Joint deformation problem of Cpt Kähler mfds and Higgs bundles

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§ Todays goal

Differential-Graded-Lie algebra

1. Introduce the DGLA which governs the joint deformation of $(\underline{X}, \underline{E, \theta})$.

Cpx mfd   Higgs bundle

2. When
   - (X, ω): Cpt kähler
   - $(E, \bar{\partial}_E, \theta)$ polystable with $c_1 = c_2 = 0$

then

$$(\mathrm{Kur}_{(X,E,\theta)}, 0) \simeq (\mathrm{Kur}_X \times \mathrm{Kur}_{(E,\theta)}, 0)$$

as germs of analytic space.

§ Deformation of Cpt cpx mfd. (Kodaira-Spencer, Kuranishi, ...)

X = Cpt cpx mfd.

$\pi : \mathcal{X} \to \Delta = \{ t \in \mathbb{C}^n ; |t| << \varepsilon \}$ is an complex analytic family of X

if
1. $\mathcal{X}$ = Cpx mfd.
2. π = holomorphic submersion.
3. $\pi^{-1}(0) = X$.

$\rightsquigarrow \{ X_t := \pi^{-1}(t) \}_{t \in \Delta}$ family of cpx mfds.

$\rightsquigarrow \{ \varphi_t \}_{t \in \Delta} \quad \varphi_t \in A^{0,1}(\underbrace{T^{1,0}X}_{\text{holomorphic tangent bundle}}) = \mathrm{Hom}(T^{0,1}X, T^{1,0}X)$

$T^{0,1}X_t = T^{0,1}X + \text{``}\varphi_t\text{''}(T^{0,1}X)$ $\quad \varphi_t$ measure the difference of X_t and X.

- φ_t has the information of the cpx str of X_t:

$$f \text{ is a hol-function of } X_t \Leftrightarrow (\bar\partial - \varphi_t \lrcorner \partial) f = 0.$$

- φ_t satisfies the Maurer-Cartan equation:

$$\bar\partial_{T_X} \varphi_t + \frac{1}{2} \underline{[\varphi_t, \varphi_t]_{SN}} = 0$$

Schouten-Nijenhuis bracket.

We have the converse:

Let $\varphi \in A^{0,1}(T^{1,0}X)$ and $\bar\partial_{T_X}\varphi + \frac{1}{2}[\varphi, \varphi]_{SN} = 0$.

Then φ defines a cpx str on X s.t.

f is a holomorphic function on $X_\varphi \Leftrightarrow (\bar\partial - \varphi \lrcorner \partial) f = 0$

$\rightsquigarrow$ Maurer-Cartan elements parametrize cpx str near to X!

- We obtain a DGLA (later).

$$(L = \oplus_i A^{0,i}(T^{1,0}X),\ [\cdot,\cdot]_{SN},\ \bar{\partial}_{T_X})$$

Since the Maurer-Cartan element of this DGLA

parametrize cpx str near to X,

We say that

$(L, [\cdot,\cdot]_{SN}, \bar{\partial}_{T_X})$ governs the deformation of X.

By using this DGLA, Kuranishi constructed

- $\exists\, \mathrm{Kur}_X \subset H^1$ [first cohomology of $(A^{0,1}(T^{1,0}X), \bar\partial_{T_X})$] analytic space

  Kuranishi space

- $\exists \{\varphi_t\}_{t \in \mathrm{Kur}\,X}$ $\varphi_t \in A^{0,1}(T^{1,0}X)$, $\bar\partial_{T_X}\varphi_t + \frac{1}{2}[\varphi_t, \varphi_t]_{SN} = 0$.

  Kuranishi family

Kuranishi $\begin{pmatrix} \text{Space} \\ \text{family} \end{pmatrix}$ contains all information of cpx str near to X.:

Let X' = near cpx mfd to X

then $\exists\, t \in \mathrm{Kur}_X$ s.t. $X' \cong X_{\varphi_t}$ //

§ Joint deformation problem 1.

X = Cpt cpx mfd.

<u>Def</u> A Higgs bundle $(E, \bar\partial_E, \theta)$ over X is a pair s.t.

1. $(E, \bar\partial_E)$ = holomorphic bundle over X.
2. $\theta \in A^{1,0}(\mathrm{End}\,E)$, $\bar\partial_{\mathrm{End}\,E}\theta = 0$, and $\theta \wedge \theta = 0$. $\boxed{\text{ } }$

We are interested in
the joint deformation problem of
$(X, \bar\partial_E, \theta)$.

We are interested in the joint deformation problem of

$$(X, \bar{\partial}_E, \theta).$$

i.e. We want to study what happens ✱

if we simultaneously deform

$$\begin{cases} \text{cpx str of } X. \\ \text{Higgs bundle str of } (\bar{\partial}_E, \theta). \end{cases}$$

✱ $\Leftrightarrow$ $\begin{cases} \cdot \text{ Find the governing DGLA} \\ \cdot \text{ Study the structure of Kuranishi Space.} \end{cases}$

§ Homotopy invariance of Kuranishi Space.

Def Differential Graded Lie Algebra (DGLA)
is a tuple $(L, [,], d)$ s.t.

1. $L = \bigoplus_{i \in \mathbb{Z}} L^i$ is a $\mathbb{Z}$-Graded $\mathbb{C}$-Vector Space.

2. $[,] : L \times L \to L$ is a bilinear map. s.t.

2.1. $[L^i, L^j] = (-1)^{ij} [L^j, L^i]$ (graded skew-symmetric)

2.2. $(-1)^{ik} [L^i, [L^j, L^k]]$ (graded Jacobi Identity).

$$+ (-1)^{ji} [L^j, [L^k, L^i]] + (-1)^{kj} [L^k, [L^i, L^j]] = 0$$

3. $d : L \to L$ is a $\mathbb{C}$-linear map

s.t. 3.1 $d \circ d = 0$

3.2 $d(L^i) \subset L^{i+1}$

3.3 $d([L^i, L^j]) = [dL^i, L^j] + (-1)^{i} [L^i, dL^j]$ ◻

<u>Def</u> Let $(L, [\cdot,\cdot], d)$ be a DGLA.

Then the Maurer–Cartan equation of L is

$$da + \frac{1}{2}[a, a] = 0 \qquad a \in L^1.$$

$$MC(L) := \{ a \in L^1, \ da + \frac{1}{2}[a, a] = 0 \}.$$ ◻

- General Picture.

X : Some Particular Object

(complex mfd, sol of Gauge theoritic eq.)

Deformation Problem of X

$\rightsquigarrow$ Find a DGLA* $(L, [\cdot,\cdot], d)$ s.t.

X' : object near to X $\subset MC(L)$

$= \{ a \in L^1 ; da + \frac{1}{2}[a, a] = 0 \}$

* L_∞-alg, these days

Deformation Problem of X

$\rightsquigarrow$ Find a DGLA $(L, [\cdot,\cdot], d)$ s.t.

X' : object near X $\subset MC(L) = \{a \in L^1 ; da + \frac{1}{2}[a,a] = 0\}$

ex) • X : cpt cpx mfd.

$\rightsquigarrow$ DGLA $\left(A^{0,\cdot}(T^{1,0}X) = \bigoplus_i A^{0,i}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_X\right)$.

• $(E, \bar\partial_E, \theta)$ = Higgs bundle

$\rightsquigarrow$ DGLA $\left(A^\cdot(\mathrm{End}E) = \bigoplus_i A^i(\mathrm{End}E), [\cdot,\cdot], \bar\partial_E + \theta\right)$.

$$[A, B] = A \wedge B - (-1)^{\deg A \cdot \deg B} B \wedge A.$$

We go back to introduce some objects related to DGLA.

<u>Def</u> Let $(L_1, [\,,]_1, d_1)$, $(L_2, [\,,]_2, d_2)$ be DGLAs.

$f : L_1 \to L_2$ is a morphism of DGLA

if

1. f is $\mathbb{C}$-linear.
2. $f(L_1^i) \subset L_2^i$.
3. $d_2 \circ f = f \circ d_1$
4. $f([\cdot, \cdot]_{L^1}) = [f\cdot, f\cdot]_{L^2}$ $\boxtimes$

Let $(L, [\,,], d)$: DGLA $\rightsquigarrow$ $H^i(L) = i$-th cohomology of (L, d).

Def Let $(L_1, [\cdot,\cdot], d_1)$, $(L_2, [\cdot,\cdot], d_2)$ be DGLAs.

- Let $f : L_1 \to L_2$: morphism of DGLA.

  f is a quasi-isomorphism if

$$\forall i \in \mathbb{Z} \quad H^i(f) : H^i(L_1) \xrightarrow{\simeq} H^i(L_2)$$

- $(L_1, [\cdot,\cdot], d_1)$, $(L_2, [\cdot,\cdot], d_2)$ is quasi-isomorphic if

  L_1 and L_2 are connected by a zigzag of quasi-isomorphisms

i.e. $\exists (W_i, [\cdot,\cdot], d_i)$: DGLAs $\exists f_i$: morphisms of DGLA.

s.t. •
$$\begin{tikzcd}[column sep=small] & W_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & W_3 \arrow[dl, "f_3"'] \arrow[dr, "f_4"] & & \cdots \arrow[dl] & & W_{n-2} \arrow[dr] & & W_n \arrow[dl, "f_n"'] \arrow[dr, "f_{n+1}"] & \\ L_1 & & W_2 & & W_4 & & & & W_{n-1} & & L_2 \end{tikzcd}$$
holds.

• f_i is a quasi-isomorphism for $\forall i$.

We denote as $(L_1, [\cdot,\cdot]_1, d_1) \simeq_{q\text{-iso}} (L_2, [\cdot,\cdot]_2, d_2)$ $\boxtimes$

Def $(L, [\cdot,\cdot], d)$: DGLA is called analytic

if
- $\forall i$ L^i has a norm $\|\cdot\|_i$
- $d, [\cdot,\cdot]$ are continuous w.r.t norms.
- $\dim H^0(L), H^1(L) < \infty$
- $\hat{L}^i$ (completion of L^i w.r.t $\|\cdot\|_i$) has "Hodge decomposition"

For analytic DGLAs, $\cdot$ (L, d) are like "elliptic complex",

$\cdot$ $\|\cdot\|_i$ are like "Sobolev norms".

Ex) (E, θ) = Higgs bundle. $(\oplus_i A^i(\mathrm{End}\,E), [\cdot,\cdot], \bar\partial_E + \theta)$

X = Cpx mfd $\quad (\oplus_i A^{0,i}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_X)$

are analytic DGLAs. ($\because$ Sobolev norms and Hodge decomposition.)

Goldman-Millson generalised the technique of Kuranishi:

$\forall\ (L, [\cdot,\cdot], d) =$ Analytic DGLA

they constructed an analytic germ $(\mathrm{Kur}_L, 0)$. $\left(\mathrm{Kur}_L \subset H^1(L) \right)$

- If $(L, [\cdot,\cdot], d) = (\oplus_i A^{0,i}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar{\partial}_{T_X})$ then, $\mathrm{Kur}_L = \mathrm{Kur}_X$.

<u>Thm</u> (Goldman-Millson '90).

Let $(L_1, [\cdot,\cdot]_1, d_1)$, $(L_2, [\cdot,\cdot]_2, d_2)$ be analytic DGLAs

If $(L_1, [\cdot,\cdot]_1, d_1) \simeq_{q\text{-iso}} (L_2, [\cdot,\cdot]_2, d_2)$

then, $(\mathrm{Kur}_{L_1}, 0) \simeq (\mathrm{Kur}_{L_2}, 0)$

as germs of analytic spaces ∎

<u>Thm</u> (Goldman-Millson '90).

Let $(L_1, [\cdot,\cdot]_1, d_1)$, $(L_2, [\cdot,\cdot]_2, d_2)$ be analytic DGLAs

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then, $(\mathrm{Kur}_{L_1}, 0) \simeq (\mathrm{Kur}_{L_2}, 0)$

as germs of analytic spaces ▪

Very Suprising!

$\because$) $L_1 \simeq_{q\text{-iso}} L_2$

$\underset{\mathrm{dt}}{\Longleftrightarrow}$ $^\exists (W_i, [\cdot,\cdot], d_i)$ = DGLAs $^\exists f_i$ = morphisms of DGLA.

s.t.

- $$\begin{tikzcd}[column sep=small] & W_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & W_3 \arrow[dl, "f_3"'] \arrow[dr, "f_4"] & & \cdots & W_{n-2} \arrow[dr] & & W_n \arrow[dl, "f_n"'] \arrow[dr, "f_{n+1}"] & \\ L_1 & & W_2 & & W_4 & \cdots & & W_{n-1} & & L_2 \end{tikzcd}$$ holds.

- f_i is a quasi-isomophism for $^\forall i$.

§ Joint deformation problem 2.

X : cpx mfd. $(E, \bar{\partial}_E, \theta)$: Higgs bundle over X.

Again,

We are interested in the joint deformation of

$$(X, \bar{\partial}_E, \theta).$$

$\rightsquigarrow$ (next page).

We are interested in the joint deformation problem of

$(X, \bar{\partial}_E, \theta)$.

i.e. We want to study what happens (*)

if we simultaneously deform

$$\begin{cases} \text{cpx str of } X. \\ \text{Higgs bundle str of } (\bar{\partial}_E, \theta) \end{cases}$$

(*) $\Leftrightarrow$ · Find the governing DGLA · Study the structure of Kuranishi Space.

We now answer this.

I want to introduce the DGLA which governs the deformation of $(X, (E, \bar\partial_E, \theta))$.

$$\underbrace{L}_{\mathbb{Z}\text{-Graded Vect Sp}}, \underbrace{[\,,\,]}_{\text{Bracket}}, \underbrace{d}_{\text{Differential.}}$$

Let
- k = Hermitian metric of E.
- ∂_k = $(1,0)$-part of the Chern connection w.r.t $\bar\partial_E, k$.
- $\theta_k^\dagger$ = formal adjoint of θ w.r.t k. $\left(k(\theta u, v) = k(u, \theta_k^\dagger v) \right)$

Graded Vector Space.

$$L^i := A^i(\mathrm{End}E) \oplus A^{0,i}(T^{1,0}X), \quad L := \bigoplus L^i$$

Bracket

$(A, \varphi) \in L^i, \quad (B, \psi) \in L^j$

$[(A,\varphi), (B,\psi)]_L$

$$:= \left(\{\partial_E, \varphi\} A - (-1)^{ij} \{\partial_E, \psi\} B - [A, B] ,\ [\varphi, \psi]_{SN} \right)$$

$\{\partial_E, \varphi\}\psi := \partial_E(\varphi \lrcorner \psi) - (-1)^i \varphi \lrcorner \partial_E$ (Generalisation of Lie derivative)

$[A, B] = A \wedge B - (-1)^{ij} B \wedge A$

Differential

Let $B_E, C_E : A^{0,i}(T^{1,0}X) \to A^{1,i}(\mathrm{End}\,E)$ be $\mathbb{C}$-linear maps

s.t. For $\varphi \in A^{0,i}(T^{1,0}X)$,

$$B_E(\varphi) := (-1)^i \varphi \lrcorner \underset{\substack{\shortparallel \\ (\partial_E + \bar\partial_E)^2}}{F_{d_E}}, \quad C_E(\varphi) := \{\partial_{E}, \varphi \lrcorner\}\Theta .$$

We define $d_L : L^i \to L^{i+1}$ as

$$d_L := \begin{pmatrix} \bar\partial_{\mathrm{End}E} & B_E \\ 0 & \bar\partial_{T_X} \end{pmatrix} + \begin{pmatrix} \Theta & C_E \\ 0 & 0 \end{pmatrix}$$

Thm (0.) $(L, [\,,\,]_L, d_L)$ is a DGLA.

$(L, [\,,\,]_L, d_L)$ governs the deformation of $(X, (E, \bar\partial_E, \theta))$:

<u>Thm (O.)</u>

- $(X', (E, \bar\partial_E', \theta'))$ triple near to $(X, (E, \bar\partial_E, \theta))$.

Then $\exists\, \eta \in MC(L) = \{ \eta \in L^1 ; d_L \eta + \frac{1}{2}[\eta, \eta] = 0 \}$.

- Let $\eta \in MC(L)$.

If η small (in Sobolev norm)

then η defines a triple $(X_\eta, (E, \bar\partial_{E_\eta}, \theta_\eta))$.

Thm (Q) Let $\eta \in MC(L)$ [i.e. $\eta \in L^1$ $d_L\eta + \frac{1}{2}[\eta, \eta]_L = 0$].

If η small (in Sobolev norm) then η defines $(X_\eta, (E, \bar\partial_{E_\eta}, \theta_\eta))$

outline of proof) Let $\eta = (A, \varphi) \in L^1 = A^1(\mathrm{End}\,E) \oplus A^{0,1}(T^{1,0}X)$.

Step 1. Construction of X_η.

Since $d_L\eta + \frac{1}{2}[\eta, \eta]_L = 0 \Rightarrow \bar\partial_{T_X}\varphi + \frac{1}{2}[\varphi, \varphi]_{SN} = 0$.

Then φ defines a cpx str on $X \rightsquigarrow X_\eta$.

Step 2. Construction of E_η.

Let $D^{0,1} := \bar\partial_E + \mathfrak{z}\,\partial_{E}, \varphi\lrcorner + A^{0,1}$. $\rightsquigarrow (D^{0,1})^2 = 0$.

By Newlander-Nirenberg Theorem, $\ker D^{0,1}$ generates $A(E)$ locally.

(i.e. $\exists\, \{e_i\}$: local frame of E s.t. $\{e_i\} \subset \ker D^{0,1}$)

For the local frame $\{e\} \subset \ker D^{0,1}$,

We define. $\bar\partial_{E_\eta}(f \otimes e) := \bar\partial_\eta f \otimes e$ ($\bar\partial_\eta = \bar\partial$-operator of X_η)

$\bar\partial_{E_\eta}$ is well-defined.

$\rightsquigarrow$ $(E, \bar\partial_{E_\eta})$ = holomorphic bundle over X_η. $\rightsquigarrow$ E_η.

Step3. Construction of θ_η

We set $\theta_\eta := \theta + A^{1,0} + \varphi s(\theta + A^{1,0})$.

Then
- $\theta_\eta \in A^{1,0}_{X_\eta}(\operatorname{End} E)$
- $\bar\partial_{E_\eta} \theta_\eta = 0$.
- $\theta_\eta \wedge \theta_\eta = 0$. [We can check them by calculation.]

$\rightsquigarrow$ $(X_\eta, E_\eta, \theta_\eta)$ is a [Cpx mfd. Higgs bundle] $\square$

The same deformation problem was studied by Elena Martinengo.

$(L, [\cdot,\cdot]_L, d_L)$ is an analytic DGLA

by Sobolev norms.

$\widetilde{\ \ }$? $\quad (\ker_{(X,E,\theta)} := \ker_L,\ 0)$

$\uparrow$

Goldman-Millson $\quad$ $\hookrightarrow$ What kind of structure does it have?

§ Harmonic bundles · metrics

Let
- (X, ω) : Cpt Kähler mfd.
- $(E, \bar\partial_E, \theta)$: Higgs bundle.
- h : Hermitian metric of E.

We set $\quad D_h' = \partial_h + \theta_h^\dagger, \quad D'' = \bar\partial_E + \theta.$

<u>Def</u> $\quad h$ is a harmonic metric if

$D := D_h' + D''$ is a flat connection.

We call $(E, \bar\partial_E, \theta, h)$: Harmonic bundle. ∎

Thm (Hitchin, Simpson)

$(E, \bar\partial_E, \theta)$ admits a harmonic metric h

iff

$(E, \bar\partial_E, \theta)$ is polystable with $c_1(E) = c_2(E) = 0$ □

From a Harmonic bundle $(E, \bar\partial_E, \theta, h)$, we obtain a DGLA

$\left(A^\bullet(\operatorname{End}E) := \bigoplus_i A^i(\operatorname{End}E),\ [\cdot,\cdot],\ D'' \right)$

$A \in A^i(\operatorname{End}E)$, $B \in A^j(\operatorname{End}E)$

$[A, B] = A \wedge B - (-1)^{ij} B \wedge A.$

Moreover, this is an analytic DGLA w.r.t Sobolev norms and Hodge decomposition.

Hence We obtain an analytic germ

$(\operatorname{Kur}_{(E,\theta)}, 0).$

§ Main Theorem

Recall (X, E, θ) = Triple of cpx mfd + Higgs bundle.

Then · $L^i = A^i(\mathrm{End}\,E) \oplus A^{0,i}(T^{1,0}X)$, $L = \bigoplus_i L^i$

· $[\cdot,\cdot]_L$ = bracket

· $$d_L = \begin{pmatrix} \bar\partial_{\mathrm{End}\,E} & (-1)^{i-1} \bullet \lrcorner F_{d_E} \\ 0 & \bar\partial_{T_X} \end{pmatrix} + \begin{pmatrix} \theta & \{\partial_E, \cdot \lrcorner\}\theta \\ 0 & 0 \end{pmatrix}$$

$(L, [\cdot,\cdot]_L, d_L)$ forms a DGLA and governs the deformation of (X, E, θ).

Recall (Goldman-Millson)

$(L_1, [\cdot,\cdot]_1, d_1)$, $(L_2, [\cdot,\cdot]_2, d_2)$ = Analytic DGLAs.

If $(L_1, [\cdot,\cdot]_1, d_1) \simeq_{q\text{-iso}} (L_2, [\cdot,\cdot]_2, d_2)$

then $(\mathrm{Kur}_{L_1}, 0) \simeq (\mathrm{Kur}_{L_2}, 0)$

Thm 1 Let • (X, ω) : cpt kähler,

• $(E, \bar\partial_E, \theta, h)$ Harmonic bundle.

Then

$$(L, [\cdot,\cdot]_L, d_L) \simeq_{q\text{-iso}} (A^\bullet(\mathrm{End}E), [\cdot,\cdot], D'') \oplus (A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_{T_X})$$

where $D'' = \bar\partial_E + \theta$. $\square$

Main Thm Let • (X, ω) = Cpt kähler,

• $(E, \bar\partial_E, \theta)$ is polystable with $c_1(E) = c_2(E) = 0$.

Then,

1. $(L, [\cdot,\cdot]_L, d_L) \simeq_{q\text{-iso}} (A^\bullet(\mathrm{End}E), [\cdot,\cdot], D'') \oplus (A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_{T_X})$

2. $(\mathrm{Kur}_{(X,E,\theta)}, 0) \simeq (\mathrm{Kur}_{(E,\theta)} \times \mathrm{Kur}_X, 0)$ as germ of analytic space. $\square$

pf) 1. Hitchin-Simpson (K-H corres) + Formality + Thm 1.

2. 1. + Goldman-Millson. $\square$

Let · M : Cpt Riemann Surface of genus $g \geq 2$.

· (V, φ) = Stable Higgs bundle of degree 0.

In this case, Kur_M, $\mathrm{Kur}_{(V,\varphi)}$ are smooths.

and $\dim \mathrm{Kur}_M = 3g-3$, $\dim \mathrm{Kur}_{(V,\varphi)} = 2+r^2(2g-2)$.

<u>Cor</u> · $\mathrm{Kur}_{(M,V,\varphi)}$ is a complex mfd

· $\dim \mathrm{Kur}_{(M,V,\varphi)} = g(2r^2+3) - 2r^2 - 1$. $\square$

pf) Main Thm + Disscussion above. $\square$

§ Formality

(X,ω) = Cpt kähler , $(E, \bar\partial_E, \theta, h)$ = Harmonic bundle.

$D_h' = \partial_h + \theta_h^\dagger$, $D'' = \bar\partial_E + \theta$

$(D_h')^*$, $(D'')^*$ = formal adjoint of D_h' , D'' w.r.t L^2-norm.

kähler Identity (Simpson)

$$(D_h')^* = \sqrt{-1}\,[\Lambda_\omega, D''], \quad (D'')^* = -\sqrt{-1}\,[\Lambda_\omega, D_h'].$$

$D_h'D''$-lemma (Simpson)

$$\ker D_h' \cap \ker D'' \cap (\operatorname{Im} D_h' + \operatorname{Im} D'') = \operatorname{Im} D_h' D''.$$

(cf. X = Cpt kähler. $\rightsquigarrow$ $\ker\partial \cap \ker\bar\partial \cap (\operatorname{Im}\partial + \operatorname{Im}\bar\partial) = \operatorname{Im}\partial\bar\partial$.)

Let $(L, [\cdot,\cdot], d)$ be a DGLA.

We obtain a DGLA $(H^\bullet(L), [\cdot,\cdot], 0)$ $H^\bullet(L) = \bigoplus_i H^i(L)$.

<u>Def</u> $(L, [\cdot,\cdot], d)$ is called formal

if $(L, [\cdot,\cdot], d) \simeq_{q\text{-iso}} (H^\bullet(L), [\cdot,\cdot], 0)$.

<u>Prop</u> Formality (Simpson '92, Goldman-Millson '88)

Let $(E, \bar\partial_E, \theta, h)$: Harmonic bundle.

- $(A^\bullet(\mathrm{End}\,E), [\cdot,\cdot], D'')$ is formal.

$$D'' = \bar\partial_E + \theta$$

$\blacksquare$

pf) Let $H^i_{D''}$: i-th cohomology of $(\oplus_i A^i(\mathrm{End}\,E), D'')$.

Let $i: \ker D'_E \to A^i(\mathrm{End}\,E)$, $q: \ker D'_E \to H^i_{D''}$ be the natural $\begin{cases}\text{inclusion}\\ \text{projection.}\end{cases}$

Then i, q are quasi-isomorphisms of the following DGLAs:

$$i: (\ker D'_E, [\cdot,\cdot], D'') \longrightarrow (A^\bullet(\mathrm{End}\,E), [\cdot,\cdot], D''),$$

$\ker D'_E \cap A^\bullet(\mathrm{End}\,E)$

$$q: (\ker D'_E, [\cdot,\cdot], D'') \longrightarrow (H^\bullet_{D''}, [\cdot,\cdot], 0).$$

(Kähler Identity and $D'_E D'$-lemma plays an essential role.)

※ Formality of Kähler manifold was proved by Deligne - Griffiths - Morgan - Sullivan.

$\underline{\text{Thm}}$ Let $\cdot (X, \omega)$ Cpt kähler, $\cdot (E, \bar\partial_E, \theta, h)$ Harmonic bundle

Then $(L, [\cdot,\cdot]_L, d_L) \simeq_{q\text{-iso}} (A^\bullet(\mathrm{End}E), [\cdot,\cdot], D'') \oplus (A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_{T_X})$ ⊡

pf) Let $\cdot$ $(H^\bullet_{D_E'}, [\cdot,\cdot], 0)$ be the DGLA associated to $(A^\bullet(\mathrm{End}E), [\cdot,\cdot], D_E')$.

- i : $\ker D_E' \to A^i(\mathrm{End}E)$ inclusion,
- q : $\ker D_E' \to H^i_{D_E}$ projection.

We can prove

- $(\ker D_E' \oplus A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_L, d_L)$ is a DGLA.

$\ker D_E' = \ker D_E' \cap A^\bullet(\mathrm{End}E)$

- $\begin{pmatrix} i & 0 \\ 0 & \mathrm{Id} \end{pmatrix} : (\ker D_E' \oplus A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_L, d_L) \to (L, [\cdot,\cdot]_L, d_L)$

is a quasi-isomorphism of DGLA.

$\cdot \begin{pmatrix} q & 0 \\ 0 & \mathrm{Id} \end{pmatrix} : (\ker D_E' \oplus A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_L, d_L)$

$$\to (H_{D_E''}, [\cdot,\cdot], 0) \oplus (A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_{T_X})$$

is a quasi-isomorphism of DGLA.

Then we have the following

$$(L, [\cdot,\cdot]_L, d_L) \overset{\begin{pmatrix} i & 0 \\ 0 & \mathrm{Id} \end{pmatrix}}{\simeq_{q\text{-iso}}} (\ker D_E' \oplus A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_L, d_L)$$

$$\overset{\begin{pmatrix} q & 0 \\ 0 & \mathrm{Id} \end{pmatrix}}{\simeq_{q\text{-iso}}} (H^\bullet_{D_E''}, [\cdot,\cdot], 0) \oplus (A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_{T_X})$$

Simpson, Goldman–Millson.

$$\simeq_{q\text{-iso}} (\ker D_E', [\cdot,\cdot], D'') \oplus (A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_{T_X})$$

$$\simeq_{q\text{-iso}} (A^\bullet(\mathrm{End}\,E), [\cdot,\cdot], D'') \oplus (A^{0,\bullet}(T^{1,0}X), [\cdot,\cdot]_{SN}, \bar\partial_{T_X}) \quad \blacksquare$$

Thank you

for listening!

Why Japanese people

say Kobayashi-Hitchin

instead of Hitchin-Kobayashi?

In alphabet order.

a b c d e f g h i j k ...

$\rightsquigarrow$ Hitchin-Kobayashi is correct.

Hiragana
In ひらがな (Japanese alphabet)

Kobayashi - Hitchin is

こばやし − ひっちん.

ひらがな

⑤	④	③	②	①
は	た	さ	か	あ
ひ	ち	し	き	い
ふ	つ	す	く	う
へ	て	せ	け	え
ほ	と	そ	こ	お

In ひらがな order

Kobayashi - Hitchin

is correct.

ひらがな

...は	た	...さ	②か	①あ
ひ	ち	し	き	い
...ふ	つ	す	く	う
へ	て	さ	け	え
ほ	と	そ	こ	お

This might be feel strange

but...

历太微以过紫宫兮、
摅幽衣而奔驰
临无地之杳冥兮、
恐舟车之覆毁.

read by
張衡
(78~139).

He also calcurated
$\pi \approx 3.16$.

Thank you

for attention !