

Symplectic leaves of meromorphic Higgs bundles (Joint w. Jia Choon Lee)

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Motivation I

- C : sm. proj. curve/ \mathbb{C} of genus $g \geq 0$,
- $(r, d) = (\text{rk}, \text{deg})$ and assume $(r, d) = 1$,
- $(E, \Phi \in H^0(C, \text{End}(E) \otimes K_C))$: K_C -valued Higgs bundle,
- \mathcal{H} : moduli space of stable K_C -valued Higgs bundles,
- $h: \mathcal{H} \rightarrow B := \bigoplus_{i=0}^r H^0(C, K_C^{\otimes i})$ Hitchin map.

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Key properties (Hitchin '87, Beauville–Narasimhan–Ramanan '89, etc.):

- 1 \mathcal{H} is holomorphic symplectic
- 2 $h: \mathcal{H} \rightarrow B$ is an algebraic integrable system: Lagrangian fibration whose generic fiber is an abelian variety ($\dim \mathcal{H} = 2 \dim B$)

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Key properties (Bottacin '95, Markman '94, etc):

- 1 \mathcal{H} is holomorphic **Poisson**
- 2 $h: \mathcal{H} \rightarrow B$ is an algebraic **Poisson** integrable system: **coisotropic** fibration whose generic fiber is an abelian variety ($\dim \mathcal{H} < 2 \dim B$)

- ④ (Symplectic leaves) Candidate: $\mathcal{H}^{\mathfrak{O}} \subset \mathcal{H}$ the locus where the residue of Higgs field $\text{res}_D \Phi \in \mathfrak{O}$ for an adjoint orbit $\mathfrak{O} \in \mathfrak{gl}_r$.
 - (Local) Infinitesimal deformation theory; smooth, $T_{(E, \Phi)} \mathcal{H}^{\mathfrak{O}}$ is characteristic distribution.
 - (Global) Connectedness.

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 - (Local) Infinitesimal deformation theory; smooth, $T_{(E,\Phi)}\mathcal{H}^\mathcal{O}$ is characteristic distribution.
 - (Global) Connectedness.
- 2 (Integrable system) Does the restriction $h|_{\mathcal{H}^\mathcal{O}} : \mathcal{H}^\mathcal{O} \rightarrow B$ carry an integrable system?
 - (Base) Describe an *affine* base $B^\mathcal{O}$ with $\dim \mathcal{H}^\mathcal{O} = 2 \dim B^\mathcal{O}$.
 - (Fiber) Spectral correspondence.

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 - $\mathcal{H}^{\overline{\mathcal{O}}}$ has symplectic singularities in the sense of Beauville.

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Goal: Answer to these questions using parabolic Higgs bundles.

Parabolic Higgs bundles

- (Combinatorial data) A partition of $r = \underline{m} = (m_1 \geq \cdots \geq m_\ell)$.
- A parabolic Higgs bundle (E, Φ, E_p^\bullet) :
 - ① (E, Φ) : a meromorphic Higgs bundle;
 - ② A parabolic structure

$$E_p^\bullet : 0 = E_p^\ell \subseteq \cdots \subseteq E_p^1 \subseteq E^0 = E|_p$$

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- To define moduli spaces, we choose a set of parabolic weights $\underline{\alpha} : 1 \geq \alpha_\ell \geq \dots \geq \alpha_1 \geq 0$.
- A parabolic degree is $\text{pardeg}(E) = \deg(E) + \alpha_j \sum \dim(E_p^{j-1}/E_p^j)$
- A parabolic Higgs bundle (E, Φ, E_p^\bullet) is $\underline{\alpha}$ -(semi)stable if for any non-trivial subbundle $F \subset E$ preserved by Φ ,

$$\frac{\text{pardeg}(F)}{\text{rk}(F)} < \frac{\text{pardeg}(E)}{\text{rk}(E)} \quad (\text{resp. } \leq)$$

Parabolic Higgs bundles II

- (★) We say $\underline{\alpha}$ generic if $\underline{\alpha}$ -semistability implies $\underline{\alpha}$ -stability.
- $\mathcal{H}^{par}(\underline{m})$: moduli space of $\underline{\alpha}$ -semistable parabolic Higgs bundles.
- $h^{par} : \mathcal{H}^{par}(\underline{m}) \rightarrow B := \bigoplus_{i=0}^r H^0(C, K_C(D)^{\otimes i})$ Hitchin map.

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Key properties (Logares-Martens 10', Su-Wen-Wang 21')

- 1 $\mathcal{H}^{par}(\underline{m})$ is holomorphic Poisson.
- 2 $h^{par} : \mathcal{H}^{par}(\underline{m}) \rightarrow B$ is a Poisson integrable system.

$\underline{\xi}$ -parabolic Higgs bundles I

- (Combinatorial data) A partition of $r = \underline{m} = (m_1 \geq \cdots \geq m_\ell)$,
a collection of complex numbers $\underline{\xi} = (\xi_1, \cdots, \xi_\ell)$
- A $\underline{\xi}$ -parabolic Higgs bundle (E, Φ, E_p^\bullet) :
 - 1 (E, Φ, E_p^\bullet) : a parabolic Higgs bundle;
 - 2 The restriction of the residue $\text{res}_p \Phi$ to the associated graded pieces $\text{gr} E_p^{i-1}/E_p^i$ acts by scalar multiplication ξ_i .

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- (Ex1) $\underline{\xi} = \underline{0}$: strongly parabolic Higgs bundles.
- (Ex2) For $r = 2, \underline{m} = (2)$, $\begin{pmatrix} \xi & * \\ 0 & \xi \end{pmatrix}$ is $\underline{\xi} = (\xi)$ -parabolic iff $* = 0$.

For $\underline{m} = (1, 1)$, $\begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix}$ is $\underline{\xi} = (\xi_1, \xi_2)$ -parabolic.

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- $\mathcal{H}^{\underline{\xi}-par}(\underline{m})$ moduli space of $\underline{\alpha}$ -semistable $\underline{\xi}$ -parabolic Higgs bundles.
- **Question:** Geometry of $\mathcal{H}^{\underline{\xi}-par}(\underline{m})$ and the restriction of the Hitchin map $h^{par}|_{\mathcal{H}^{\underline{\xi}-par}(\underline{m})} : \mathcal{H}^{\underline{\xi}-par}(\underline{m}) \rightarrow B$.

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- (Su-Wen-Wang '21) For $\underline{\xi} = \underline{0}$ the base becomes

$$B(\underline{m})_{\underline{0}} := \bigoplus_{i=1}^r H^0(C, K_C(D)^{\otimes i}(-\gamma(i)))$$

where γ is the level function of \underline{m} .

- For example, when $\underline{m} = \underline{1}$, then $B/B(\underline{m})_{\underline{0}}$ parametrizes sets of unordered eigenvalues. But in general, this is no longer true.

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- For example, when $\underline{m} = \underline{1}$, then $B/B(\underline{m})_{\underline{0}}$ parametrizes sets of unordered eigenvalues. But in general, this is no longer true.
- Unless $\underline{m} = (1, \dots, 1)$ is the full flag, $\cup \mathcal{H}^{\underline{\xi}-par}(\underline{m}) \neq \mathcal{H}^{par}(\underline{m})$
- $P : \mathcal{H}(\underline{m}) \rightarrow N(\underline{m})$ relative moduli problem of over the space of $\underline{\xi}$'s; i.e. $P^{-1}(\underline{\xi}) = \mathcal{H}^{\underline{\xi}-par}(\underline{m})$.

$\underline{\xi}$ -parabolic Higgs bundles III

Theorem (Lee, L. 24')

$P : \mathcal{H}(\underline{m}) \rightarrow N(\underline{m})$ is smooth. In particular, every $\mathcal{H}^{\underline{\xi}-par}(\underline{m})$ is smooth.

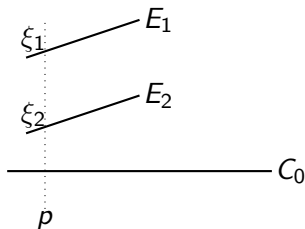
Theorem (Lee, L. 25', Spectral Correspondence)

For each $\underline{\xi}$, we have a commutative diagram

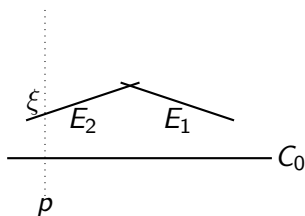
$$\begin{array}{ccc} \mathcal{H}^{\underline{\xi}-par}(\underline{m}) & \xrightarrow{\text{cld.emb}} & \mathcal{M}_{S_{\underline{\xi}}}(\Sigma(\underline{m})) \\ \downarrow h^{par} & & \downarrow \text{supp} \\ B & \longleftrightarrow & B(\underline{m})_{\underline{\xi}} \end{array}$$

- $S_{\underline{\xi}}$ holomorphic symplectic surface obtained by blow-up of $\text{Tot}(K_C(D))$,
- $\mathcal{M}_{S_{\underline{\xi}}}(\Sigma(\underline{m}))$ moduli of pure dim one sheaves with $\text{ch}_1 = \Sigma(\underline{m})$,
- $B(\underline{m})_{\underline{\xi}} \subset |\Sigma(\underline{m})|$ affine subspace away from infinity.

- $r = 2$, $\underline{m} = (1, 1)$, $\underline{\xi} = (\xi_1, \xi_2)$ and $\xi_1 \neq \xi_2$. Then $\Sigma(\underline{m}) = 2\pi^* C_0 - E_1 - E_2$;



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Denote the restriction of the Hitchin map by $h(\underline{m})_{\underline{\xi}} : \mathcal{H}^{\underline{\xi}-par}(\underline{m}) \rightarrow B(\underline{m})_{\underline{\xi}}$.

Theorem (Lee, L. *In progress*)

- $\mathcal{H}^{\underline{\xi}-par}(\underline{m})$ is a closed symplectic leaf of $\mathcal{H}^{par}(\underline{m})$.
- $h(\underline{m})_{\underline{\xi}} : \mathcal{H}^{\underline{\xi}-par}(\underline{m}) \rightarrow B(\underline{m})_{\underline{\xi}}$ carries an integrable system structure.

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- For connectedness, we use $P : \mathcal{H}(\underline{m}) \rightarrow N(\underline{m})$ is \mathbb{C}^* -equivariant and make $\mathcal{H}(\underline{m})$ semi-projective. Thus, every fiber has the same cohomology.
- Over the locus $B^{int}(\underline{m})_{\underline{\xi}}$ of $B(\underline{m})_{\underline{\xi}}$ where the spectral curve in $S_{\underline{\xi}}$ is integral, if non-empty, then spectral correspondence is an isomorphism.

Main results I

- Given an adjoint orbit $\mathcal{O} \in \mathfrak{gl}_r$, we assign combinatorial data $(\underline{m}, \underline{\xi})$ where \underline{m}^t is a collection of size of Jordan blocks. Denote $\mathcal{O} = \mathcal{O}(\underline{m}, \underline{\xi})$.
- For $\mathcal{O} = \begin{pmatrix} \xi & * \\ 0 & \xi \end{pmatrix}$, $\underline{m} = (1, 1)$, $\underline{\xi} = (\xi, \xi)$ when $* = 1$, and $\underline{m} = (2)$, $\underline{\xi} = (\xi)$ when $* = 0$.
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- We describe the Hitchin base of the restriction $h|_{\mathcal{H}^{\bar{\mathcal{O}}}}$ by $B^{\bar{\mathcal{O}}} := B(\underline{m})_{\underline{\xi}}$; $h^{\bar{\mathcal{O}}} : \mathcal{H}^{\bar{\mathcal{O}}} \rightarrow B^{\bar{\mathcal{O}}}$.

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Remark

We don't know how to describe the Hitchin base for $\mathcal{H}^{\mathcal{O}}$.

For small enough parabolic weights $\underline{\alpha}$, we have the forgetful map

$$F : \mathcal{H}^{\underline{\xi}-par}(\underline{m}) \rightarrow \mathcal{H}^{\bar{0}}$$

Proposition

The forgetful map F satisfies the following properties:

- 1 surjective and projective,
- 2 isomorphism over $\mathcal{H}^{\bar{0}}$. In fact this is compatible with symplectic forms.
- 3 compatible with Hitchin maps.

Theorem (Lee, L. *In preparation*)

- ① $\mathcal{H}^{\mathcal{O}}$ is a symplectic leaf of \mathcal{H} .
 - ② $\mathcal{H}^{\mathcal{O}} = \overline{\mathcal{H}^{\mathcal{O}}}$. In fact, F gives a symplectic resolution of $\mathcal{H}^{\mathcal{O}}$.
- If we drop $(r, d) = 1$, then one needs to work with $\mathcal{H}^{\mathcal{O}, ss}$, but surjectivity is missing.
 - C : Elliptic curve, $(r, d) = (r, 0)$, \mathcal{O} nilpotent orbit whose JNF size $(2, 1, \dots, 1)$. Then $\underline{m} = (r - 1, 1)$, $\underline{\xi} = (0, 0)$. In this case, we have

$$\begin{array}{ccc}
 \mathrm{Hilb}^r(T^*C) & \xrightarrow{F} \twoheadrightarrow & \mathrm{Sym}^r(T^*C) \\
 & \searrow h(\underline{m})_{\underline{\xi}} \circ F & \downarrow h^{\mathcal{O}} \\
 & & B^{\mathcal{O}} = \mathbb{A}^r
 \end{array}$$

Thank you for listening!