A Closed-Time-Path look at neutrino dynamics in the early universe

Yannis Georis (ヤニス・ジョリス)

Based on various works in collaboration with T. Binder, M. Drewes, A. Finke, M. Klasen, Y. Li, G. Pierobon, L. Wiggering and Y.Y.Y. Wong.

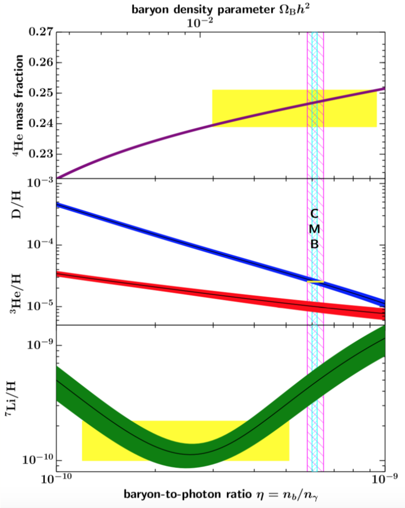
See arXiv:2402.18481, arXiv:2411.14091, arXiv:26XX.XXXXX

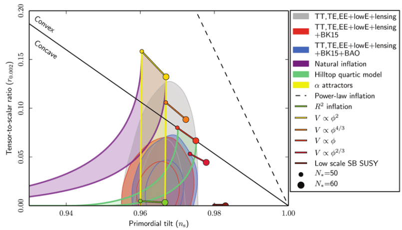
Quantum crossroads: Interdisciplinary, hybrid workshop at the interface of open quantum systems, high-energy physics, and quantum science

August 4, 2026

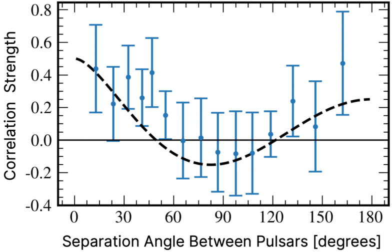
Entering the precision cosmology era

[Credit: Particle Data Group]



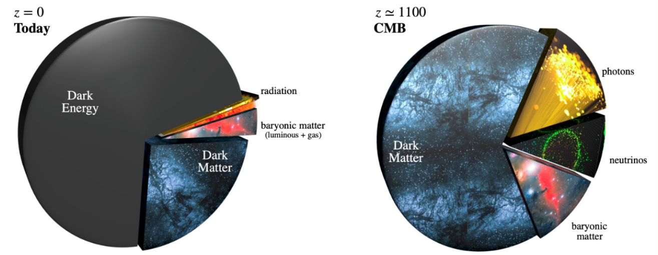


[Credit: Planck]



[Credit: Nanograv]

- Increasing precision and number of observables help building a consistent picture of the early universe history!



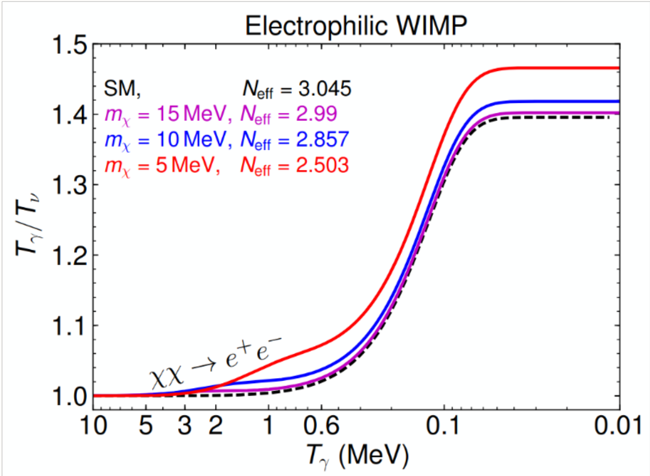
[Credit: Cirelli/Strumia/Zupan; 2406.01705]

- Amount of radiation parametrised by

$$\rho_{\text{rad}} \equiv \rho_\gamma + \rho_\nu + \text{BSM} = \frac{\pi^2 T^4}{15}\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{\text{eff}}\right)$$

- Effective number of neutrinos: $N_{\text{eff}} \simeq 3$ in the SM due to 3 generations of neutrinos.
  $\longrightarrow$ Sensitive probe of new light physics!

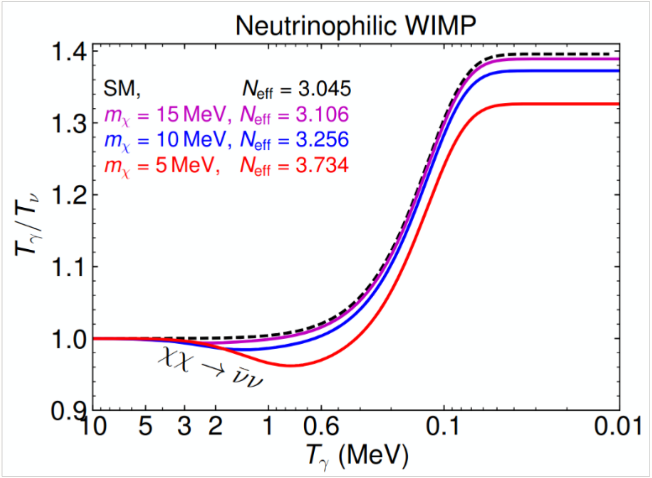
$N_{\rm eff}$ as a probe of new physics



[Credit: M. Escudero]

- Light BSM physics can lead to larger deviation from $N_{\rm eff} = 3$.

$N_{\rm eff}$ as a probe of new physics



[Credit: M. Escudero]

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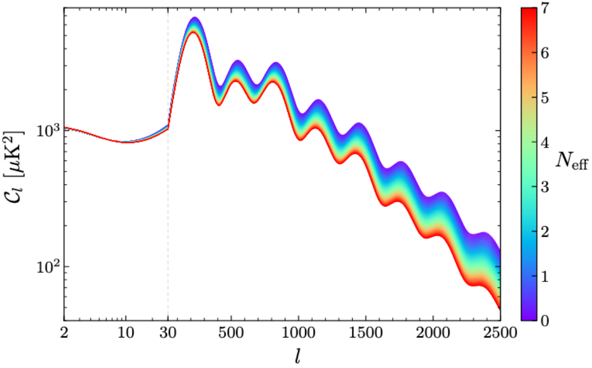
Measuring N_{eff}



[Credit: Baumann; 1807.03098]

- Planck measurement (+BAO) leads to the constraint

$$N_{\text{eff}}^{\text{obs}} = 2.99 \pm 0.17 \text{ (68\% C.L.)}$$

- Future experiments (Simons Observatory, CMB-S4/-HD (?), ...) expected to improve to $\sigma(N_{\text{eff}}) \sim \mathcal{O}(10^{-2})$
  $\longrightarrow$ Timely to improve accuracy of theoretical predictions!

Outline

Question: How well do we know N_{eff} theoretically?

1. What physical effects impact N_{eff}?
2. Impact of NLO corrections to the electron-neutrino interaction rates
   [Drewes/YG/Klasen/Wiggering/Wong, 2402.18481], [Drewes/YG/Finke/Klasen/Li/Wong, 26XX.XXXXX]
3. Impact of bound-state production on N_{eff}
   [Binder/Drewes/YG/Klasen/Pierobon/Wong, 2411.14091]
4. Take-home message

Physical effects impacting N_{eff}

Tracking N_{eff} in the SM

In the SM:

- Continuity equation

$$\frac{\mathrm{d}\rho_{\mathrm{tot}}}{\mathrm{d}t}+3H(\rho_{\mathrm{tot}}+P_{\mathrm{tot}})=0 \longleftrightarrow \frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\tau J_+(\tau)-1/(2z^3)\mathrm{d}\bar{\rho}_\nu/\mathrm{d}x+G_1(\tau)}{\tau^2 J_+(\tau)+Y_+(\tau)+2\pi^2/15+G_2(\tau)}$$

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- Necessity to simultaneously track the time evolution of neutrinos

Motivations

Need systematic way to study particle properties and production in the early universe:

- S-matrix approach does not apply: No asymptotic states!
- Boltzmann ansatz not always consistent (double counting) and misses some aspects (e.g., screening).

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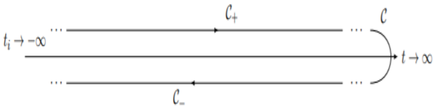
- Powerful approach: CTP formalism.



- Derive evolution equations for correlation functions

$$iS^{++}_{\alpha\beta}(x,y) \equiv \langle T_{\mathcal{C}}\left[\psi_\alpha(x)\bar{\psi}_\beta(y)\right]\rangle$$

through controlled approximations.

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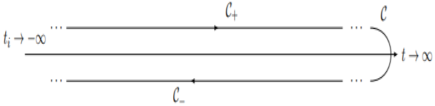

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$$iS^{++}_{\alpha\beta}(x,y) \equiv \langle T_{\mathcal{C}} \left[\psi_\alpha(x) \bar{\psi}_\beta(y) \right] \rangle$$
through controlled approximations.
- Broad range of applications! Dark matter, baryogenesis, inflation/reheating, neutrino decoupling, ...

Deriving the ν quantum kinetic equations from first principles

- Starting point: Kadanoff-Baym equations

Spectral
$$\left(i\not\partial_{x_1} - M\right) S^{\mathcal{A}}(x_1,x_2) = \int d^4x' \left[\not\Sigma^{H}(x_1,x')S^{\mathcal{A}}(x',x_2) + \not\Sigma^{\mathcal{A}}(x_1,x')S^{H}(x',x_2)\right],$$

Statistical
$$\left(i\not\partial_{x_1} - M\right) S^{+}(x_1,x_2) = \int d^4x' \left[\not\Sigma^{+}(x_1,x')S^{H}(x',x_2) + \not\Sigma^{H}(x_1,x')S^{+}(x',x_2)\right] + \frac{1}{2}\int d^4x' \left[\not\Sigma^{>}(x_1,x')S^{<}(x',x_2) - \not\Sigma^{<}(x_1,x')S^{>}(x',x_2)\right].$$

ν propagators

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ν propagators

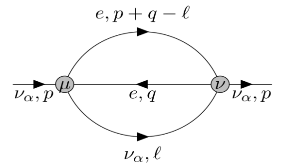
- Inject knowledge of physical system:
  - Homogeneous isotropic universe
  - Equilibrium of all non-ν species (trace out environment)
  - No chemical potential (see e.g. Li & Yu, 2409.08280)
  - Gradient expansion $\partial_t \ll E$ (separation of scales)

$$\longrightarrow \partial_t \delta S = 2\frac{\partial_t f_F}{1-2f_F}\,\overline{S}^{+} + i\left[\overline{\mathcal{H}}, \delta S\right] - \left\{\overline{\mathcal{G}}, \delta S\right\}$$

$\frac{i}{2}\left(\bar{\Sigma\!\!\!/}^{>} - \bar{\Sigma\!\!\!/}^{<}\right)\gamma^0$

$i\gamma^0\left(S^{+} - \overline{S}^{+}\right)$ $\left(p\!\!\!/ - \bar{\Sigma\!\!\!/}^{H} - M\right)\gamma^0$

Deriving the ν quantum kinetic equations fr



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$i\gamma^0\left(S^+ - \overline{S}^+\right)$ $\qquad \left(\not{p} - \bar{\not{\Sigma}}^H - M\right)\gamma^0$ $\qquad \frac{i}{2}\left(\bar{\not{\Sigma}}^> - \bar{\not{\Sigma}}^<\right)\gamma^0$

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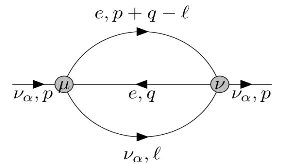

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$$\longrightarrow \partial_t \delta S = \cancel{2\frac{\partial_t f_F}{1-2f_F}\overline{S}^+} + i\left[\overline{\mathcal{H}}, \delta S\right] - \left\{\overline{\mathcal{G}}, \delta S\right\}$$

Relativistic νs

$i\gamma^0\left(S^+ - \overline{S}^+\right)$

$\left(\not p - \bar{\not\Sigma}^H - M\right)\gamma^0$

$\frac{i}{2}\left(\bar{\not\Sigma}^> - \bar{\not\Sigma}^<\right)\gamma^0$

Deriving the ν quantum kinetic equations from first principles

- ν's close to equilibrium: Kadanoff-Baym/quasiparticle ansatz for

$$\text{Propagator} \longleftrightarrow \text{Distribution function}$$

$$iS^{+}_{\alpha\beta}(k) = (1-2\varrho_{\alpha\beta})S^{\mathcal{A}} \propto (1-2\varrho_{\alpha\beta})\delta(k^2), \quad \delta\mathcal{S}_{\alpha\beta} \propto \delta\varrho_{\alpha\beta}\delta(k^2)$$

$\longrightarrow$ Valid for a broad range of non-equilibrium situation where particles are feebly interacting

- Including the expansion of the universe

$$\partial_t \varrho - pH\partial_p \varrho = -\mathrm{i}[\boxed{H_\nu}, \varrho] + \boxed{\mathcal{I}_\nu[\varrho]}$$

  - Effective Hamiltonian: Vacuum + thermal masses (modified dispersion relation).
  - Collision integral: Neutrino annihilation and scatterings ($\nu - \nu$, $\nu - e$). Non-linear in ϱ.

Tracking $N_{\rm eff}$ in the SM

In the SM:

- Continuity equation

$\propto T_\gamma/T_\nu$

Neutrinos Non-ideal terms

$$\frac{\mathrm{d}\rho_{\rm tot}}{\mathrm{d}t}+3H(\rho_{\rm tot}+P_{\rm tot})=0 \longleftrightarrow \frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\tau J_+(\tau)-1/(2z^3)\mathrm{d}\bar{\rho}_\nu/\mathrm{d}x+G_1(\tau)}{\tau^2 J_+(\tau)+Y_+(\tau)+2\pi^2/15+G_2(\tau)}$$

Electrons Photons

$\propto a$

- Necessity to simultaneously track the time evolution of neutrinos

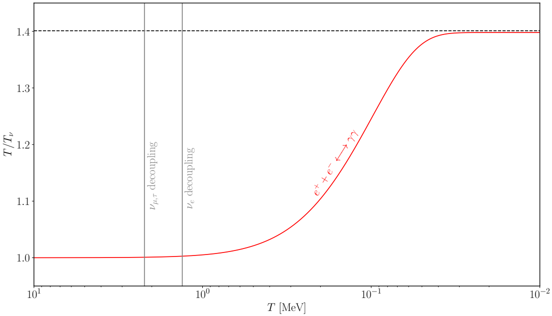
$$\partial_t \varrho - pH\partial_p \varrho = -\mathrm{i}[H_\nu,\varrho]+\mathcal{I}_\nu[\varrho]$$

Oscillations

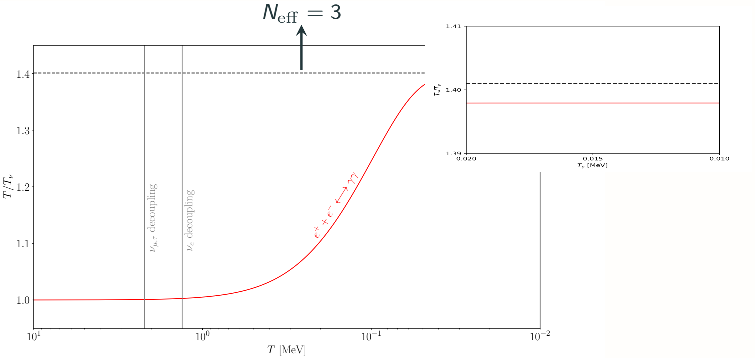
ν density matrix

$\nu-\nu$, $\nu-e$
scatterings/annihilations

Neutrino decoupling



Neutrino decoupling



- Deviations from $N_{\rm eff} = 3$ from

Standard-model corrections to $N_{\rm eff}^{\rm SM}$	Leading-digit contribution
m_e/T_d correction	$+0.04$
$\mathcal{O}(e^2)$ FTQED correction to the QED EoS	$+0.01$
Non-instantaneous decoupling+spectral distortion	-0.005
$\mathcal{O}(e^3)$ FTQED correction to the QED EoS	-0.001
Flavour oscillations	$+0.0005$
Type (a) FTQED corrections to the weak rates	$\lesssim 10^{-4}$

[Bennett et al, '20]

N_{eff} in the SM: Theoretical status

- Current theoretical reference:

$$N_{\text{eff}} = 3.0440 \pm 0.0002$$

[Bennett et al '20, Froustey/Pitrou/Volpe '20, Akita/Yamaguchi '20]

Is it the ultimate result?

1. Obtained for tree-level $\nu - e$ interaction rates. What about NLO?
   Recent debate in the literature: 3.043 or 3.044?
2. What about non-perturbative effects, i.e. bound-state formation?

Useful/Desirable to keep theoretical error well below future experimental sensitivities in order to avoid to propagate errors.

NLO corrections to the $\nu - e$ interaction rates

N_{eff} in the SM at NLO

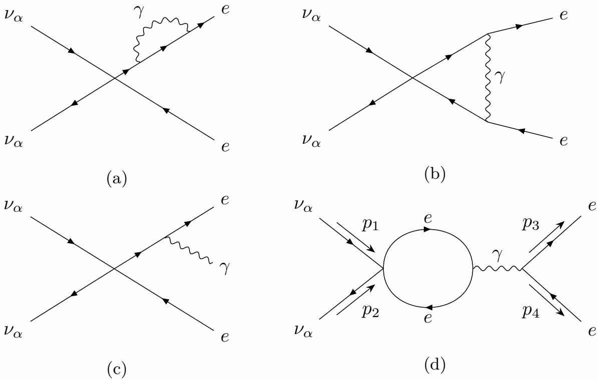
Recent spike of interest in N_{eff} but contradictory results:

- [Cielo et al '23] claims $3.044 \to 3.043$ but
    1. Do not account for Pauli blocking
    2. Neglect NLO corrections to $\nu e \leftrightarrow \nu e$.
    3. Missing one diagram

- [Laine/Jackson '23, '24] observe a $\mathcal{O}(.1\%)$ correction to the $\nu - e$ rates but set $m_e = 0$
  $\longrightarrow$ Inconsistent with [Cielo et al '23] + no estimate of N_{eff} *(fixed in '24)*.

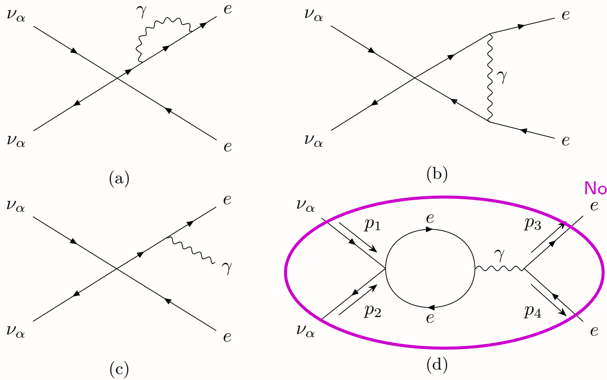
Need to clarify the situation!

NLO QED correction to $\nu - e$ interaction rates



- Diagrams contributing at NLO to the $\nu - e$ interaction rates.

NLO QED correction to $\nu - e$ interaction rates



Not accounted for by [Cielo et al, '23]!

- Diagrams contributing at NLO to the $\nu - e$ interaction rates.
- t-channel enhancement: expect to be the dominant NLO contribution
  $\longrightarrow$ Confirmed for $m_e = 0$ by [Laine/Jackson, '23]

Thermal equilibrium

- Close to equilibrium, relevant quantities can be computed through use of modified Feynman rules.
- Replace vacuum propagators by

$$iS_\Psi(p) = \begin{pmatrix} \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} & 0 \\ 0 & -\frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 - i\epsilon} \end{pmatrix}$$

$$- 2\pi(\not{p} + m_\psi)\delta(p^2 - m_\psi^2) \begin{pmatrix} f_F(|p^0|) & f_F(|p^0|) - \theta(-p^0) \\ f_F(|p^0|) - \theta(p^0) & f_F(|p^0|) \end{pmatrix}$$

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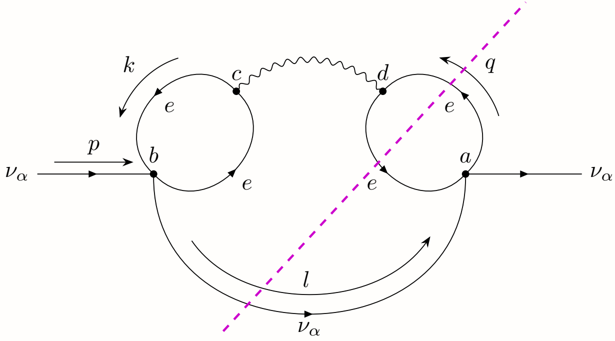
Usual Feynman contribution

$$
iS_\Psi(p) = \boxed{\begin{pmatrix} \dfrac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} & 0 \\ 0 & -\dfrac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 - i\epsilon} \end{pmatrix}}
$$

$$
- \boxed{2\pi(\not{p} + m_\psi)\delta(p^2 - m_\psi^2) \begin{array}{c} \begin{array}{cc} ++ & \qquad\qquad +- \end{array} \\ \begin{pmatrix} f_F(|p^0|) & f_F(|p^0|) - \theta(-p^0) \\ f_F(|p^0|) - \theta(p^0) & f_F(|p^0|) \end{pmatrix} \\ \begin{array}{cc} -+ & \qquad\qquad -- \end{array} \end{array}}
$$

On-shell plasma contribution

Computing the $\nu - e$ interaction rates

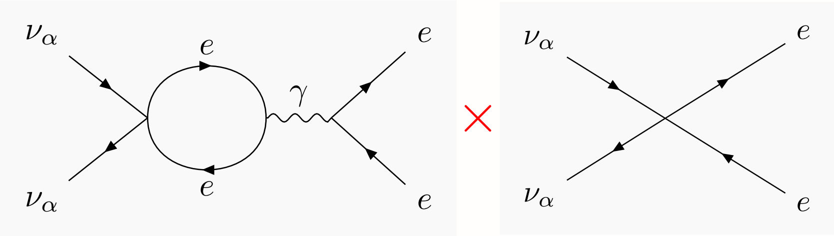


- Optical theorem allows to extract relevant interaction rate from the above self-energy

$$\frac{\mathrm{d}f_\nu^\alpha}{\mathrm{d}t} = -\Gamma_\alpha (f_\nu^\alpha - f_\nu^{\mathrm{eq}})$$

Computing the $\nu - e$ interaction rates

- Equivalently:



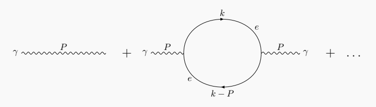
- In the CTP formalism

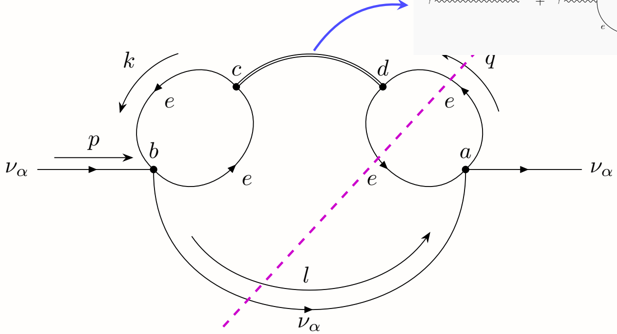
$$\Gamma_\alpha \propto \frac{1}{2E_p} \sum_{c,d=\pm} \mathrm{sign}(cd)(ie)^2 \left(\frac{i4G_F}{\sqrt{2}}\right)^2 \times$$

$$\times \int_{l,q,k} \mathrm{Tr}\left[iS_e^{+d}(q)\gamma_\rho(P_L g_L^\alpha + P_R g_R) iS_e^{d+}(q+p-l)\gamma^\mu\right] iD_{\mu\nu}^{cd}(p-l)$$

$$\times \mathrm{Tr}\left[\not{p}\gamma^\rho P_L iS_{\nu_\alpha}^{-+}(l)\gamma^\sigma P_L\right] \mathrm{Tr}\left[iS_e^{c-}(k)\gamma^\nu iS_e^{-c}(k+p-l)\gamma_\sigma(P_L g_L^\alpha + P_R g_R)\right].$$

Computing the $\nu - e$ interaction rates



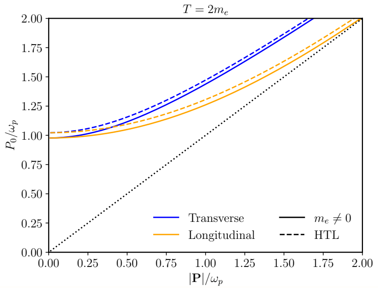


- Optical theorem allows to extract relevant interaction rate from the above self-energy

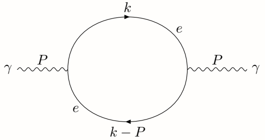
$$\frac{\mathrm{d}f_\nu^\alpha}{\mathrm{d}t} = -\Gamma_\alpha (f_\nu^\alpha - f_\nu^{\mathrm{eq}})$$

- To avoid t-channel divergence, need to 1PI-resum photon propagator

Resummed photon propagator



[Drewes/YG/Klasen/Wiggering/Wong, 2402.18481]



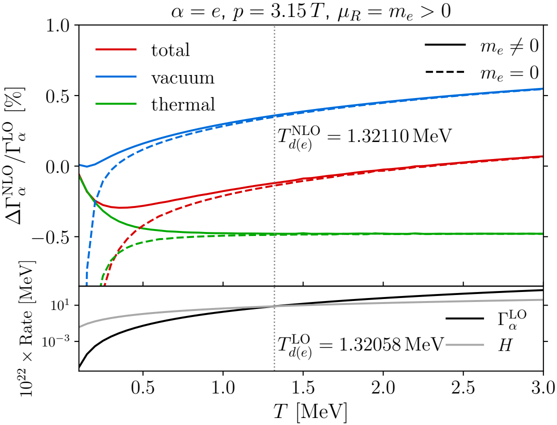
$$\Pi^{\mu\nu}_{ab}(P) = \mathrm{sign}(ab) ie^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[iS^{ab}_e(k)\gamma^\mu iS^{ba}_e(k-P)\gamma^\nu \right] ,$$

$$m_D = \sqrt{-\,\mathrm{Re}\,\Pi^L_{++}\,(q_0 = 0,\, |\mathbf{q}| \to 0)} \longrightarrow \frac{eT}{\sqrt{3}}\ (T \gg m_e)$$

At finite temperature, QED interaction leads to

- Photon thermal mass
- Debye screening
- Plasmon process ($\gamma \to \nu\bar{\nu}$): well known in astrophysical media
- ...

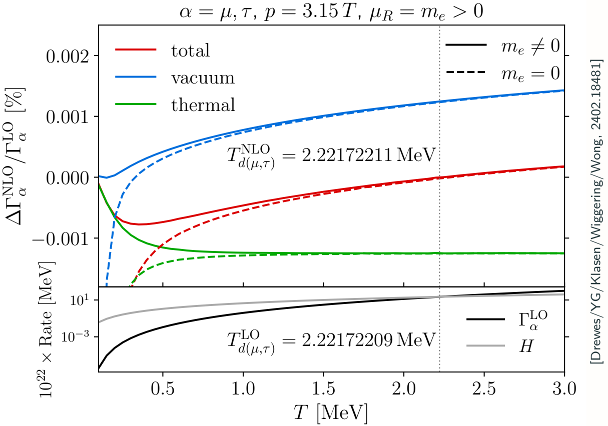
Correction to the LO interaction rates



[Drewes/YG/Klasen/Wiggering/Wong, 2402.18481]

- $\mathcal{O}(.1\%)$ corrections in the electron flavour: Consistent with [Laine/Jackson, '23]
- Partial cancellation between vacuum and thermal NLO corrections at $T = T_d$

Correction to the LO interaction rates



- $\mathcal{O}(.001\%)$ corrections in the μ/τ flavours: strong flavour dependence of the results due to the different chiral structure!
- Strong cancellation between vacuum and thermal NLO corrections at $T = T_d$

Estimating N_{eff}

Method	$\delta N_{\text{eff}}/N_{\text{eff}}^{\text{LO}}$	$\delta N_{\text{eff}}^{m_e=0}/N_{\text{eff}}^{\text{LO}}$
Entropy conservation (common decoupling)	-1.1×10^{-5}	-1.2×10^{-5}
Entropy conservation (flavour-dependent decoupling)	-5.4×10^{-6}	-6.1×10^{-6}
Boltzmann damping approximation (mean momentum)	-7.8×10^{-6}	-1.6×10^{-5}
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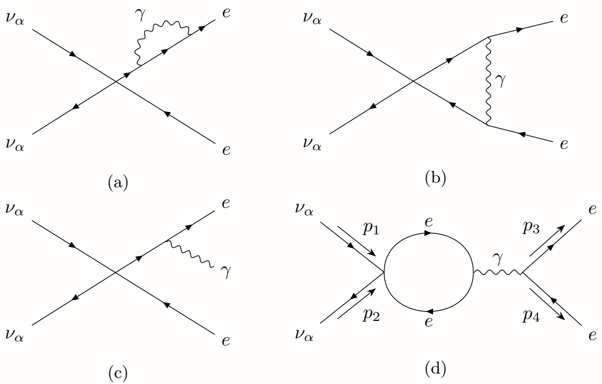
But 3.044 remains for now the reference result!

What about other diagrams?



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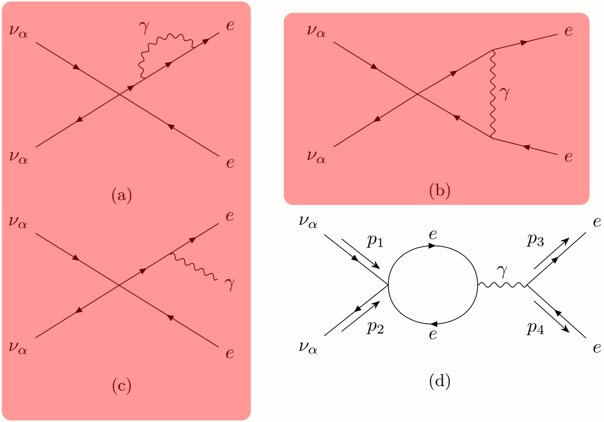

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Impact of bound-state formation

Positronium and $N_{\rm eff}$

- QED bath is made of e^+, e^- and γ at $T \sim$ MeV
  $\longrightarrow$ Can $e^+ - e^-$ bound states form?!

Property	Expression at $T = 0$	Numerical value
Binding energy of energy level n	$\Delta E_n = m_e \alpha_{\rm EM}^2/(4n^2)$	$6.8/n^2$ eV
Mass	$m_{\rm Ps} = 2m_e - \Delta E_1$	1.022 MeV
Bohr radius	$a_0 = 2/(m_e \alpha_{\rm EM})$	$536\ \mathrm{MeV}^{-1}$
Lifetime (for $n = 1$):		
• Para-positronium ($s = 0$)	$\tau_0 = 2/(m_e \alpha_{\rm EM}^5)$	0.1244 ns
• Ortho-positronium ($s = 1$)	$\tau_1 = 9\pi/[2(\pi^2 - 9) m_e \alpha_{\rm EM}^6]$	138.7 ns

- QED equation of state sensitive to the **total energy and pressure densities**

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\left(\bar{\rho}_{\rm tot} - 3\bar{P}_{\rm tot}\right)/x - \partial\bar{\rho}_{\rm tot}/\partial x}{\partial\bar{\rho}_{\rm tot}/\partial z}$$

Can Positronium formation change the prediction for $N_{\rm eff}$?

How can we model positronium formation?

- **Complete first-principles study of positronium is complicated:**
  - Requires tracking out-of-equilibrium evolution of 2- and 4-point correlation function in non-relativistic QED
- Therefore, we rely on **two different phenomenological modellings/ansatz** for positronium formation:
  1. Out-of-equilibrium (instantaneous vs non-instantaneous formation)
     $\rightarrow$ Positronium formation is associated with entropy production
  2. Complete thermodynamic equilibrium
     $\rightarrow$ All thermodynamic quantities remain smooth

When does positronium form?

Two physical effects can prevent bound-state formation (in QED):

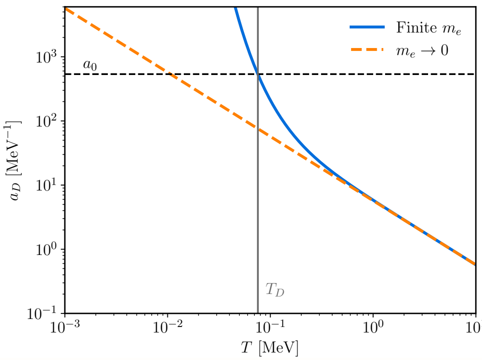
1. Debye screening rendering QED too short range to host a bound state

$$V(r) \sim -\frac{\alpha_{\mathrm{EM}}}{r} e^{-m_D r}$$

2. Dissociation from real scatterings giving a large thermal width to the bound state

$$\Gamma = \alpha_{\mathrm{EM}} T \int \mathrm{d}^3\mathbf{r}\ \phi(r/a_D) |\Psi_{1s}^{\mathrm{Ps}}(\mathbf{r})|^2$$

Debye screening



[Binder/Drewes/YG/Klasen/Pierobon/Wong, 2411.14091]

- Simple criterion: Debye screening length should be larger than Positronium's Bohr radius

$$a_D \equiv \frac{1}{m_D} > a_0 = \frac{8\pi}{m_e e^2} \simeq 536 \text{ MeV}^{-1}$$

$$\longrightarrow T < T_D \simeq 75.6 \text{ keV}$$

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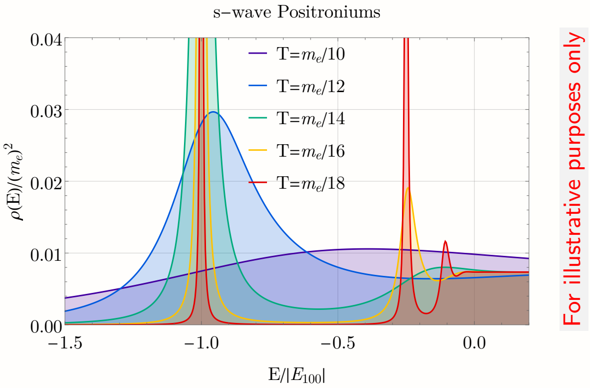
Dissociation from scatterings



[Credit: T. Binder]

- Simple criterion: Thermal width should be smaller than the separation from the continuum/binding energy

$$\Gamma = \alpha_{\rm EM}\, T \int \mathrm{d}^3\mathbf{r}\ \phi(r/a_D)|\Psi_{1s}^{\rm Ps}(\mathbf{r})|^2 < \Delta E_n = \frac{m_e \alpha_{\rm EM}^2}{4n^2}$$

$$\longrightarrow T < T_{\rm dis} \simeq 45 \text{ keV}$$

Positronium equilibration rate

Once allowed to form, how fast does positronium equilibrate?

- Deep in the NR regime, can be described by standard Boltzmann equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(n_{\mathrm{Ps}}x^3) = -\frac{1}{Hx}\left[\Gamma^{\mathrm{dec}} + \Gamma^{\mathrm{ion}}(T)\right]\left(n_{\mathrm{Ps}}x^3 - n_{\mathrm{Ps}}^{\mathrm{eq}}x^3\right)$$

- Rates are well known

$$\Gamma^{\mathrm{dec}} = \begin{cases} m_{\mathrm{r}}\alpha_{\mathrm{EM}}^5, & \text{for } s=0, \\ \frac{4(\pi^2-9)}{9\pi} m_{\mathrm{r}}\alpha_{\mathrm{EM}}^6, & \text{for } s=1, \end{cases} \qquad \begin{array}{l} \mathrm{Ps} \leftrightarrow 2\gamma \\ \mathrm{Ps} \leftrightarrow 3\gamma \end{array}$$

$$\Gamma^{\mathrm{ion}}(T) = \frac{m_{\mathrm{r}}\alpha_{\mathrm{EM}}^5}{8\pi}\int_0^{+\infty}\frac{\mathrm{d}\xi\ \xi^{-4}}{\exp\left[\left(\frac{1}{n^2}+\frac{1}{\xi^2}\right)\frac{m_e\alpha_{\mathrm{EM}}^2}{4T}\right]-1}S^{\mathrm{BSF}}(\xi), \qquad \mathrm{Ps}+\gamma \leftrightarrow e^+e^-$$

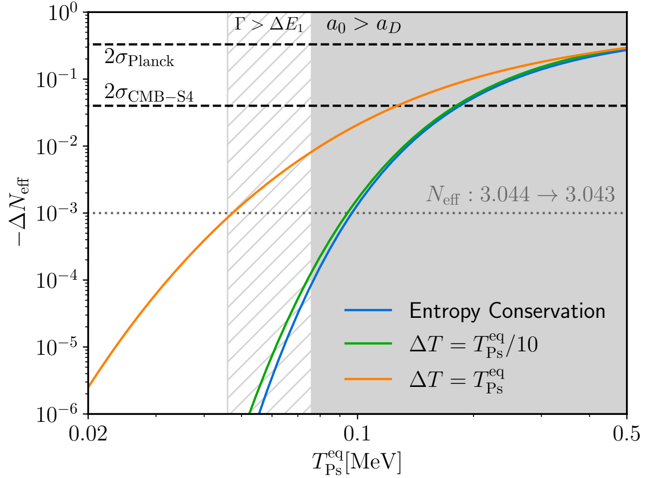
- At formation temperature:

$$\left.\frac{\Gamma^{\mathrm{dec}}+\Gamma^{\mathrm{ion}}}{H}\right|_{T\leq T_D} \gtrsim 10^{16} \gg 1$$

- Consistent with dynamical Saha equilibrium.

Limiting factor is not equilibration!

Out-of-equilibrium positronium formation



- Impact of positronium is most likely negligible.

[Binder/Drewes/YG/Klasen/Pierobon/Wong, 2411.14091]

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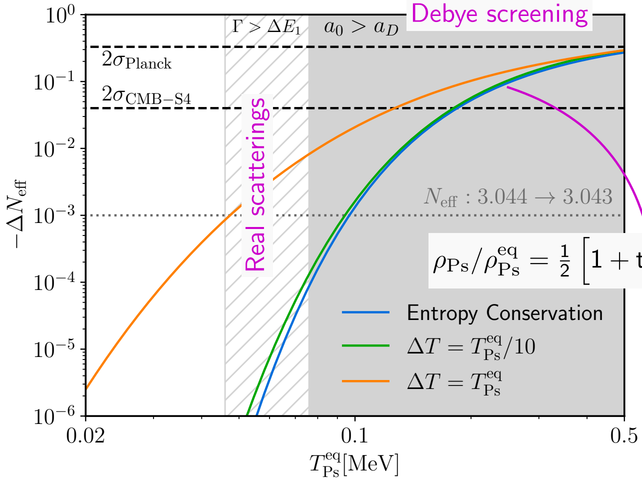


$$\rho_{\mathrm{Ps}}/\rho_{\mathrm{Ps}}^{\mathrm{eq}} = \frac{1}{2}\left[1 + \tanh\left(\frac{T_{\mathrm{Ps}}^{\mathrm{eq}} - T}{\Delta T}\right)\right]$$

$$V(r) = -\alpha m_D - \frac{\alpha}{r} e^{-m_D r} - \mathrm{i}\alpha T \phi(m_D r)$$

- Impact of positronium is most likely negligible.

[Binder/Drewes/YG/Klasen/Pierobon/Wong, 2411.14091]

- Direct relation between the change in N_{eff} and entropy production in Approach 1:

$$\Delta N_{\text{eff}} = N_{\text{eff}} - 3 = 3\left[\left(1 + \frac{\delta s}{s_{2-}}\right)^{-4/3} - 1\right] \simeq -4\frac{\delta s}{s_{2-}}$$

What if Positronium formation happens in complete thermodynamic equilibrium?

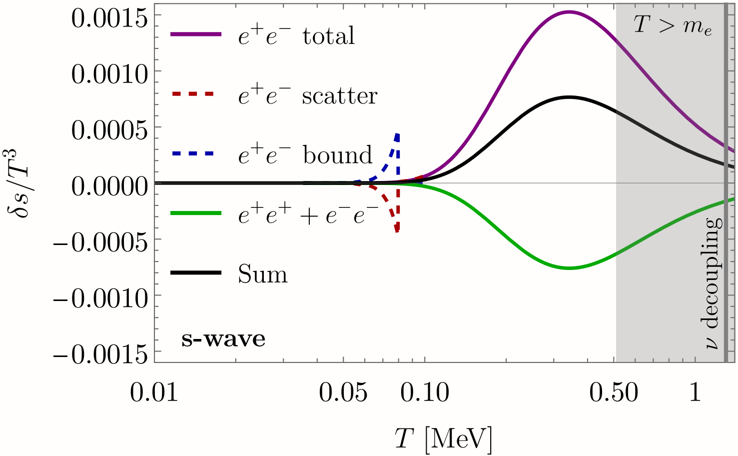
- All thermodynamic quantities remain smooth: Cancellation between bound state and scattering state contribution.
- Pressure can be derived from Beth-Uhlenbeck formula

$$\beta(P - P^{(0)}) = 4\left(\frac{2m_e T}{2\pi}\right)^{3/2} e^{-\beta 2m_e} \sum_{\ell=0}^{\infty}(2\ell+1)b_\ell$$

where

$$b_\ell = \sum_{n\geq\ell+1} e^{-\beta E_{n\ell}} + \tfrac{1}{\pi}\int_0^\infty \mathrm{d}E e^{-\beta E}\tfrac{\mathrm{d}}{\mathrm{d}E}\left\{\delta_\ell^{e^+e^-} + \left(\tfrac{1}{2} - \tfrac{(-1)^\ell}{4}\right)\left(\delta_\ell^{e^+e^+} + \delta_\ell^{e^-e^-}\right)\right\}$$

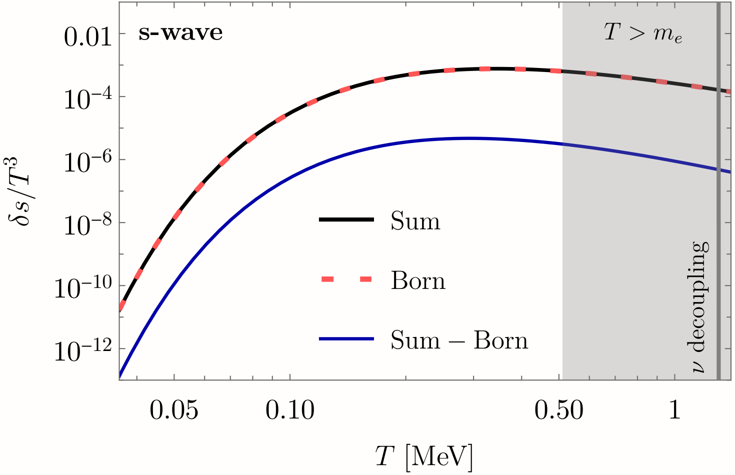
Equilibrium positronium formation



- Cancellation between bound and scattering states.

[Binder/Drewes/YG/Klasen/Pierobon/Wong, 2411.14091]

Equilibrium positronium formation



- At $\mathcal{O}(e^2)$, coincide with perturbative result.
- Impact of positronium is most likely negligible. But further investigation needed.

[Binder/Drewes/YG/Klasen/Pierobon/Wong, 2411.14091]

Conclusion

Conclusion and outlook

N_{eff} provides a key cosmological test of the SM

- Below $\mathcal{O}(.1\%)$ correction at NLO
  $\longrightarrow$ benchmark result remains 3.044
- Impact of m_e on NLO corrections non-negligible
- A full evaluation of all the NLO diagrams remains to be done for a definitive answer
- Bound states unlikely to matter in the SM but further investigations into the modelling of their formation would be useful.

CTP formalism key ingredient for accurate description of neutrino dynamics at higher order.

Thanks for your attention!