ} \psi_J$$

$$K = \ln[(1 - C_L)/C_L]$$

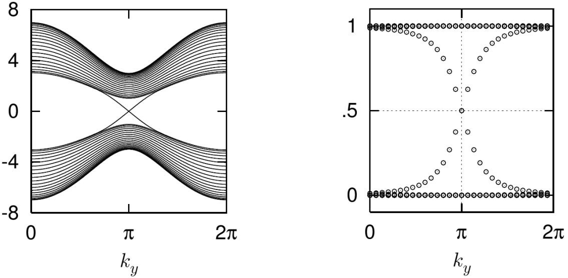
The integer quantum Hall effect

- A prototype of topological phases

- Characterized by quantized Hall conductance $\sigma_{xy} = \frac{e^2}{h} \times (\text{integer})$

- Gapped bulk, gapless edge

- Robust against disorder and interactions



Physical spectrum v.s. (single-particle) ES

[SR-Hatsugai(06)]

Entanglment spec. and non-spatial symmetries

- How about symmetry ? [Ryu-Hatsugai(06)]

- Corr. matrix inherits symmetries of the Hamiltonian

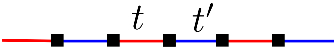
$$\psi_I \to U_{IJ}\psi_J \qquad H_{phys} \to U^\dagger H_{phys} U = H_{phys}$$

$$Q \to U^\dagger Q U = Q$$

- Non-spatial symmetry: $Q_L \to U^\dagger Q_L U = Q_L$

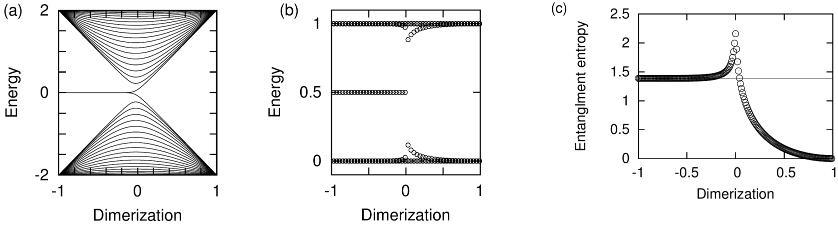
- E.g. 1d lattice fermion model ("SSH" model)

$$H = t\sum_i (a_i^\dagger b_i + h.c.) + t'\sum_i (b_i^\dagger a_{i+1} + h.c.)$$



symmetry: $a_i \to a_i^\dagger \quad b_i \to -b_i^\dagger$

phase diagram:



$$S_A \sim (1/6)\log(\xi/a_0) + \ln 2$$

(Entanglment Spec)^2 and SUSY QM

[Turner-Zhang-Vishwanath,
Hughes-Prodan-Bervevig
Chang-Mudry-SR]

- $C^2 = C$

$$
\begin{aligned}
C_L^2 - C_L &= -C_{LR} C_{RL}, \\
Q_L C_{LR} &= -C_{LR} Q_R, \\
C_{RL} Q_L &= -Q_R C_{RL}, \\
C_R^2 - C_R &= -C_{RL} C_{LR},
\end{aligned}
$$

$$Q_{IJ} := 1 - 2C_{IJ}$$

- Introduce

$$\mathcal{S} = \begin{pmatrix} 1 - Q_L^2 & 0 \\ 0 & 1 - Q_R^2 \end{pmatrix} \quad \mathcal{Q} = \begin{pmatrix} 0 & 2C_{LR} \\ 0 & 0 \end{pmatrix} \quad \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ 2C_{RL} & 0 \end{pmatrix}$$

- SUSY algebra:

$$
\begin{aligned}
&[\mathcal{S}, \mathcal{Q}] = [\mathcal{S}, \mathcal{Q}^\dagger] = 0, \\
&\{\mathcal{Q}, \mathcal{Q}^\dagger\} = \mathcal{S}, \quad \{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = 0.
\end{aligned}
$$

Entanglment spec. and spatial symmetries

[Turner-Zhang-Vishwanath,
Hughes-Prodan-Bervevig
Chang-Mudry-SR]

- L/R = fermionic/bosonic sector

$$\mathcal{H}_L \underset{C_{RL}}{\overset{C_{LR}}{\rightleftarrows}} \mathcal{H}_R$$

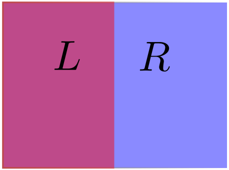
- Spatial symmetry O: choose bipartition s.t.

$$\mathcal{O} : \mathcal{H}_L \leftrightarrow \mathcal{H}_R$$

$$O = \left(\begin{array}{cc} 0 & O_{LR} \\ O_{RL} & 0 \end{array} \right), \quad O_{LR} O^{\dagger}_{LR} = O_{RL} O^{\dagger}_{RL} = 1$$

- Symmetry in entanglement Hamiltonian: e.g.

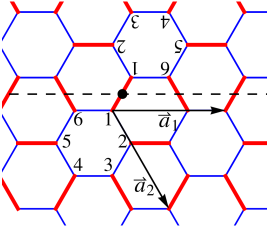
$$Q_L C_{LR} O^{\dagger}_{LR} = C_{LR} O^{\dagger}_{LR} Q^{*}_{L},$$
$$Q_R C_{RL} O^{\dagger}_{RL} = C_{RL} O^{\dagger}_{RL} Q^{*}_{R}.$$

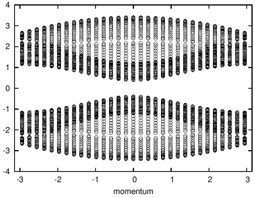


Graphene with Kekule order

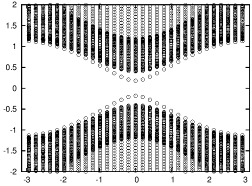
[Chang-Mudry-SR]

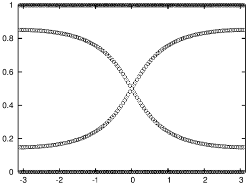
- Kekule distortion in graphene





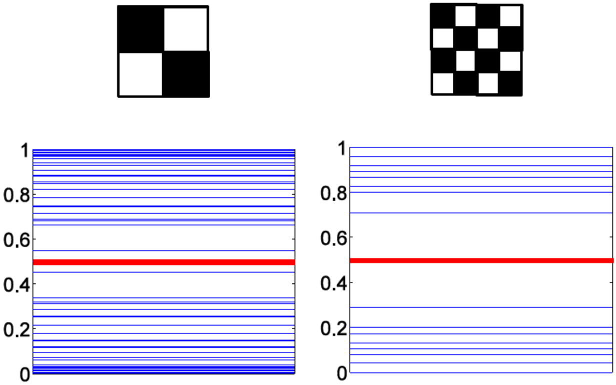
- Degeneracy protected by inversion





- Entanglement spec. is more useful than physical spec.

[Fang-Gilbert-Bernevig(12)]



1d Kitaev chain with TRS

- Studying single particle entanglement Hamiltonian is not enough.

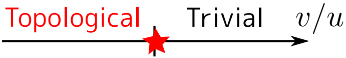
- 1d Kitaev chain w/ TRS (integer classification):

Fidkowski-Kitaev

$$H = uH_1 + vH_2$$

$$H_1 = \sum_j (-c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j^\dagger + h.c.)$$

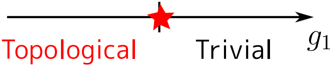
$$H_2 = \sum_j (c_j^\dagger c_j - 1/2)$$



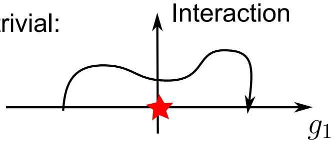
$$H = \sum_{a=1}^{N_f} (uH_{1a} + vH_{2a})$$

- Make Nf copies (topological invariant = Nf):

- Add interactions to go from topological to trivial:



Two distinct phases ?



Single phase!

- Possible only when Nf = 8 (Nf = 0 mod 8)

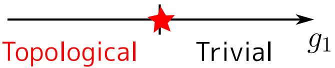
Other known "collapses" of non-interacting classification

- 1d TSC (Kitaev chain w/ TRS) : Z --> Z_8 [Fidkowski-Kitaev (10)]

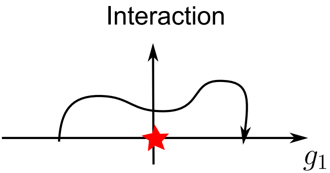
- 2d TSC w/ Z_2 symmetry: Z --> Z_8 [Qi (12), SR-Zhang (12)]
- 2d TSC with reflection symmetry: Z --> Z_8 [Yao-SR (12)]

- 3d TSC (3He-B): Z --> Z_{16} [Fidkowski et al (13), Metlitski et al (14), Wang-Senthil (14)]

- 3d crystalline TSC: Z --> Z_{16} [Hsieh-Cho-SR (15) See Poster presentation!]
- 3d crystalline TI: Z --> Z_8 [Isobe-Fu (15)]



Two distinct phases ?



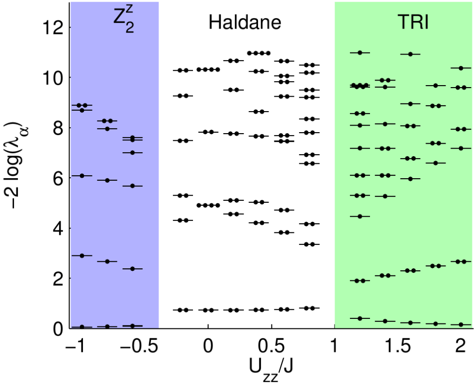
Single phase!

(1+1)d SPT phases

- (1+1)d SPT phase:
E.g. the Haldane phase, the Kitaev chain

- Symmetry-protected degenearcy in ES:

$$H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + U_{zz} \sum_i (S_j^z)^2$$



[Pollmann-Berg-Turner-Oshikawa (10)]

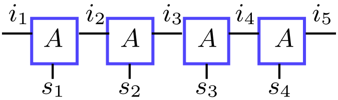
MPSs for gapped states

Matrix Product State (MPS):

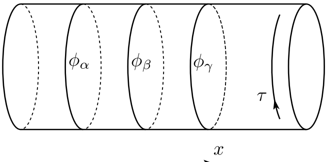
physical Hilbert space

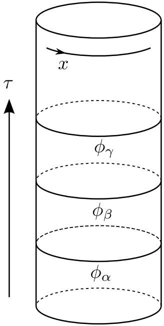
$$|\Psi\rangle = \sum_{\{s_a\}} \sum_{\{i_n=1,\cdots,\chi\}} A^{s_1}_{i_1,i_2} A^{s_2}_{i_2,i_3} A^{s_3}_{i_3,i_4} A^{s_4}_{i_4,i_5} \cdots |s_1, s_2, s_3, s_4 \ldots\rangle$$

auxiliary Hilbert space



path integral picture

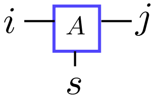




Classification of (1+1)d SPT phases via MPSs

- Symmetry acting on physical Hilbert space: $G = \{g, h, \cdots\}$

$$|s\rangle \to \hat{U}(g)_s^{s'}|s'\rangle \qquad \hat{U}(g)\hat{U}(h) = \hat{U}(gh)$$



- Induced action on auxiliary Hilbert space

$$\hat{U}(g)_s^{s'}A^s = \hat{V}^{-1}(g)A^{s'}\hat{V}(g)e^{i\theta_g}$$

- Aux spac may be a projective representation of G:

$$\hat{V}(g)\hat{V}(h) = e^{i\alpha(g,h)}\hat{V}(gh)$$

- 1d gapped states with symmetry G can be classified by $H^2(G, U(1))$

[Chen-Gu-Wen,
Pollmann-Turner-Berg-Oshikawa,
Schuch-Perez-Garcia-Cirac]

- Higher dimensions ?

SPTs and BCFTs

- Continuum field theory? How can MPS picture arise ?

- ES near critical point [Baxter, Peschel, Date-Jimbo-Miwa-Okado, ...]

$$\rho_A \propto \exp(-H_e)$$

$$H_e = const. \frac{L_0}{\log(\xi/a_0)}$$

- ES for gapped phase is given by nearby boundary conformal field

ξ correlation length

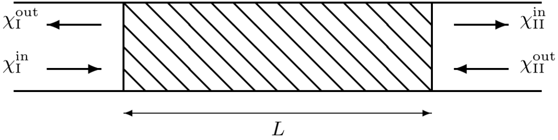
- Can be understood by putting system on Rindler spacetime [Swingle]

- Which boundary condition ? <----> Which gapped phase?

Scattering off from SPTs

- Scattering matrix formulation of topological invariants: [Akhmerov et al]

Ideal lead SPT (gapped) Ideal lead



$$\begin{pmatrix} \chi_{\mathrm{I}}^{\mathrm{out}} \\ \chi_{\mathrm{II}}^{\mathrm{out}} \end{pmatrix}_E = S_E \begin{pmatrix} \chi_{\mathrm{I}}^{\mathrm{in}} \\ \chi_{\mathrm{II}}^{\mathrm{in}} \end{pmatrix}_E \equiv \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}_E \begin{pmatrix} \chi_{\mathrm{I}}^{\mathrm{in}} \\ \chi_{\mathrm{II}}^{\mathrm{in}} \end{pmatrix}_E$$

- E.g.: $\mathrm{sgn}\ \det r = \pm 1$

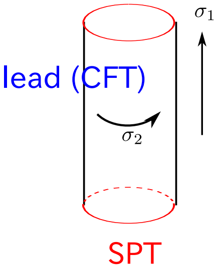
Anomalous boundary states

[Work in progress with Gil Young Cho and Andreas Ludwig]

- Ideal lead obeys B.C. set by SPT

$$\Phi(\sigma_2) - U \cdot \Phi(\sigma_2) = 0$$

$$[\Phi(\sigma_2) - U \cdot \Phi(\sigma_2)]|B\rangle = 0$$



- Symmetry G acts on fundamental fields

$$\mathcal{G} \cdot \Phi(\sigma_2) \cdot \mathcal{G}^{-1} = U_G \cdot \Phi(\sigma_2)$$

- B.C. is invariant under G:

$$\mathcal{G}\left[\Phi - U \cdot \Phi\right]\mathcal{G}^{-1} = U_G \cdot \Phi - U_G \cdot U \cdot \Phi$$

- But boundary state may not be:

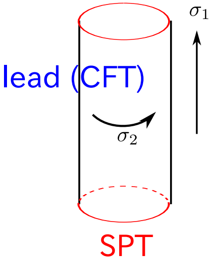
$$\mathcal{G} \cdot |B\rangle \neq |B\rangle$$

- Z_8 classification of TRS Kitaev chain, Haldane phase

Analysis and result: Fidkowski-Kitaev problem

- Ideal lead

$$H = \sum_{a=1}^{N_f} \int_0^\ell dx \, [\psi_L^a(-vi\partial_x)\psi_L^a + \psi_R^a(+vi\partial_x)\psi_R^a]$$



- Symmetry group: $\{T, G_f, T \times G_f\}$

- Boundary states

$$[\psi_L(\sigma_2) - i\eta_1\psi_R(\sigma_2)] \, |B(\eta_1, \eta_2)\rangle = 0$$
$$[\psi_R(\sigma_2) + i\eta_2\psi_L(\sigma_2)] \, |B(\eta_1, \eta_2)\rangle = 0$$

- Symmetry action on fermion number parity:

$$G_f|B(\eta_1 = -\eta_2)\rangle = |B(\eta_1 = -\eta_2)\rangle$$
$$G_f|B(\eta_1 = \eta_2)\rangle = (-1)^{N_f}|B(\eta_1 = \eta_2)\rangle$$

Anomalous relative sign goes away for 2N copies --> Z2

- Time reversal:

$$T|B(\eta_1 = \eta_2)\rangle = e^{i\pi N_f/4}|B(\eta_1 = \eta_2)\rangle$$

Comments

- Maybe related to Tadashi's talk today.

- {SPTs} = {BCFTs} ?? Maybe not ..
  SPTs are classified but boundary states are not.

- Similar analysis can be used to study (2+1)d SPTs with
  parity symmetry by using crosscap states.
  [Yao-SR (12), Cho-Hsieh-Morimoto-SR (15)]