Calabi–Yau Geometry Behind Tensor-Product BPS Quivers

Philsang Yoo

joint work in progress with Sangjin Lee

Seoul National University

The Geometry and Physics of Higgs Bundles
Kavli IPMU
June 24th, 2026

Introduction

Where Higgs bundles come from in Physics

- Higgs bundles enter physics through at least two large gateways.
  - Geometric Langlands: $d = 4$, $\mathcal{N} = 4$ supersymmetric Yang–Mills theories
  - (parabolic/meromorphic) Hitchin system: $d = 4$, $\mathcal{N} = 2$ class $\mathcal{S}$ theories $\mathcal{S}[G, C, D]$

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- This talk has more to do with Higgs bundles from class $\mathcal{S}$ theories:

physics	math
$d = 4$, $\mathcal{N} = 2$ theory $\mathcal{T}$ on $\mathbb{R}^2 \times \mathbb{C}$	3-Calabi–Yau category $\mathcal{C}$
Hilbert space of quantum mechanics on $\mathbb{R} \subset \mathbb{R}^2$	$\mathcal{H}_{\mathcal{C}}$ cohomological Hall algebra
category $\mathcal{L}(\mathcal{T})$ of line defects	$\mathcal{H}_{\mathcal{C}}\text{-mod}_{\mathbb{E}_1\text{-chiral}}$
Coulomb branch $\mathcal{M}_C(\mathcal{T})_{\mathrm{generic}}$	$\mathrm{HH}_\bullet(\mathcal{H}_{\mathcal{C}}\text{-mod}_{\mathbb{E}_1\text{-chiral}})$
$\mathcal{M}_C(\mathcal{S}[G, C, D])_{\mathrm{special}} \cong \{G\text{-Higgs bundles on } (C, D)\}$	???

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Goal of Today's talk

Understand $\mathcal{C}$ better to understand $\mathcal{T}$ mathematically.

Rashomon [Akira Kurosawa, 1950]

1. Physics (BPS quivers of $d = 4$, $\mathcal{N} = 2$ theories from string theory)
2. Algebra (Calabi–Yau associated to tensor product of path algebras)
3. Geometry (symplectic topology of Lefschetz fibrations)

PART I

Physics

From Calabi–Yau singularities to BPS quivers

$d = 4$, $\mathcal{N} = 2$ SCFTs from Type IIB String Theory

[Cecotti–Neitzke–Vafa, 2010]

CNV introduced and studied a family of $d = 4$, $\mathcal{N} = 2$ SCFTs geometrically engineered by Type IIB string theory near the isolated Calabi–Yau threefold singularity $X_{G,G'}$ for ADE types G, G'.

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G	$f_G(x,y)$
A_k	$x^{k+1} + y^2$
D_k	$x^{k-1} + xy^2$
E_6	$x^4 + y^3$
E_7	$x^3y + y^3$
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Slogan for $d = 4$, $\mathcal{N} = 2$ SCFTs from String Theory

Everything about $\mathcal{T}[X_{G,G'}]$ is encoded by $X_{G,G'}$.

Class $\mathcal{S}$ and $\mathcal{T}[X_{G,G'}]$

Class $\mathcal{S}$ theories

$$\begin{array}{c} \text{6d } \mathcal{N} = (2,0) \text{ theory} \quad \mathfrak{X}[\mathfrak{g}] \\ \Big\downarrow {\scriptstyle (C,D)} \\ \text{4d } \mathcal{N} = 2 \text{ theory} \quad \mathcal{S}[\mathfrak{g}, C, D] \end{array}$$

where $\mathfrak{g}$ is an ADE Lie algebra, C is a closed Riemann surface, and D is a decoration at marked points of C.

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The (A_k, G')-theories are known to be class $\mathcal{S}$ theories for any k. For example, if $G = A_1$,

G'	$\mathfrak{g}$	C	D in terms of quadratic differential φ
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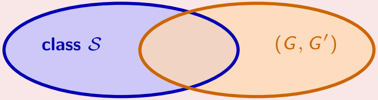
Relationship between class $\mathcal{S}$ and (G, G') theories



BPS Quiver

"Definition" of BPS Quiver

For a $d = 4$ $\mathcal{N} = 2$ theory $\mathcal{T}$ and a choice of vacuum u, a BPS quiver $Q^u_{\mathrm{BPS}}(\mathcal{T})$ is a finite quiver with potential whose vertices are elementary BPS charges and whose signed arrows record the Dirac pairing.

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CNV predicted that the BPS-quiver mutation class of $\mathcal{T}[X_{G,G'}]$ contains the triangle-product quiver with potential $(\Gamma_G \boxtimes \Gamma_{G'}, W_{G,G'})$, where $\Gamma_G \boxtimes \Gamma_{G'}$ is called the *triangle product* of quivers.

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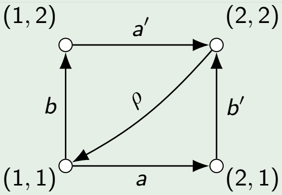

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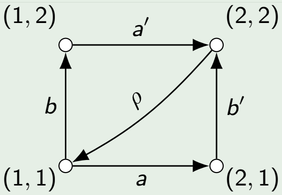

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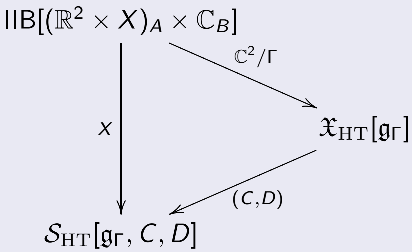
Question

Can one recover $(\Gamma_G \boxtimes \Gamma_{G'}, W_{G,G'})$ directly from Calabi–Yau geometry of $X_{G,G'}$?

Twisted String Theory to Fukaya Categories

Twisted Approach to Class $\mathcal{S}$ theory

There is a version of supersymmetric field theory and string theory



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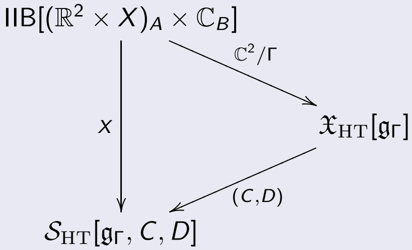

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Corollary of Slogan

Everything about $\mathcal{T}_{\mathrm{HT}}[X_{G,G'}]$ is encoded by "$\mathrm{Fuk}(X_{G,G'})$".

Main Result for Physics

Physics "Result"

Let T, T' be trees. We construct a smooth CY 3-fold $Y_{T,T'}$ and hence $\mathcal{T}_{\mathrm{HT}}[Y_{T,T'}]$ such that

- if $(T, T') = (\Gamma_G, \Gamma_{G'})$, then $Y_{T,T'} = \widetilde{X_{G,G'}}$ is a resolution of $X_{G,G'}$;

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The construction extends to thickened trees $\mathbb{T}, \mathbb{T}'$: finite quivers whose underlying simple unoriented graph is a tree, but in which an edge may be replaced by several parallel arrows, all in one direction.

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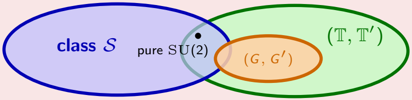

PART II

Algebra

From square relations to Calabi–Yau geometry

Calabi–Yau completion

Definition [Keller, 2009]

Let A be a homologically smooth DG algebra or DG category. Let $A^e = A \otimes A^{\mathrm{op}}$ and $\Theta_A = \mathbf{R}\mathrm{Hom}_{A^e}(A, A^e)$, where Θ_A is the *inverse dualizing bimodule*. The *n-Calabi–Yau completion of A* is

$$\Pi_n(A) = T_A(\Theta_A[n-1]) = A \oplus \Theta_A[n-1] \oplus (\Theta_A[n-1] \otimes_A \Theta_A[n-1]) \oplus \cdots .$$

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Examples

d-dimensional input A	Calabi–Yau completion Π_{d+1}
$A = \Bbbk Q$: path algebra of quiver Q	$\Pi_2(\Bbbk Q)$ is the derived preprojective algebra of Q
$\mathrm{Perf}(A) \simeq \mathrm{Perf}(X)$	$\Pi_{d+1}(A)$ models the category of coherent sheaves of $\mathrm{Tot}(K_X)$
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Idea

A concrete way to produce a 3-Calabi–Yau category is to start from a smooth algebra of global dimension 2 and apply Π_3.

Tensor Product of Path Algebras

Let Q, Q' be quivers.

Tensor product of path algebras

The "2-dimensional" algebra $A = \Bbbk Q \otimes \Bbbk Q'$ can be presented by a quiver with

- vertices (i, j);
- horizontal arrows $(\alpha, j) : (i, j) \to (i', j)$ for arrows $\alpha : i \to i'$ in Q;
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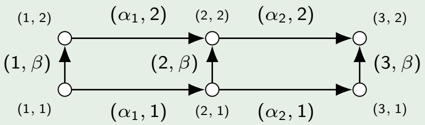

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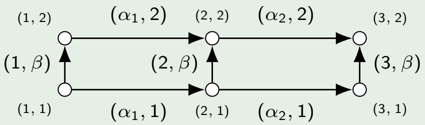

Question

What is $\Pi_3(\Bbbk Q \otimes \Bbbk Q')$?

3-Calabi–Yau Completion of the Tensor Product

3-Calabi–Yau completion of $A = \mathbb{k}Q \otimes \mathbb{k}Q'$ [Keller]

The completion $\Pi_3(A)$ is described by a Ginzburg DG algebra of a quiver with potential: add one arrow dual to each square relation and use the corresponding cubic potential.

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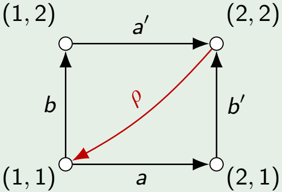

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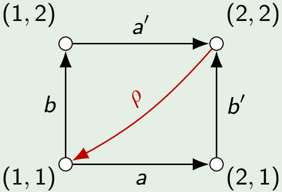

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Geometric question

Can this algebraically defined 3-Calabi–Yau completion be realized by a concrete Calabi–Yau 3-fold?

Examples from Algebraic Geometry

Tensor Product

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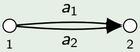

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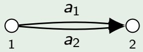

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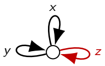
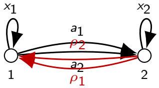
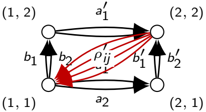
	$\mathbb{A}^1 \times \mathbb{A}^1$	$\mathbb{A}^1 \times \mathbb{P}^1$	$\mathbb{P}^1 \times \mathbb{P}^1$
Tensor-product algebra	$A = \Bbbk[x] \otimes \Bbbk[y] \cong \Bbbk[x, y]$	$A = \Bbbk[x] \otimes \Bbbk\,\mathrm{Kr}_2$	$A = \Bbbk\,\mathrm{Kr}_2 \otimes \Bbbk\,\mathrm{Kr}_2$
Quiver with potential			
Relations and potential	$r = xy - yx$, $W = zr$	$r_i = x_2 a_i - a_i x_1, \; i = 1, 2$ $W = \rho_1 r_1 + \rho_2 r_2$	$r_{ij} = b'_j a_i - a'_i b_j, \; i, j = 1, 2$ $W = \sum_{i,j} \rho_{ij} r_{ij}$
3-CY geometry	$\mathrm{Tot}(K_{\mathbb{A}^2}) \cong \mathbb{A}^3$	$\mathrm{Tot}\big(K_{\mathbb{A}^1 \times \mathbb{P}^1}\big) \cong \mathbb{A}^1 \times T^*\mathbb{P}^1$	$\mathrm{Tot}\big(K_{\mathbb{P}^1 \times \mathbb{P}^1}\big) = \mathrm{Tot}\big(\mathcal{O}(-2, -2)\big)$

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Generalization (work in progress)

The construction extends to thickened trees $\mathbb{T}, \mathbb{T}'$: finite quivers whose underlying simple unoriented graph is a tree, but in which an edge may be replaced by several parallel arrows, all in one direction.

PART III

Geometry

From vanishing cycles to tensor-product quivers

Lefschetz Fibrations

A complex-valued Morse function

A *(complex) Lefschetz fibration* is a map $f : X^n \longrightarrow \mathbb{C}$ with finitely many critical points, locally modeled by $f(z_1, \ldots, z_n) = f(p) + z_1^2 + \cdots + z_n^2$. For a regular value t_0, the fiber $F = f^{-1}(t_0)$ is a smooth $(2n-2)$-manifold.

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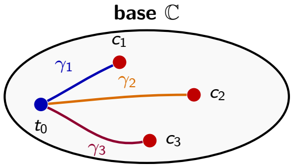
Vanishing cycles

Choose a path γ_i from t_0 to a critical value c_i. Parallel transport along γ_i picks out a Lagrangian sphere $V_i \cong S^{n-1} \subset F$ which collapses at the fiber over c_i.



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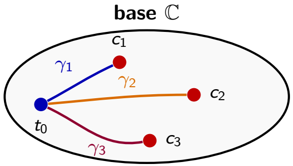


Abstract Lefschetz data

After choosing a basepoint and distinguished vanishing paths, a *marked* exact Lefschetz fibration is encoded, up to deformation, by

$$(F; V_1, \dots, V_m),$$

where $V_i \subset F$ are ordered exact Lagrangian spheres. Changing the distinguished paths changes this collection by Hurwitz moves involving Dehn twists.

The A_1 local model

Lefschetz fibration

Consider $f : \mathbb{C}^2 \to \mathbb{C}$ given by $f(x, y) = x^2 + y^2$.

Fibers

Set $u = x + iy$, $v = x - iy$ so that $f = uv$. Then

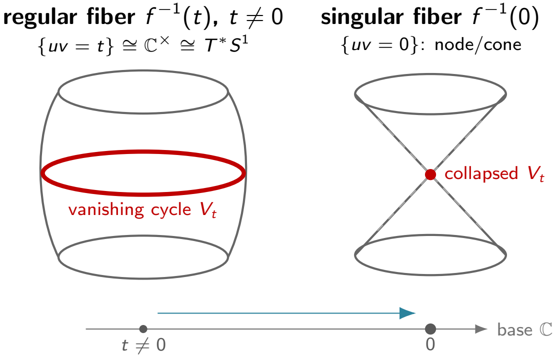
$$f^{-1}(t) = \{uv = t\} \cong \mathbb{C}^\times \cong T^*S^1 \quad (t \neq 0),$$
$$f^{-1}(0) = \{uv = 0\}.$$

Vanishing cycle

For example, for fixed $t = r^2 > 0$,

$$V_t = \{u = re^{i\theta},\ v = re^{-i\theta}\} \cong S^1.$$

As $t \to 0$, V_t shrinks to the critical point $(0, 0)$.



The A_2 local model

Vanishing paths

Consider $f(x,y) = x^3 - 3x + y^2$. Its critical values are 2 and -2. Take the regular value $t_0 = 0$ and the real vanishing paths $\gamma_1 : [0,2]$ and $\gamma_2 : [0,-2]$.

Regular fiber as a double cover

As $F = f^{-1}(0) = \{y^2 = 3x - x^3\}$, projection $\pi(x,y) = x$ is a double cover of $\mathbb{C}_x$ branched at $q_1 = -\sqrt{3}$, $q_2 = 0$, $q_3 = \sqrt{3}$. Topologically, F is a once-punctured torus.

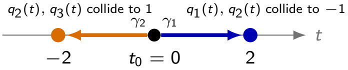
Vanishing circles

$$V_1 = \left\{ \left(x, \pm i\sqrt{x^3 - 3x}\right) : -\sqrt{3} \leq x \leq 0 \right\},$$

$$V_2 = \left\{ \left(x, \pm \sqrt{3x - x^3}\right) : 0 \leq x \leq \sqrt{3} \right\}.$$

Equivalently, $V_1 = \pi^{-1}([q_1, q_2])$, $V_2 = \pi^{-1}([q_2, q_3])$. They meet transversely once at $(0,0)$.

base of f



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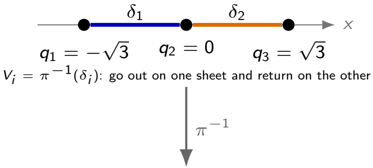

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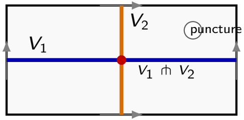

Plumbing cotangent bundles

Local exact-symplectic gluing

Start with two disk cotangent bundles $P_1 = D^*S^n$, $P_2 = D^*S^n$. Choose small disks in their zero sections and cotangent coordinates $T^*D^n \simeq D^n_q \times D^n_p$, $T^*D^n \simeq D^n_Q \times D^n_P$. The *plumbing identification* is $\phi(q, p) = (Q, P) = (p, -q)$.

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Geometric meaning

The first zero section $V_1 = \{p = 0\}$ is sent to the cotangent-fiber direction $\phi(V_1) = \{Q = 0\}$ in the second chart. It therefore meets the second zero section $V_2 = \{P = 0\}$ transversely at one point.

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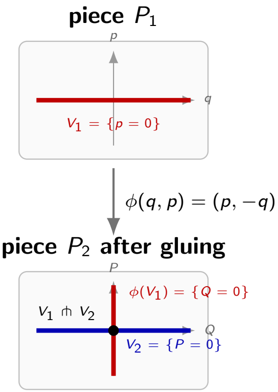

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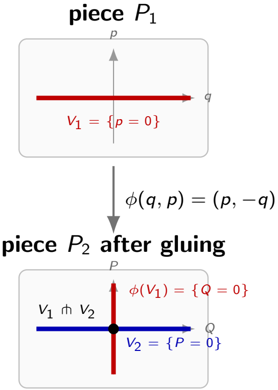

Plumbing along a graph

For a quiver Q, take one copy of D^*S^n for each vertex and perform one plumbing for each edge. The resulting space F_Q together with core spheres $\{V_v\}_{v \in Q_0}$ determines a Lefschetz fibration f_Q. If $Q = \Gamma_{\mathrm{ADE}}$, then $f_{\Gamma_{\mathrm{ADE}}} = f_\Gamma : \mathbb{C}^2 \to \mathbb{C}$ recovers the singularity associated to Γ.

Main Result

Theorem [Lee–Y.]

Let T, T' be trees. There is an equivalence of categories

$$\mathsf{FinDim}(Q_{T,T'}, W_{T,T'}) \simeq \mathrm{Fuk}_{\mathrm{van}}(Y_{T,T'})$$

where the left-hand side refers to the category of finite-dimensional DG modules over the Ginzburg algebra, $Y_{T,T'}$ is a generic fiber of the Thom–Sebastiani sum $f_T \boxplus f_{T'} : \mathbb{C}^4 \to \mathbb{C}$, and $\mathrm{Fuk}_{\mathrm{van}}(Y_{T,T'})$ refers to the full subcategory generated by vanishing cycles.

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Generalization (work in progress)

For a thickened tree $\mathbb{T}$, the domain of the corresponding Lefschetz fibration $f_{\mathbb{T}}$ is not necessarily $\mathbb{C}^2$.

Three Lessons

1. **For physics: the BPS quiver has a geometric origin.**
   For ADE pairs, the tensor-product quiver with potential predicted by Cecotti–Neitzke–Vafa is realized by vanishing 3-spheres in the smoothing of the corresponding Calabi–Yau singularity. The same mathematical construction extends beyond ADE to (thickened) trees.

2. **For algebra: the Calabi–Yau completion is geometric.**
   In this family, Keller's abstract 3-Calabi–Yau completion $\Pi_3\big(\mathbb{k}T \otimes_{\mathbb{k}} \mathbb{k}T'\big)$ admits a concrete Fukaya-theoretic model: every commuting-square relation produces a reverse diagonal arrow and a corresponding cubic potential term.

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Summary

In this family, tensor-product BPS quivers, 3-Calabi–Yau completions of a tensor product of path algebras, and Fukaya categories of Thom–Sebastiani fibers are three faces of one construction.