The Dolbeault geometric Langlands conjecture for type A groups beyond the elliptic locus

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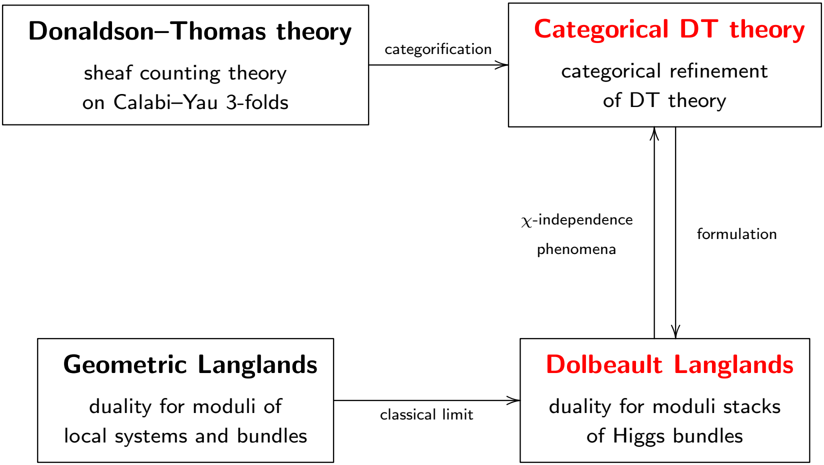
The Geometry and Physics of Higgs Bundles
June 2026

Why am I interested in Higgs bundles?



1. Dolbeault geometric Langlands correspondence

Higgs bundles

Let C be a smooth projective curve of genus g over k, and let G be a reductive group with Langlands dual ${}^L G$.

A G-Higgs bundle on C consists of

$$(E, \theta), \ \theta \in \Gamma((E \times^G \mathrm{Ad}(\mathfrak{g})) \otimes \Omega_C)$$

where $E \to C$ is a G-bundle, and $\mathfrak{g}$ is the Lie algebra of G.

We denote by

$$\mathrm{Higgs}_G (\simeq \Omega_{\mathrm{Bun}_G})$$

the derived moduli stack of G-Higgs bundles on C, *without stability condition*.

Dual fibrations

Donagi–Pantev proved that the Hitchin fibrations

$$\begin{array}{ccc} \mathrm{Higgs}_{^L G} & & \mathrm{Higgs}_G \\ & {}^L h \searrow \quad \swarrow h & \\ & B & \end{array}$$

are dual abelian fibrations over a dense open subset $B' \subset B$.
Then there is a Fourier–Mukai equivalence at the generic point $\eta \in B$

$$\Phi_\eta \colon \mathrm{IndCoh}_{\mathcal{N}}({}^L h^{-1}(\eta)) \simeq \mathrm{IndCoh}_{\mathcal{N}}(h^{-1}(\eta)).$$

‘Conjecture’ (Donagi–Pantev)

The equivalence Φ_η extends to a B-linear equivalence

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_{^L G}) \simeq \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_G).$$

Classical limit of Geometric Langlands correspondence

The Donagi–Pantev proposal is motivated by a classical limit of the geometric Langlands correspondence, and is called *Dolbeault geometric Langlands conjecture*:

$$
\begin{array}{ccc}
\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{{}^L G}) & \xrightarrow[\text{GLC (Gaitsgory et al.)}]{\simeq} & \text{D-mod}(\mathrm{Bun}_G) \\
\text{non-abelian Hodge} \Big\downarrow & & \Big\downarrow \text{classical limit of deformation quantization} \\
\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_{{}^L G}) & \xrightarrow[\text{Dolbeault Langlands}]{?\ \simeq\ ?} & \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_G).
\end{array}
$$

Example: $G = \mathrm{GL}_r$

For $G = \mathrm{GL}_r$, giving a Higgs bundle is equivalent to

$$(F, \theta), \; F \to C, \theta \colon F \to F \otimes \Omega_C$$

where F is a rank r vector bundle. The Hitchin map is

$$h \colon \mathrm{Higgs}_{\mathrm{GL}_r} \to B = \bigoplus_{i=1}^{r} H^0(\Omega_C^i)$$

given by $h(F, \theta) = (\mathrm{tr} \wedge^i \theta)_{1 \le i \le r}$. Each point $b \in B$ corresponds to the *spectral curve*

$$\mathcal{C}_b = \{(x, \lambda) \in \mathrm{Tot}_C(\Omega_C) : \det(\lambda - \theta_x) = 0\}$$

and $h^{-1}(b)$ is identified with the moduli stack of torsion-free sheaves on $\mathcal{C}_b$ of fundamental cycle $[\mathcal{C}_b]$.

Elliptic locus

For $G = \mathrm{GL}_r$, the *elliptic locus* is defined by the open subset

$$B^{\mathrm{ell}} := \{b \in B : \mathcal{C}_b \text{ is integral}\} \subset B.$$

For $b \in B^{\mathrm{ell}}$, the Hitchin fiber is the compactified Jacobian

$$h^{-1}(b) = \overline{J}_{\mathcal{C}_b}.$$

Arinkin proved that the Fourier–Mukai transform is extended to compactified Jacobians of integral curves. In particular:

Theorem (Arinkin, 2013)

The equivalence Φ_η extends over B^{ell}, i.e. the Dolbeault geometric Langlands holds for GL_r over B^{ell}.

Issue of Dolbeault geometric Langlands

However proving a desired equivalence

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_{^L G}) \simeq \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_G).$$

beyond the elliptic locus seems hopeless at this moment (and probably it does not hold as it is).

Indeed, there has been no result beyond B^{ell} because, over any open subset $B^{\circ} \subset B$ which strictly contains B^{ell}, we have

- the stack Higgs_G is a singular stack (not a scheme);
- each component of Higgs_G is not quasi-compact unless G is a torus, which yields that $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_G)$ is *not* compactly generated (while the dg-categories in GLC are compactly generated).

Better degeneration?

On the spectral side, we have a better degeneration to *semistable* Higgs bundles

$$\mathrm{LocSys}_G \rightsquigarrow \mathrm{Higgs}_G^{\mathrm{ss}}$$

Each connected component of $\mathrm{Higgs}_G^{\mathrm{ss}}$ is quasi-compact, and

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_G^{\mathrm{ss}})$$

is compactly generated.

Question

Is there a better degeneration on the automorphic side to a compactly generated dg-category

$$\text{D-mod}(\mathrm{Bun}_G) \quad \rightsquigarrow \quad ?$$

Beyond the elliptic locus via limit categories

I will propose an ansatz for this question via a limit category

$$\text{D-mod}(\mathrm{Bun}_G) \stackrel{?}{\rightsquigarrow} \mathrm{IndL}_{\mathcal{N}}(\mathrm{Higgs}_G)$$

motivated by categorical DT theory, and give a new formulation of the Dolbeault geometric Langlands conjecture of the form

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}^{\mathrm{ss}}_{^L G}) \simeq \mathrm{IndL}_{\mathcal{N}}(\mathrm{Higgs}_G).$$

An upshot is a proof of this equivalence beyond the elliptic locus:

Main result

There is an open subset $B^{\circ} \subset B$ which strictly contains the elliptic locus such that the Dolbeault geometric Langlands conjecture (via limit categories) holds.

2. Formulation via limit categories

Limit category

Let $\mathfrak{M}$ be a quasi-smooth derived stack with self-dual cotangent complex $\mathbb{L}_{\mathfrak{M}}$, and let $\delta \in \mathrm{Pic}(\mathfrak{M})_{\mathbb{R}}$. Then the *limit category*

$$\mathrm{IndL}_{\mathcal{N}}(\mathfrak{M})_{\delta} \subset \mathrm{IndCoh}_{\mathcal{N}}(\mathfrak{M})$$

is defined so that its locally compact objects satisfy a suitable $\mathbb{G}_m$-weight condition with respect to all maps $\nu \colon B\mathbb{G}_m \to \mathfrak{M}$. For a perfect complex $\mathcal{E}$, the weight condition is

$$\mathrm{wt}(\nu^*\mathcal{E}) \subset \left[\frac{1}{2}c_1(\nu^*\mathbb{L}_{\mathfrak{M}}^{<0}), \frac{1}{2}c_1(\nu^*\mathbb{L}_{\mathfrak{M}}^{>0})\right] + c_1(\nu^*\delta)$$

Here for $A = \oplus_w A_w \in \mathrm{Coh}(B\mathbb{G}_m)$, $c_1(A^{<0}) := \sum_{w<0} w \cdot \dim A_w$.

Relations to D-modules?

We expect that the limit category provides a better degeneration of D-modules for a smooth stack $\mathcal{X}$

$$\text{D-mod}(\mathcal{X}) \rightsquigarrow \mathrm{IndL}_{\mathcal{N}}(\Omega_{\mathcal{X}})_{\delta=\omega_{\mathcal{X}}^{1/2}}$$

Examples

(1) For $\mathcal{X} = BG$ for a reductive G, ($n = \dim \mathfrak{h}$, $\deg x_i = i$)

$$\text{D-mod}(BG) \simeq \mathrm{IndL}_{\mathcal{N}}(\Omega_{BG})_{\frac{1}{2}} \simeq \mathrm{QCoh}(k[x_{-1}, x_{-3}, \ldots, x_{1-2n}]).$$

(2) For $\mathcal{X} = \mathbb{A}^1/\mathbb{G}_m$,

$$\text{D-mod}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{IndL}_{\mathcal{N}}(\Omega_{\mathbb{A}^1/\mathbb{G}_m})_{\frac{1}{2}} = \langle \mathrm{QCoh}(k[-1]), \mathrm{QCoh}(\mathrm{pt}) \rangle.$$

Where does the weight condition come from?

The weight condition

$$\mathrm{wt}(\nu^*\mathcal{E}) \subset \left[\frac{1}{2}c_1(\nu^*\mathbb{L}_{\mathfrak{M}}^{<0}),\ \frac{1}{2}c_1(\nu^*\mathbb{L}_{\mathfrak{M}}^{>0})\right] + c_1(\nu^*\delta)$$

is a reformulation/generalization of ideas outside the research areas of Higgs bundles, geometric Langlands, etc:

- Spenko-Van den Bergh's construction of non-commutative (crepant) resolutions of GIT quotients;
- Halpern-Leistner-Sam's magic window theory to prove D/K hypothesis in categorical birational geometry;
- Categorification of BPS invariants in Donaldson-Thomas theory (Pădurariu–T).

Why is the weight condition natural for Higgs bundles?

Proposition

For $b \in B$ and the spectral curve $\mathcal{C}_b \subset S$, let

$$\mathcal{P} \to \mathcal{P}ic(\mathcal{C}_b) \times h^{-1}(b)$$

be the line bundle, defined by the Deligne pairing

$$\mathcal{P}|_{(E,F)} = \det \chi(E \otimes F) \otimes \det \chi(E)^{-1} \otimes \det \chi(F)^{-1} \otimes \det \chi(\mathcal{O}_{\mathcal{C}_b}).$$

Then for a line bundle E on $\mathcal{C}_b$ of degree w, it corresponds to a semistable Higgs bundle if and only if the associated line bundle $\mathcal{P}_E$ on $h^{-1}(b)$ satisfies the following: for any map $\nu \colon B\mathbb{G}_m \to h^{-1}(b)$, we have

$$\mathrm{wt}(\nu^* \mathcal{P}_E) \subset \left[\frac{1}{2} c_1(\nu^* \mathbb{L}^{<0}_{h^{-1}(b)}), \frac{1}{2} c_1(\nu^* \mathbb{L}^{>0}_{h^{-1}(b)}) \right] + c_1(\nu^* \delta_w).$$

Dolbeault geometric Langlands conjecture

Conjecture (Pădurariu–T, arXiv:2508.19624)

For $\chi \in \pi_1(G)$ and $w \in Z(G)^\vee$, there is a B-linear equivalence of compactly generated dg-categories

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_{^L G}(w)^{\mathrm{ss}})_{-\chi} \simeq \mathrm{IndL}_{\mathcal{N}}(\mathrm{Higgs}_G(\chi))_w.$$

Here $\mathrm{Higgs}_G(\chi) \subset \mathrm{Higgs}_G$ is the connected component, $Z(G)$ is the center of G and $(-)_w$ means the central weight w part.

3. Key properties on spectral/automorphic sides

IndCoh for semistable Higgs moduli (spectral side)

The spectral side is

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_{{}^L G}(w)^{\mathrm{ss}})_{-\chi}.$$

- It is compactly generated (Drinfeld-Gaitsgory).
- For $G = \mathrm{GL}_r$, it is equivalent to the category of matrix factorizations

$$\mathrm{MF}_{\mathcal{N}}(\mathcal{Y}, f)$$

for a function f on a smooth stack $\mathcal{Y}$ whose critical locus is the moduli stack of semistable sheaves on the non-compact CY 3-fold

$$X = \mathrm{Tot}_C(\Omega_C \oplus \mathcal{O}_C).$$

It is understood as a categorical DT theory for X.

Limit categories for full Higgs moduli (automorphic side)

The automorphic side is

$$\mathrm{IndL}_{\mathcal{N}}(\mathrm{Higgs}_G(\chi))_w.$$

We have developed its general theory, which indicates that this should be regarded as an algebro-geometric category of A-branes:

- It is compactly generated with compact objects $\mathrm{L}_{\mathcal{N}}(\mathrm{Higgs}_G(\chi))_w$ (Păduraru–T).
- They satisfy functorial properties with respect to (some) Lagrangian correspondences. In particular, they admit Hecke operators.
- There is a !-push-forward $f_!$ for some class of non-proper maps f, which does not exist for IndCoh or QCoh.

Semiorthogonal decompositions

Theorem (Pădurariu–T, Bu–Pădurariu–T, arXiv:2605.25976)

The compact objects on both sides admit semiorthogonal decompositions

$$\langle \mathbb{T}_M(\chi_M)_{w_M} : G \supset P \twoheadrightarrow M, \cdots \rangle$$

where $G \supset P$ is a parabolic subgroup with Levi quotient M, and

$$\mathbb{T}_G(\chi)_w := \mathrm{L}_{\mathcal{N}}(\mathrm{Higgs}_G(\chi)^{\mathrm{ss}})_w$$

is the quasi-BPS category; for $G = \mathrm{GL}_r$ and (r, χ, w) primitive, it is smooth, proper and CY over B, and categorifies the BPS invariant on the non-compact CY 3-fold

$$X = \mathrm{Tot}(\Omega_C \oplus \mathcal{O}_C).$$

4. Dolbeault geometric Langlands beyond the elliptic locus

Dolbeault geometric Langlands beyond B^{ell}

Theorem (T for $G = \mathrm{GL}_2$, arXiv:2602.09359; T, to appear)

For $G = \mathrm{GL}_r$, SL_r or PGL_r, there is an equivalence

$$\Phi \colon \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Higgs}_{{}^L G}(w)^{\mathrm{ss}})_{-\chi} \xrightarrow{\sim} \mathrm{IndL}_{\mathcal{N}}(\mathrm{Higgs}_G(\chi))_w$$

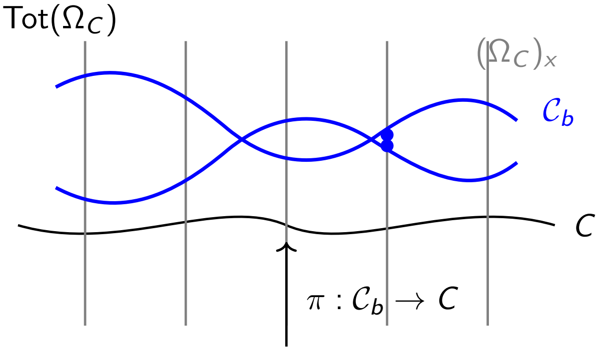
over the open subset

$$B^{\circ} = B^{\mathrm{ell}} \cup B^A \subset B$$

where B^A corresponds to spectral curves $\mathcal{C}_b$ with at worst A-type singularities $y^2 = x^m$ (with projection to C given by $(x, y) \mapsto x$), allowing any number of irreducible components $\leq r$.

Corollary

For $G = \mathrm{GL}_2$, SL_2 or PGL_2, the Dolbeault geometric Langlands holds over $B^{\mathrm{red}} \subset B$, corresponding to reduced spectral curves.



Beyond the elliptic locus

Note that over B^{ell}, we have

$$\mathrm{Higgs}_G \times_B B^{\mathrm{ell}} = \mathrm{Higgs}_G^{\mathrm{ss}} \times_B B^{\mathrm{ell}} = \mathrm{Higgs}_G^{\mathrm{st}} \times_B B^{\mathrm{ell}}$$

and each component of Higgs_G is quasi-compact over B^{ell}.

However over B° (with $B^{\mathrm{ell}} \subsetneq B^{\circ}$), we have

$$\mathrm{Higgs}_G \times_B B^{\circ} \supsetneq \mathrm{Higgs}_G^{\mathrm{ss}} \times_B B^{\circ} \supsetneq \mathrm{Higgs}_G^{\mathrm{st}} \times_B B^{\circ}$$

and each component of Higgs_G is not quasi-compact over B°.

The theory of limit categories is essential both to the formulation and to the proof of Dolbeault geometric Langlands over B°.

Key ingredients of the proof of the main theorem

(i) Limit categories on non-quasi-compact moduli stacks.

(ii) Left adjoint for non-proper maps (e.g. Hitchin section s) in the limit category.

(iii) Fourier–Mukai transform via Arinkin's Cohen–Macaulay sheaf

(iv) Whittaker normalization: $\mathcal{O}_{\mathrm{Higgs}_G^{\mathrm{ss}}} \mapsto s_! \mathcal{O}_B$

(v) Wilson/Hecke compatibility

(vi) Compact generation by Wilson/Hecke operators

Summary and future directions

Summary

- A new formulation of the Dolbeault geometric Langlands conjecture, inspired by categorical DT theory;
- A proof of this conjecture beyond the elliptic locus, for type A groups.

Future directions

- Extend the proof to the full Hitchin base;
- Develop the meromorphic and parabolic versions;
- Formulate Dolbeault-type geometric Langlands conjectures for more general Calabi–Yau 3-folds, related to categorical χ-independence phenomena.