An Introduction to Open Quantum Systems

Yukinao Akamatsu (National Institute of Technology, Matsue College)

Quantum Crossroads @Kavli IPMU, U. Tokyo

August 3-7, 2026

Why Open Quantum Systems?

In Matsue, there is a lake called Lake Shinji
Lake Shinji is not an isolated lake; it is connected to the ocean through Lake Nakaumi

- The salinity is about 10% of that of seawater, creating a unique ecosystem (e.g. Shijimi)

A unique nonequilibrium steady state is realized

Why Open Quantum Systems?

In Matsue, there is a lake called Lake Shinji
Lake Shinji is not an isolated lake; it is connected to the ocean through Lake Nakaumi

- The salinity is about 10% of that of seawater, creating a unique ecosystem (e.g. Shijimi)

A unique nonequilibrium steady state is realized

And Why Open Quantum Systems?

In some systems, the coupling to the environment is inevitable

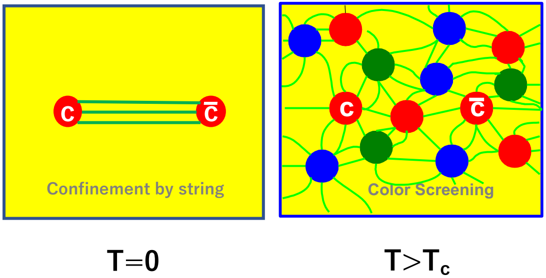
- Brownian motion
- Quarkonium in Quark-Gluon Plasma
- Primordial fluctuations during inflation

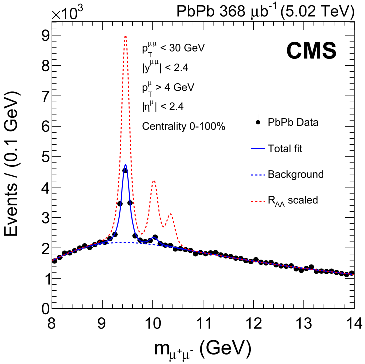
In other systems, the coupling to environment can be engineered

- Cavity QED
- Trapped ions
- Cold atoms

Brief Introduction to Quarkonium Suppression

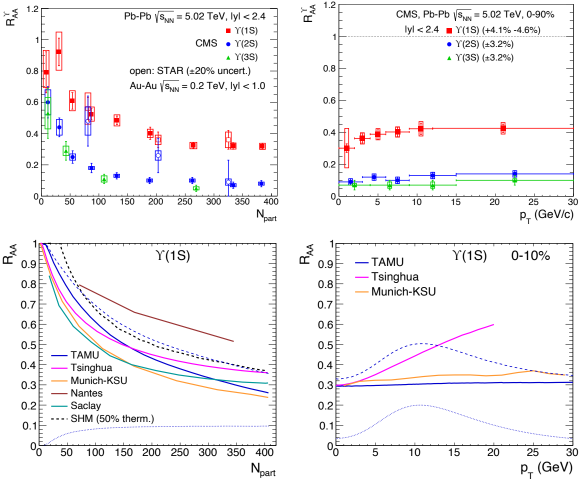
- Quark-Gluon Plasma (QGP) is a deconfined matter made of quarks and gluons
- J/ψ is no longer bound in QGP $\rightarrow$ Suppression signals the QGP creation? [Matsui-Satz (86)]





J/ψ and Υ is an open quantum system in QGP environment

Experimental Data and Simulations

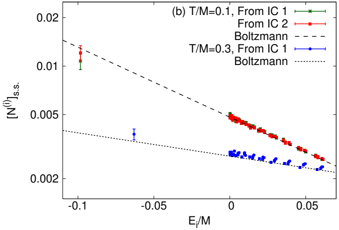


Some of the simulations are based on Stochastic Unraveling of Lindblad equation

Theoretical Results

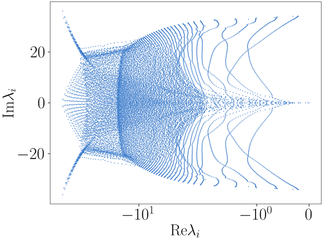
Thermalization

[Miura et al (22)]



Spectrum

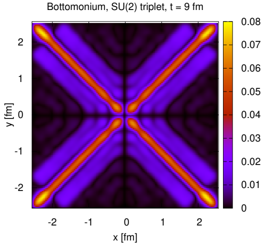
[Brambilla et al (26)]



Courtesy of Tom Magorsch

Strong Symmetry

[Akamatsu et al (22)]



Let us share the common tools and concepts across the fields!

Basics of Open Quantum Systems

Reduced Density Matrix

System $+$ Environment

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_S \otimes \mathcal{H}_E, \qquad H_{\text{tot}} = H_S + H_E + H_{SE}, \qquad \rho_{\text{tot}} = \sum_\alpha w_\alpha |\alpha\rangle\langle\alpha|$$

System observables $\mathcal{O}_S \otimes I_E \rightarrow$ Reduced density matrix $\rho_S := \mathrm{Tr}_E \rho_{\text{tot}}$

$$\langle \mathcal{O}_S \otimes I_E \rangle = \mathrm{Tr}_{\text{tot}} \left[\rho_{\text{tot}} \left(\mathcal{O}_S \otimes I_E\right)\right] = \mathrm{Tr}_S \Big[\underbrace{\left(\mathrm{Tr}_E \rho_{\text{tot}}\right)}_{=:\, \rho_S} \mathcal{O}_S\Big] = \mathrm{Tr}_S \left[\rho_S \mathcal{O}_S\right]$$

Time evolution of ρ_S

$$\rho_{\text{tot}}(t) = e^{-iH_{\text{tot}}t} \rho_{\text{tot}}(0)\, e^{iH_{\text{tot}}t} \qquad \longrightarrow \qquad \rho_S(t) = \mathrm{Tr}_E \left[e^{-iH_{\text{tot}}t} \rho_{\text{tot}}(0)\, e^{iH_{\text{tot}}t}\right]$$

Everything well-defined at the level of the total closed system

Dynamical Map and Kraus Representation

Initial condition: no correlation, fixed $\rho_E(0)$, and arbitrary $\rho_S(0)$

- A dynamical map $\mathcal{V}_{\mathrm{dyn}}(t) : \rho_S(0) \to \rho_S(t)$

$$
\begin{tikzcd}
\rho_S(0) \otimes \rho_E(0) = \rho_{\mathrm{tot}}(0) \arrow[r, "\mathrm{Tr}_E"] \arrow[d, "\text{Unitary evolution}"'] & \rho_S(0) \arrow[d, "{\text{Dynamical map} \atop \mathcal{V}_{\mathrm{dyn}}(t)}"'] \\
\rho_{\mathrm{tot}}(t) \arrow[r, "\mathrm{Tr}_E"] & \rho_S(t)
\end{tikzcd}
$$

Kraus representation for the dynamical map $\mathcal{V}_{\mathrm{dyn}}(t)$

$$
\rho_S(t) = \sum_\mu K_\mu(t) \rho_S(0) K_\mu^\dagger(t), \quad \sum_\mu K_\mu^\dagger(t) K_\mu(t) = I_S
$$

- Can be explicitly proved by diagonalizing $\rho_E(0)$

What Characterizes the Kraus Representation?

$$\rho_S(t) = \sum_\mu K_\mu(t)\rho_S(0)K_\mu^\dagger(t), \quad \sum_\mu K_\mu^\dagger(t)K_\mu(t) = I_S$$

Kraus representation guarantees

- Trace preservation

$$\mathrm{Tr}_S\left[\rho_S(t)\right] = \mathrm{Tr}_S\left[\rho_S(0)\right] = 1$$

- Preservation of Hermiticity and positivity

$$\begin{array}{lcl} \rho_S(0) = \rho_S^\dagger(0) & \longrightarrow & \rho_S(t) = \rho_S^\dagger(t) \\ \rho_S(0) \geq 0 & \longrightarrow & \rho_S(t) \geq 0 \end{array}$$

The converse is not true

- e.g., $\rho_S(0) \to \rho_S(0)^T$ satisfies both conditions, but does not admit a Kraus representaion

Complete Positivity

Consider extending the system by adding an ancilla

$$\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_S \otimes \mathcal{H}_E \quad \longrightarrow \quad \mathcal{H}'_{\mathrm{tot}} = (\mathcal{H}_S \otimes \mathcal{H}_A) \otimes \mathcal{H}_E$$

$$H_{\mathrm{tot}} = H_S + H_E + H_{SE} \quad \longrightarrow \quad H'_{\mathrm{tot}} = \underbrace{(H_S + H_A)}_{\text{no interaction}} + H_E + H_{SE}$$

Extended dynamical map $\mathcal{V}'_{\mathrm{dyn}}(t)$ (for $H_A = 0$)

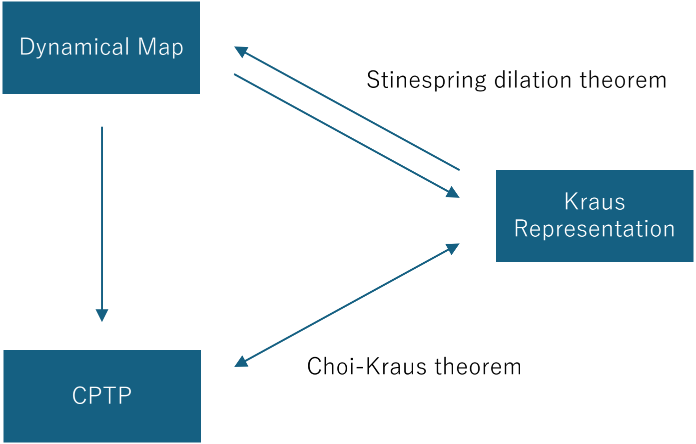
$$\mathcal{V}'_{\mathrm{dyn}}(t) = \mathcal{V}_{\mathrm{dyn}}(t) \otimes e^{-iH_A t}[\cdot]e^{iH_A t} = \mathcal{V}_{\mathrm{dyn}}(t) \otimes \mathcal{I}_A$$

Positive for any ancilla

- This property indicates that $\mathcal{V}_{\mathrm{dyn}}(t)$ is “Completely Positive”

Dynamical map is completely positive and trace preserving (CPTP)

Relations among Dynamical Maps, Kraus Representations, and CPTP Maps



Quantum Master Equation

Dynamical map and Kraus representation

$$\rho_S(t) = \mathcal{V}_{\rm dyn}(t)[\rho_S(0)] = \sum_\mu K_\mu(t)\rho_S(0)K_\mu^\dagger(t), \quad \sum_\mu K_\mu^\dagger(t)K_\mu(t) = I_S$$

Markov approximation

- Assumes a separation of time scales: $\tau_E \ll \tau_R$
  τ_E = environment correlation time, τ_R = system relaxation time

Dynamical maps form a semigroup

$$\forall t_1, t_2 \geq 0, \quad \mathcal{V}_{\rm dyn}(t_1+t_2) = \mathcal{V}_{\rm dyn}(t_1)\mathcal{V}_{\rm dyn}(t_2) \qquad \longrightarrow \qquad \underbrace{\mathcal{V}_{\rm dyn}(t) = e^{\mathcal{L}t}}_{\text{generator}}$$

Quantum master equation

$$\rho_S(t+dt) = \mathcal{V}_{\rm dyn}(t+dt)[\rho_S(0)] = \mathcal{V}_{\rm dyn}(dt)[\rho_S(t)] = e^{\mathcal{L}dt}[\rho_S(t)], \quad \frac{d}{dt}\rho_S(t) = \mathcal{L}[\rho_S(t)]$$

Lindblad Equation

Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation, or Lindblad equation

[Gorini-Kossakowski-Sudarshan (76), Lindblad (76)]

$$\frac{d}{dt}\rho_S(t) = -i[H'_S, \rho_S] + \sum_\mu L_\mu \rho_S L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu \rho_S - \frac{1}{2}\rho_S L_\mu^\dagger L_\mu$$

- Lindblad equation generates a CPTP semigroup
  Conversely, every Markovian CPTP semigroup has a generator of Lindblad form
- In general, $H'_S = H_S + \Delta H_S$, where ΔH_S is the Lamb-shift correction
- Decomposition into non-Hermitian effective Hamiltonian and jump terms

$$\frac{d}{dt}\rho_S(t) = -iH_{\mathrm{eff}}\rho_S + i\rho_S H_{\mathrm{eff}}^\dagger + \underbrace{\sum_\mu L_\mu \rho_S L_\mu^\dagger}_{\text{jump}} \qquad \underbrace{H_{\mathrm{eff}} = H'_S - \frac{i}{2}\sum_\mu L_\mu^\dagger L_\mu}_{\text{non-Hermitian}}$$

Perturbative Derivation of the Lindblad Equation

Master Equation in the Born-Markov Approximation (I)

1. Hamiltonian

$$H_{\text{tot}} = H_S \otimes I_E + I_S \otimes H_E + \underbrace{\sum V_S^{(i)} \otimes V_E^{(i)}}_{\equiv V}, \qquad \rho_{\text{tot}}(0) = \rho_S(0) \otimes \underbrace{\rho_E(0)}_{\propto e^{-\beta H_E}}$$

2. von Neumann equation in the interaction picture

$$\frac{d}{dt}\rho_{\text{tot}}(t) = -i[V(t), \rho_{\text{tot}}(t)]$$

3. Substitute the formal solution to the right-hand side

$$\rho_{\text{tot}}(t) = \rho_{\text{tot}}(0) - i\int_0^t ds[V(s), \rho_{\text{tot}}(s)],$$

$$\frac{d}{dt}\rho_{\text{tot}}(t) = -i[V(t), \rho_{\text{tot}}(0)] - \int_0^t ds[V(t), [V(s), \rho_{\text{tot}}(s)]]$$

Master Equation in the Born-Markov Approximation (II)

4. Trace out environment, Born & Markov approximations (Redfield equation)

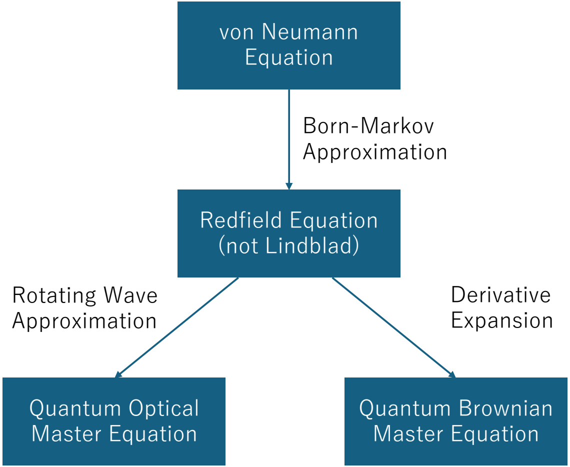
$$\frac{d}{dt}\rho_S(t) \simeq -\int_0^\infty ds \mathrm{Tr}_E\left[V(t), \left[V(t-s), \rho_S(t)\otimes\rho_E(0)\right]\right]$$

$$= \sum_{i,j}\int_0^\infty ds \underbrace{\left\langle V_E^{(i)}(s)\, V_E^{(j)}(0)\right\rangle}_{\text{environment correlator}} \left[V_S^{(j)}(t-s)\rho_S(t)\, V_S^{(i)}(t) - V_S^{(i)}(t)\, V_S^{(j)}(t-s)\rho_S(t)\right] + h.c.$$

- Assume $\mathrm{Tr}_E(\rho_E(0)\, V_E^{(i)}(t)) = 0$
- Born approximation: $\rho_{\text{tot}}(s) \simeq \rho_S(s)\otimes\rho_E(0)$
- Markov approximation ($\tau_E \ll \tau_R$): $\rho_S(s) \simeq \rho_S(t)$, $\int_0^t ds \simeq \int_0^\infty ds$

Close to but not the Lindblad equation

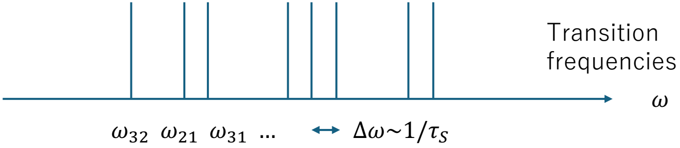
Two Regimes for Master Equation



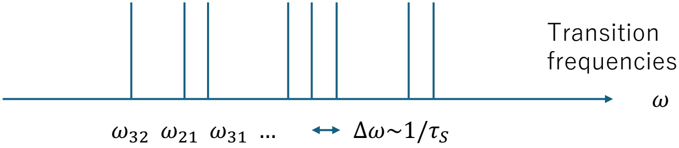
Quantum Optical Regime

Time scales for rotating wave approximation

- Markov approximation $\tau_E \ll \tau_R$
- Assumes another separation of scale $\tau_S \ll \tau_R$
  $\tau_S =$ system time scale $\sim$ inverse of spectral difference ($\sim$ energy gap)
- Rotating wave approximation: $e^{-i\tau_R \Delta\omega} \sim e^{-i\tau_R/\tau_S} \sim 0$



Jump Terms in Quantum Optical Regime



Jump terms in the eigenbasis of the system ($H_S|n\rangle = \epsilon_n|n\rangle$, $\omega_{nk} = \epsilon_n - \epsilon_k$)

$$\langle n| V_S^{(j)}(t-s)\rho_S(t) V_S^{(i)}(t)|m\rangle = \sum_{kl} \underbrace{e^{i(t-s)\omega_{nk} - it\omega_{ml}}}_{\propto e^{it(\omega_{nk}-\omega_{ml})} \sim \delta_{\omega_{nk},\omega_{ml}}} \langle n| V_S^{(j)}|k\rangle\langle k|\rho_S(t)|l\rangle\langle l| V_S^{(i)}|m\rangle$$

e.g. $\rho_S = |1\rangle\langle 1| \xrightarrow{V_S} \frac{1}{2}(|2\rangle + |3\rangle)(\langle 2| + \langle 3|) \xrightarrow{\text{RWA}} \frac{1}{2}(|2\rangle\langle 2| + |3\rangle\langle 3|)$

Jump operators are organized by discrete transition frequencies ω

$$V_S^{(i)}(\omega = \omega_{nk}) = |n\rangle\langle n| V_S^{(i)}|k\rangle\langle k|$$

Quantum Optical Master Equation

Environmental correlator

$$\int_0^\infty ds e^{i\omega s} \langle V_E^{(i)}(s) V_E^{(j)}(0) \rangle = \underbrace{\frac{1}{2}\gamma_{ij}(\omega)}_{\gamma(\omega) \geq 0} + iS_{ij}(\omega)$$

Lindblad equation

$$\frac{d}{dt}\rho_S(t) = -i\left[\Delta H_S, \rho_S\right] + \sum_\omega \sum_{i,j} \gamma_{ij}(\omega) \left[V_S^{(j)}(\omega)\rho_S V_S^{(i)\dagger}(\omega) - \frac{1}{2}\left\{ V_S^{(i)\dagger}(\omega) V_S^{(j)}(\omega), \rho_S \right\} \right]$$

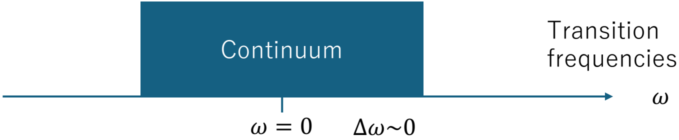
$$\Delta H_S \equiv \sum_\omega \sum_{i,j} S_{ij}(\omega) V_S^{(i)\dagger}(\omega) V_S^{(j)}(\omega)$$

- Lindblad equation is obtained after diagonalizing $\gamma_{ij}(\omega)$
- Steady state $\rho_S^{ss} \propto e^{-\beta H_S}$ ($\because$ detailed balance $\gamma_{ij}(\omega) = e^{\beta\omega}\gamma_{ji}(-\omega)$)

Quantum Brownian Regime

Time scales for derivative expansion

- Markov approximation $\tau_E \ll \tau_R$
- Assumes another separation of scale $\tau_E \ll \tau_S$
  $\tau_S =$ system time scale $\sim$ inverse of spectral difference ($\sim$ energy gap)
- Derivative expansion $V_S(t-s) \sim V_S(t) - s\dot{V}_S(t) + \cdots$



Jump Terms in Quantum Brownian Regime

Derivative expansion of the jump term

$$V_S^{(j)}(t-s)\rho_S(t)\,V_S^{(i)}(t) = \left[V_S^{(j)}(t) - s\,\dot{V}_S^{(j)}(t) + \cdots\right]\rho_S(t)\,V_S^{(i)}(t)$$

Convoluted with environmental correlator

$$\int_0^\infty ds\langle\, V_E^{(i)}(s)\,V_E^{(j)}(0)\rangle\, s^n = \left(-i\frac{d}{d\omega}\right)^n \left[\frac{1}{2}\gamma_{ij}(\omega) + iS_{ij}(\omega)\right]_{\omega=0}$$

- Assume $\gamma_{ij}(\omega)$ is analytic around $\omega = 0$ (no long-time tail $\leftrightarrow$ finite τ_E)
- Kubo-Martin-Schwinger relation

$$\gamma'_{ij}(0) = \frac{1}{2T}\gamma_{ij}(0) \quad (\neq 0 \text{ for Ohmic bath})$$

Quantum Brownian Master Equation

Lindblad equation up to first order derivative expansion

$$\frac{d}{dt}\rho_S(t) = -i\,[\Delta H_S, \rho_S] + \sum_{i,j}\gamma_{ij}(0)\left[\tilde{V}_S^{(j)}\rho_S\,\tilde{V}_S^{(i)\dagger} - \frac{1}{2}\left\{\tilde{V}_S^{(i)\dagger}\,\tilde{V}_S^{(j)}, \rho_S\right\}\right]$$

$$\Delta H_S \equiv \sum_{i,j} S_{ij}(0)\,V_S^{(i)}\,V_S^{(j)} + \frac{1}{8\,T}\sum_{i,j}\gamma_{ij}(0)\{\,V_S^{(i)},\,\dot{V}_S^{(j)}\}$$

$$\tilde{V}_S^{(i)} = V_S^{(i)} + \frac{i}{4\,T}\,\dot{V}_S^{(i)}$$

- Lindblad equation is obtained after diagonalizing $\gamma_{ij}(0)$
- Steady state is expected to be close to $\rho_S^{ss} \propto e^{-\beta H_S}$
- "Lindbladized" Caldeira-Legget master equation $L \propto x + \frac{i}{4T}p$ [Gao (97)]

Quantum Brownian Master Equation for Quarkonium

Hamiltonian of quarkonium in the dipole limit

$$H_S = \frac{p^2}{M} \underbrace{- \frac{(N_c^2-1)\alpha_s}{2N_c r}|s\rangle\langle s| + \frac{\alpha_s}{2N_c r}\sum_a |a\rangle\langle a|}_{\text{attractive (repulsive) for singlet (octet)}}$$

$$H_I = -\sum_{i,a} r_i \Big[\underbrace{\sqrt{\frac{1}{2N_c}}\,(|a\rangle\langle s| + |s\rangle\langle a|)}_{\text{singlet} \leftrightarrow \text{octet}} + \underbrace{\frac{1}{2}d^{abc}|b\rangle\langle c|}_{\text{octet} \leftrightarrow \text{octet}}\Big] \otimes gE_i^a(\boldsymbol{R}) \underbrace{+ (T_A^a)_{bc}|b\rangle\langle c| \otimes gA_0^a(\boldsymbol{R})}_{\text{removed in ``field redefinition''}}$$

- Lindblad operators take into account color-dependent potential energy $\Delta E = \alpha_s N_c/2r$

$$\tilde{V}_S^{so/os} \propto -\left(r_i + \frac{ip_i}{2MT} \mp \frac{N_c}{8T}\frac{\alpha_s r_i}{r}\right), \quad \tilde{V}_S^{oo} \propto -\left(r_i + \frac{ip_i}{2MT}\right)$$

- For $N_c = 2$, $d^{abc} = 0$ and $H_S + H_I$ is invariant under $r_i \to -r_i$, $|s\rangle \to -|s\rangle$, $|a\rangle \to |a\rangle$ $\to$ $\tilde{V}_S$ is also invariant = strong symmetry

Detailed Balance in Quantum Brownian Master Equation

Transition rates between system eigenstates $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$

$$\Gamma_{1\to 2} = \gamma(0)\left|\langle\epsilon_2|\,\tilde{V}_S|\epsilon_1\rangle\right|^2 = \gamma(0)\left|\langle\epsilon_2|\,V_S - \frac{1}{4T}[H_S,\,V_S]|\epsilon_1\rangle\right|^2 \propto \left(1 - \frac{\epsilon_2 - \epsilon_1}{4T}\right)^2$$

Ratio of $\Gamma_{1\to 2}$ and $\Gamma_{2\to 1}$

$$\frac{\Gamma_{1\to 2}}{\Gamma_{2\to 1}} = \left(\frac{1 - \frac{\epsilon_2 - \epsilon_1}{4T}}{1 - \frac{\epsilon_1 - \epsilon_2}{4T}}\right)^2 \simeq e^{-(\epsilon_2 - \epsilon_1)/T}$$

$$\because \quad \left(\frac{1 - x/4}{1 + x/4}\right)^2 \simeq 1 - x + \frac{1}{2}x^2 - \underbrace{\frac{3}{16}}_{\simeq 1/6} x^3 + \cdots \simeq e^{-x}$$

- When the energy gap is small ($\lesssim T$) , detailed balance holds approximately
- Without $\frac{i}{4T}\dot{V}_S$ correction of the jump operator, system reaches $T = \infty$

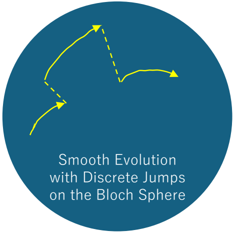
Stochastic Unraveling

Stochastic Unraveling of the Lindblad Equation

Solution of the Lindblad equation by stochastic sampling of the states

$$\rho_S(t) = \lim_{N_{\textsf{sample}}\to\infty} \frac{1}{N_{\textsf{sample}}} \sum_{i=1}^{N_{\textsf{sample}}} |\psi_i(t)\rangle\langle\psi_i(t)|$$

- $\rho_S(t) \to \{|\psi_i(t)\rangle\}$ is not unique, so the stochastic dynamics is not unique
- Quantum trajectory [Dalibard-Castin-Mølmer (92), Dum-Zoller-Ritsch (92)] : Discontinuous
- Quantum state diffusion [Gisin-Percival (92)]: Continuous



Density Matrix in Quantum Trajectory

One time step by Lindblad equation

$$\rho_S(t+dt) = \Big(1 - idtH_{\text{eff}}\Big)\rho_S(t)\left(1 + idtH_{\text{eff}}^\dagger\right) + dt\sum_\mu L_\mu \rho_S(t) L_\mu^\dagger$$

$$H_{\text{eff}} = H_S' - \frac{i}{2}\sum_\mu \underbrace{L_\mu^\dagger L_\mu}_{=\Gamma_\mu} = H_S' - \frac{i}{2}\underbrace{\sum_\mu \Gamma_\mu}_{=\Gamma} = H_S' - \frac{i}{2}\Gamma$$

Update of a pure state to a mixed state

$$\rho_S(t+dt) = \Big(1 - idtH_{\text{eff}}\Big)|\psi(t)\rangle\langle\psi(t)|\left(1 + idtH_{\text{eff}}^\dagger\right) + dt\sum_\mu L_\mu |\psi(t)\rangle\langle\psi(t)| L_\mu^\dagger$$

Pure State in Quantum Trajectory

Update of a pure state with probabilistic interpretation

$$\rho_S(t+dt) = \left(1 - idtH_{\text{eff}}\right)|\psi(t)\rangle\langle\psi(t)| \left(1 + idtH_{\text{eff}}^\dagger\right) + dt\sum_\mu L_\mu|\psi(t)\rangle\langle\psi(t)|L_\mu^\dagger$$

$$\text{1st term} = \underbrace{(1 - dt\langle\Gamma\rangle_\psi)}_{\text{no-jump rate}} \frac{\left(1 - idtH_{\text{eff}}\right)|\psi(t)\rangle\langle\psi(t)| \left(1 + idtH_{\text{eff}}^\dagger\right)}{1 - dt\langle\Gamma\rangle_\psi}$$

$$\text{2nd term} = \sum_\mu \underbrace{dt\langle\Gamma_\mu\rangle_\psi}_{\text{jump rate by } L_\mu} \frac{L_\mu|\psi(t)\rangle\langle\psi(t)|L_\mu^\dagger}{\langle\Gamma_\mu\rangle_\psi}$$

Quantum trajectory of $|\psi\rangle$

- Non-Hermitian evolution to $\propto (1 - idtH_{\text{eff}})|\psi(t)\rangle$ with probability $(1 - dt\langle\Gamma\rangle_\psi)$
- Jump to $\propto L_\mu|\psi(t)\rangle$ with probability $dt\langle\Gamma_\mu\rangle_\psi$
- Proportionality fixed by normalization $\langle\psi|\psi\rangle = 1$

Waiting Time Approach in Quantum Trajectory Algorithm

Norm of a state under non-Hermitian evolution

$$P(\tau;\psi) = \left\| e^{-iH_{\mathrm{eff}}\tau}|\psi\rangle \right\|^2 = \text{Probability to evolve until } \tau \text{ w/o jump}$$

Waiting time algorithm (repeat 1 $\to$ 3)

1. Sample a uniform random number $0 < p < 1$
2. Non-Hermitian evolution until $P(\tau;\psi(t)) = p$ (waiting for a jump)

$$|\psi'(t+\tau)\rangle = e^{-iH_{\mathrm{eff}}\tau}|\psi(t)\rangle \qquad \xrightarrow{\text{normalize}} \qquad |\psi(t+\tau)\rangle$$

3. Jump by L_μ with a probability $\langle\Gamma_\mu\rangle_{\psi(t+\tau)}/\langle\Gamma\rangle_{\psi(t+\tau)}$

$$|\psi'(t+\tau+dt)\rangle = L_\mu|\psi(t+\tau)\rangle \qquad \xrightarrow{\text{normalize}} \qquad |\psi(t+\tau+dt)\rangle$$

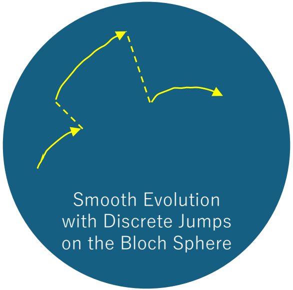
La Roja (Spain National Football Team) Plays a Beautiful Quantum Trajectory

Dribble $\rightarrow$ Pass $\rightarrow$ Dribble $\rightarrow$ Pass $\rightarrow \cdots$

Champions at Soccer World Cup 2026

The ball jumps from one player to another

Evolution $\rightarrow$ Jump $\rightarrow$ Evolution $\rightarrow$ Jump $\rightarrow \cdots$



Basic Idea of Quantum State Diffusion

1. Determine a pure state (drift term) closest to the Lindblad evolution
2. Randomly distribute around the pure state by Gaussian noise

Drift in the Quantum State Diffusion

Minimize the distance to the Lindblad step by Hilbert-Schmidt norm [Gisin-Rigo (95)]

$$\min_{\delta\psi} \| P_{\psi+\delta\psi} \underbrace{- P_\psi - dt\mathcal{L}[P_\psi]}_{\text{Lindblad step}} \|_{\mathrm{HS}}, \quad P_\psi = |\psi\rangle\langle\psi|$$

$$\longrightarrow \delta P_\psi = P_{\psi+\delta\psi} - P_\psi = [P_\psi, [P_\psi, \mathcal{L}[P_\psi]]]\, dt$$

Evolution of the optimized pure state

$$\frac{d}{dt}|\psi\rangle = \Big[\mathcal{L}[|\psi\rangle\langle\psi|] - \underbrace{\langle\mathcal{L}[|\psi\rangle\langle\psi|]\rangle_\psi}_{\text{c-number}}\Big]|\psi\rangle = \Big[-iH_{\mathrm{eff}} + \sum_\mu \langle L_\mu^\dagger\rangle_\psi L_\mu \underbrace{+ \text{ c-number}}_{\text{to keep the norm}}\Big]|\psi\rangle$$

- Relation to pointer state by einselection is discussed [Busse-Hornberger (09)]

Nonlinear Stochastic Schrödinger Equation for Quantum State Diffusion

Add Gaussian noise term to reproduce the Lindblad evolution (Ito discretization)

$$d|\psi(t)\rangle = \left[-iH_{\text{eff}} + \sum_\mu \langle L_\mu^\dagger\rangle_\psi L_\mu\right]|\psi\rangle\,dt + \sum_\mu L_\mu|\psi\rangle\,d\xi_\mu + \underbrace{[\text{(random) c-number}]|\psi\rangle}_{\text{to keep the norm}}$$

$$\langle d\xi_\mu^* d\xi_\nu\rangle = \delta_{\mu\nu}\,dt$$

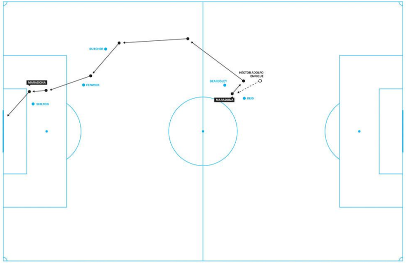
Quantum state diffusion with unnormalized states

$$d|\phi(t)\rangle = \left[-iH_{\text{eff}} + \sum_\mu \langle L_\mu^\dagger\rangle_\phi L_\mu\right]|\phi\rangle\,dt + \sum_\mu L_\mu|\phi\rangle\,d\xi_\mu, \quad \langle L_\mu^\dagger\rangle_\phi = \frac{\langle\phi|L_\mu^\dagger|\phi\rangle}{\langle\phi|\phi\rangle}$$

- Normalize the state $|\phi(t)\rangle$ when calculating density matrix and expectation values

Diego Maradona Played a Fantastic Quantum State Diffusion

Dribble $\to$ Dribble $\to$ Dribble $\to \cdots$



Goal of the Century

The ball never leaves Maradona's feet

Random evolution lead by the drift

Steady States and Symmetry Breaking

Long-Time Behavior of Reduced Density Matrix

Lindbladian $\mathcal{L}$ is a linear superoperator in the operator space $\mathcal{B}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^*$

$$\mathcal{L} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$$

What is the spectrum of the Lindbladian $\mathcal{L}$?

$$\mathcal{L}[\sigma_a] = \lambda_a \sigma_a$$

- Steady state (zero eigenmode) in thermal environment
  - Quantum optical master equation $\rho_{\mathrm{ss}} = e^{-\beta H_S}/\mathrm{Tr}_S[e^{-\beta H_S}]$
  - Quantum Brownian master equation suggests $\rho_{\mathrm{ss}} \simeq e^{-\beta H_S}/\mathrm{Tr}_S[e^{-\beta H_S}]$
- Properties of spectrum and eigenmodes
- Phase transition in a dissipative environment

Lindbladian as a Non-Hermitian Matrix

Lindbladian using the basis $|i\rangle$ of $\mathcal{H}$:

$$(\mathcal{L}[\rho])_{ij} = \sum_{kl} \mathcal{L}_{ij,kl}\rho_{kl},$$

$$\mathcal{L}_{ij,kl} = \underbrace{-i(H'_S)_{ik}\delta_{lj} + i\delta_{ik}(H'_S)_{lj}}_{\text{anti-Hermitian}} + \sum_\mu \underbrace{(L_\mu)_{ik}(L^\dagger_\mu)_{lj}}_{\text{mixed}} \underbrace{-\frac{1}{2}(L^\dagger_\mu L_\mu)_{ik}\delta_{lj} - \frac{1}{2}\delta_{ik}(L^\dagger_\mu L_\mu)_{lj}}_{\text{Hermitian}}$$

- $\mathcal{L}$ is neither Hermitian nor anti-Hermitian $\mathcal{L}_{ij,kl} \neq \pm\mathcal{L}^*_{kl,ij}$
- Eigenvalues of non-Hermitian operators are complex and eigenmodes are not orthogonal

$$\sum_{ij} (\sigma_a)^*_{ij}(\sigma_b)_{ij} = \mathrm{Tr}_S\left[\sigma_a^\dagger \sigma_b\right] \neq \delta_{ab}$$

- Non-Hermitian operator is not always diagonalized $\to$ Jordan blocks

Spectral Properties of the Lindbladian

Lindbladian is a CPTP map

- Preserves the Hermiticity $(\mathcal{L}[\rho])^\dagger = \mathcal{L}[\rho^\dagger]$

$$\mathcal{L}[\sigma_i] = \lambda_i \sigma_i, \quad \mathcal{L}[\sigma_i^\dagger] = \lambda_i^* \sigma_i^\dagger \quad \longrightarrow$$

Eigenvalues are paired (λ_i, λ_i^*)

Can choose Hermitian modes for real eigenvalues

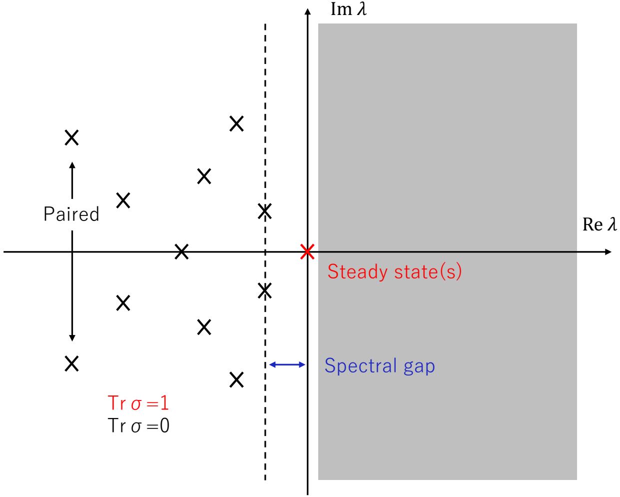
- Preserves the trace $\mathrm{Tr}_S(\mathcal{L}[\rho]) = 0$

$$\lambda_i \mathrm{Tr}_S[\sigma_i] = 0 \qquad \longrightarrow \qquad \mathrm{Tr}_S[\sigma_i] = 0 \text{ for non-zero modes } \lambda_i \neq 0$$

- Contraction property of CPTP (Trace norm $\mathrm{Tr}_S\sqrt{\sigma^\dagger \sigma}$ decreases for Hermitian operators)
  - Real part of λ_i is nonpositive $(\mathrm{Re}\lambda_i \leq 0)$
  - Subspace for zero modes is diagonalizable (No Jordan blocks)
- There is at least one zero mode for finite systems: long-time average

$$\bar{\rho} = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} d\tau\, e^{\mathcal{L}\tau} \rho$$

Schematic Spectrum of a Lindbladian



Symmetry classifications

- Hermitian systems $=$ 10 fold
  [Altland-Zirnbauer (97)]
- non-Hermitian systems $=$ 38 fold
  [Kawabata et al (19)]
- Lindbladian open systems $=$ 38 fold
  [Kawabata et al (23)]

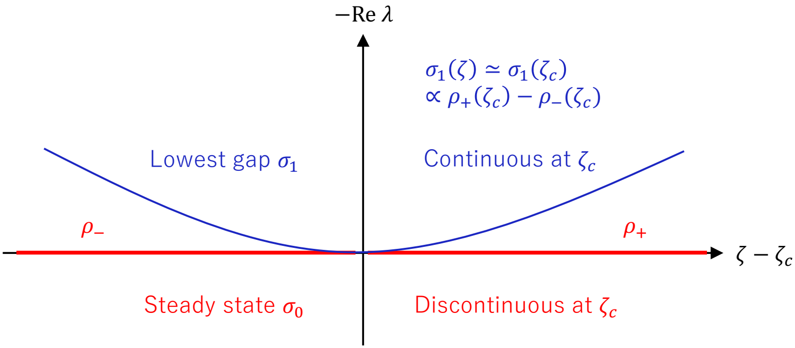
Dissipative Phase Transition by the Steady State of Open Many-Body Systems

[Minganti et al (18)]

Definition of the M-th order dissipative phase transition

$$\lim_{\zeta\to\zeta_c}\left\|\frac{\partial^M}{\partial\zeta^M}\lim_{N\to\infty}\rho_{\mathrm{ss}}(\zeta,N)\right\|=\infty$$

Typical spectral change at 1st order phase transition



Symmetry of a Lindbladian

Symmetry transformation by a unitary operator $U(g)$ with $g \in G$

$$\mathcal{U}(g)[\rho] = U(g)\rho U(g)^\dagger$$

Lindbladian $\mathcal{L}$ has a (weak) symmetry G when it commutes with all $\mathcal{U}(g)$

$$\mathcal{L}[U(g)\rho U(g)^\dagger] = U(g)\mathcal{L}[\rho]U(g)^\dagger \quad \longleftrightarrow \quad \mathcal{L}\mathcal{U}(g) = \mathcal{U}(g)\mathcal{L}$$

- $\mathcal{L}$ is block diagonal in the irreducible representations R of G

$$\mathcal{L} = \mathcal{L}_{R_1} \oplus \mathcal{L}_{R_2} \oplus \mathcal{L}_{R_3} \oplus \cdots$$

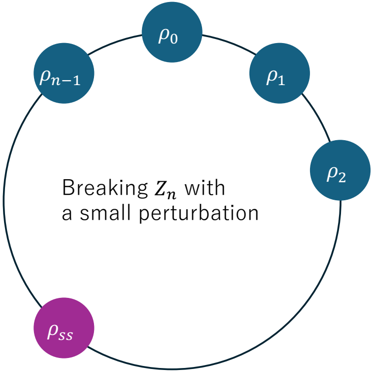
- $\mathcal{U}(g)$ preserves the trace

$$\mathrm{Tr}_S[\mathcal{U}(g)[\sigma_a^{[R]}]] = \sum_b [D(g)]_{ab}^{[R]} \mathrm{Tr}_S[\sigma_b^{[R]}] = \mathrm{Tr}_S[\sigma_a^{[R]}]$$

$\longrightarrow$ Traceful zero modes must be in trivial representation

Symmetry Breaking in Dissipative Phase Transition: Z_n as an example

Steady states in broken phase



$$\mathrm{Tr}_S[\rho_k] = 1$$

Zero modes in each representation

$$\sigma_j \propto \sum_k (z_j^*)^k \rho_k, \quad Z_n[\sigma_j] = z_j \sigma_j, \quad z_j = e^{i2\pi j/n}$$

$$\underbrace{\mathrm{Tr}_S[\sigma_0] \neq 0,}_{\text{traceful}} \quad \mathrm{Tr}_S[\sigma_{j\neq 0}] = 0$$

Unbroken phase

- Only one zero mode $\sigma_0 \propto \rho_{\mathrm{ss}}$
- The other modes $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ are gapped
- Steady state is continuous because $\rho_0 \approx \rho_1 \approx \cdots \approx \rho_{n-1}$ near the phase transition

Schwinger-Keldysh Effective Field Theory for Open Quantum Systems

Effective action on a closed-time path $S_{\text{eff}}(\phi_1, \phi_2)$

- Unitarity condition
- Dynamical Kubo-Martin-Schwinger symmetry (for thermal initial condition)
- Symmetry of the microscopic theory

Symmetries for open systems

- Weak symmetry G_{diag}: environment degrees of freedom also have the charge

$$\mathcal{L}[U(g)\rho U(g)^\dagger] = U(g)\mathcal{L}[\rho]U(g)^\dagger$$

- Strong symmetry $G_1 \times G_2$: environment degrees of freedom do not have the charge

$$\mathcal{L}[U(g_1)\rho U(g_2)^\dagger] = U(g_1)\mathcal{L}[\rho]U(g_2)^\dagger$$

- Symmetry for a closed total system is always strong: $\mathcal{L}_{\text{tot}} = -i[H_{\text{tot}}, *]$

Symmetry Breaking in Open Quantum Systems [Gu et al (2025)]

Strong symmetry is always broken, at least to weak symmetry (strong-to-weak SSB)

e.g., $\mathrm{U}(1)$ case: $\rho_{\mathrm{ss}} = \sum_{n_1 n_2} c_{n_1 n_2} |n_1\rangle\langle n_2|, \quad N|n\rangle = n|n\rangle, \quad c_{n_1 n_2} = c^*_{n_2 n_1}$

$$\underbrace{e^{i\theta_1 N} \rho_{\mathrm{ss}} e^{-i\theta_2 N}}_{\mathrm{U}(1)_1 \times \mathrm{U}(1)_2} = \sum_{n_1 n_2} c_{n_1 n_2} e^{i\theta_1 n_1 - i\theta_2 n_2} |n_1\rangle\langle n_2| \neq \rho_{\mathrm{ss}}$$

Symmetry breaking patters for continuous symmetries

Pattern of SSB	Low-energy dynamics
$G_1 \times G_2 \to G_{\mathrm{diag}}$	Charge conservation (diffusion)
$G_1 \times G_2 \to H_{\mathrm{diag}}$	Dissipative NG modes with charge conservation (diffusion)
$G_{\mathrm{diag}} \to H_{\mathrm{diag}}$	Dissipative NG modes without charge conservation

Summary

Summary

I talked about

- Lindblad equation as a CPTP generator
- Perturbative derivation of the Lindblad equation
- Stochastic unraveling for simulating the Lindblad equation
- Spectral properties of the Lindbladian and dissipative phase transitions

Topics that I did not cover are

- Entropy production and relation to information theory
- Non-Markovian dynamics
- Petz recovery map
- . . .

The END

Strong Symmetry for $N_c = 2$ [Kajimoto's PhD Thesis, Akamatsu et al (22)]

Relative wave function of a quarkonium

$$\Psi_i^j(\boldsymbol{r}, t) = \left\langle \psi_i(\boldsymbol{r}/2)\chi^{j\dagger}(-\boldsymbol{r}/2) \middle| \Psi(t) \right\rangle$$

Gluons cannot distinguish color and anti-color due to the pseudo-reality of $\mathrm{SU}(2)$

$$\Psi_i^j(\boldsymbol{r}, t) \xleftrightarrow{\text{same for gluons}} -\epsilon_{ik}\epsilon^{jl}\Psi_l^k(-\boldsymbol{r}, t) \equiv \Psi_i^{Sj}(\boldsymbol{r}, t), \quad S^2 = 1 \quad \therefore S = \pm 1$$

- "Spinor conjugation" $S = Ce^{i\pi T_2}$ c.f. G-parity $G = Ce^{i\pi I_2}$

Symmetry transformations on density matrix

$$\rho_{ij,kl}(\boldsymbol{r}, \boldsymbol{r}', t) = \underbrace{\rho^S_{ij,kl}(\boldsymbol{r}, \boldsymbol{r}', t) = \rho^{S^*}_{ij,kl}(\boldsymbol{r}, \boldsymbol{r}', t)}_{\text{strong symmetry}} = \underbrace{\rho^{SS^*}_{ij,kl}(\boldsymbol{r}, \boldsymbol{r}', t)}_{\text{weak symmetry}}$$

- Initially, singlet ground state $\to S = -1$ conserved (singlet = parity $+$, triplet = parity $-$)
- Off-diagonal element remains $\rho \sim$ "X"

$$\underbrace{\rho_s(\boldsymbol{r}, \boldsymbol{r}', t) = \rho_s(-\boldsymbol{r}, \boldsymbol{r}', t) = \rho_s(\boldsymbol{r}, -\boldsymbol{r}', t)}_{\text{singlet}}, \quad \underbrace{\rho_t(\boldsymbol{r}, \boldsymbol{r}', t) = -\rho_t(-\boldsymbol{r}, \boldsymbol{r}', t) = -\rho_t(\boldsymbol{r}, -\boldsymbol{r}', t)}_{\text{triplet}}$$