Quantum Information and the Physical Environment

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Outline

- Open systems: channels vs. the physical environment
- Cosmological decoherence from thermal gravitons
- Thermal reconstruction of chaotic many-body systems
- Dynamics beyond Lindblad: or, the revenge of the physical environment

Why I'm here

- My background: originally early-universe cosmology. Try to understand: why does the universe look classical in the first place? Why does space(time) exist?
  - Need to understand how to pass from quantum→ effective classical behavior. Study decoherence, equilibration, etc…lots of papers with the word "entropy"

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  - Need to understand how to pass from quantum → effective classical behavior. Study decoherence, equilibration, etc…lots of papers with the word "entropy"
- Toy model for this sort of problem: finite-dimensional Hilbert space! Take a closed system, look at coarse-grained behavior when tracing out degrees of freedom
  - This is a spin chain, AKA a "quantum computer"
- These "toy models" are interesting in their own right! And the field is young – easy to come up with simple, interesting questions that haven't been answered yet.
  - Maximally entangled states? Entropy inequalities? Thermalization?
  - Lots of room in the QI playground for cosmologists and effective field theorists!

Textbook QI: quantum physical evolutions

- I have some "quantum stuff." Qubits, etc. I want to do something with them, which means the degrees of freedom aren't isolated. Also making qubits is hard, so there is probably noise I don't control.
  - One approach: start with unitary QM (Schrodinger equation) on the total Hilbert space of…well, everything. Then trace out all but the degrees of freedom I care about. Top-down approach.
    - This isn't crazy, but it sounds *hard*.

- Quantum information theorists therefore tend to take the opposite, bottom-up approach. Postulate reasonable-sounding *axioms* for the evolution of quantum states of a subsystem, write down all possible maps compatible with these axioms, pick some nice parameterization of the space of maps, study the behavior of these parameters.

Axioms for QPE

- 0) Quantum mechanics is linear.
  - Pure quantum states are complex norm-one vectors $|\psi\rangle$. Mixed quantum states are positive-semidefinite matrices, whose eigenvectors are outer products of quantum states: density matrices $\hat{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$. Normalization means $\sum_i p_i = 1$, so we can interpret a density matrix as a probability distribution over its eigenvectors.
  - Any physical evolution $\mathcal{N}$ must, given as input a density matrix $\hat{\rho}$, output a density matrix $\mathcal{N}(\hat{\rho})$ ("trace preservation", "positivity")
  - A physical evolution applied to a probability distribution over states is the same as the weighted sum of the physical evolution applied to each individual state:

    $$\mathcal{N}(p\hat{\rho} + (1-p)\hat{\sigma}) = p\mathcal{N}(\hat{\rho}) + (1-p)\mathcal{N}(\hat{\sigma})$$

    ("convex linearity")

- 1) Nothing breaks when I add extra degrees of freedom.
  - If $\mathcal{N}$ is a physical evolution acting on states in $\mathcal{H}_A$, $\mathcal{N} \circ \mathrm{Id}_B$ is a physical evolution acting on states in the composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ for *any* choice of reference system $\mathcal{H}_B$ ("complete positivity")
- That's it! These requirements completely define the action of quantum physical evolutions. In fact, they uniquely definite their action not just on states but on general operators: "quantum channels" *are* CPTP linear maps in full generality.

Three nicer characterizations of channels

- These definitions (in particular CP) seem pretty awful for characterization, since they're extremely abstract.
- Thankfully, the math people got to this problem well before QI was a real field. We have several very nice characterizations dating from the 1950s-1980s.

The Kraus picture

- Kraus: every channel application can be written as a linear combination of conjugation by linear operators:
$$\mathcal{N}(\hat{\rho}) = \sum_i E_i \, \hat{\rho} E_i^\dagger$$
where the "Kraus operators" obey a completeness relation $\sum_i E_i^\dagger E_i = I$. At most there are $|\mathcal{H}|^2$ Kraus operators.
- Much nicer to work with operators: they're just normal matrices. Price: many sets of Kraus ops can describe $\mathcal{N}$
- Very nice interpretation of a measurement channel: the Kraus operators are just the projectors onto the eigenspaces of the measurement operator. More general picture: POVMs "positive-operator valued measurements"

The Choi picture

- Matrices are vectors! Just write all the elements in a single column rather than in a rectangle.
- So the space of operators acting on a Hilbert space is itself a "doubled" Hilbert space: turn matrix elements $|i\rangle\langle j|$ into basis states $|i\rangle \otimes |j\rangle$
- Choi: $\mathcal{N}$ is CP iff the operator on the doubled Hilbert space whose blocks are $\mathcal{N}(|i\rangle\langle j|)$, the "Choi matrix", is positive semi-definite.
- Easily checkable, basis-independent way to check if a map is a good channel. Cost is working with quadratically more degrees of freedom.

The Stinespring picture

- Stinespring: any channel has an “environment model.”
  - For any channel on $\mathcal{H}_A$, there exists an auxiliary Hilbert space $\mathcal{H}_E$, of dimension at most $|\mathcal{H}_A|^2$, a “ready state” $|0\rangle_E$, and a unitary “Stinespring dilation” U such that
$$\mathcal{N}(\hat{\rho}) = \mathrm{tr}_E\big(U(\hat{\rho} \otimes |0\rangle_E\langle 0|)U^\dagger\big)$$
for all states $\hat{\rho}$.
  - Physical intuition for a measurement channel: the unitary evolves the system as
$$U: |\psi\rangle_A \otimes |0\rangle_E \to \sum_i c_i\, |i\rangle_S \otimes |e_i\rangle_E$$
Decoherence, with pointer states $|e_i\rangle_E$!

- It’s tempting to ask: when is the Stinespring dilation a unitary corresponding to a reasonable Hamiltonian? When is the auxiliary environment a physical environment?
- Good question! I’ll return to it at the end of the talk.

Lindblad evolution

- For now let me answer a much simpler question: when can a channel describe infinitesimal time evolution?

$$N\big(\rho(t)\big) \simeq \rho(t + dt)$$

- One easy example: unitary evolution. Described by the von Neumann equation:

$$i\frac{d}{dt}\hat{\rho} = [H, \hat{\rho}]$$

- Working through a very short calculation, we see that the channel is described by a single Kraus operator,

$$E = I - iHdt + \mathcal{O}(dt^2)$$

- More generally, we can add additional Kraus operators $L_i\sqrt{dt}$, which describe "jumps" a system undergoes when interacting with an environment. To preserve the completeness relation, we need to modify E to

$$E' = I + \left(-iH + \frac{1}{2}\sum_i L_i^\dagger L_i\right)dt + \mathcal{O}(dt^2)$$

- This gives a channel in "Lindblad form":
$\frac{d}{dt}\hat{\rho} = \mathcal{L}(\hat{\rho}) = -i[H,\hat{\rho}] + \sum_i (L_i\hat{\rho}L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \hat{\rho}\})$
- By construction, this time evolution is Markovian! So it doesn't describe the most general time evolution of a reduced system, but only the most general time evolution generated infinitesimally by a channel.

The physical environment

- I won’t describe this in detail, since I’m assuming this audience will be more familiar with the standard ways of describing environments in cosmology/QFT/etc, e.g. from earlier talks this week.
- Let me just remark that Markovianity, in particular, is much more fraught. In fact, all of the ways I know to do it involve *both* an infinite environment *and* tuned interactions between the system and environment. (Exercise left to the reader: what does this imply about Rindler/de Sitter space?)

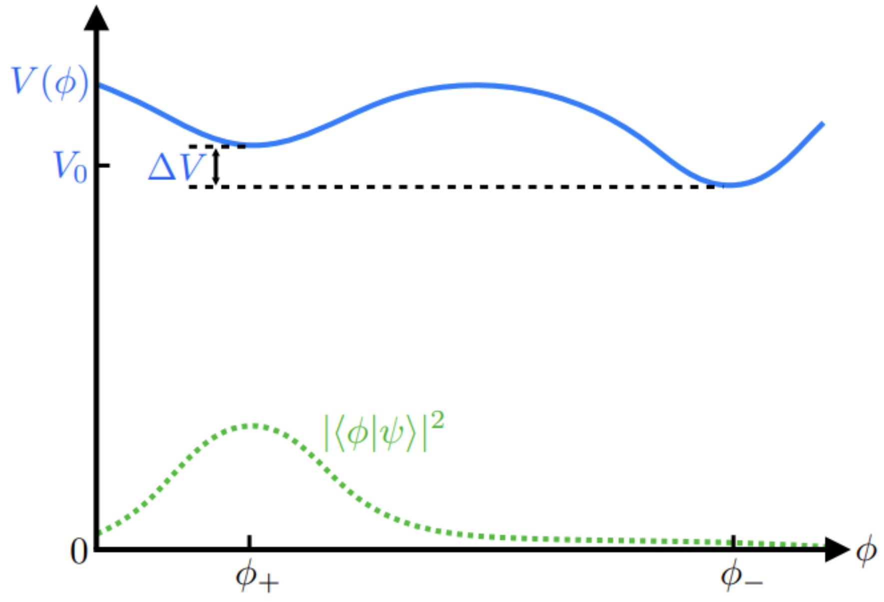
- Next, I want to give two examples of where I think the simplified QI notion of the environment is helpful in physical systems: one in gravity and one in many-body physics.
- I'll close with some in-progress work in the opposite direction: re-importing the notion of a physical environment back into the general open quantum systems picture.

Example One: Gravitational Decoherence

Cosmological Decoherence from Thermal Gravitons

Ning Bao,[a,b] Aidan Chatwin-Davies,[c] Jason Pollack,[d] and Grant N. Remmen[a]

- The setting: a cosmological landscape with multiple vacua.



- There's a classic story due to Coleman-De Luccia. First recall the radioactive atom in QM:

$$\Gamma_{\rm QM} = a e^{-b/\hbar} \left[1 + O(\hbar)\right].$$

$$b = 2 \int_{\vec{q}_+}^{\vec{\sigma}} \mathrm{d}\vec{q} \sqrt{2[V(\vec{q}) - V(\vec{q}_+)]}$$

  - Integrate over a path through the potential minimizing the action.
- Equivalent story in gravity, evaluating the Euclidean action of an instanton (spherically symmetric, thin wall)

- Scalar field coupled to gravity

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]$$

- Now WKB gives rate per time per unit volume:

$$\Gamma/\mathcal{V} = Ae^{-B/\hbar}\left[1 + O(\hbar)\right]$$

$$B = S_E(\phi_*, g_*) - S_E(\phi_+, g_+)$$

- Parametrically:

$$B \sim \Delta\phi^4(V_{\mathrm{t}} - V_0)^2/\Delta V^3$$

(lots of hard work to find A)

- This is a QFT in curved-space calculation. That is, the only degrees of freedom are the scalar field modes.
- However, gravity is present, and so the scalar field will source, and couple to, gravitons. So we should really describe the state of the scalar field as a *mixed* state, after integrating out the gravitons.
- We considered the special case of homogenous field configurations and a thermal graviton bath.

- Why? Our intuition: graviton bath could act as a decohering environment. Gravity couples to stress-energy, so it could, essentially, continually “measure” the field value. This could give a quantum Zeno effect! Suppress decay by repeated measurement. This would mean that the usual C-DL picture might be dramatically modified.

- Since we are now doing field theory, instead of tracing out the environment we integrate it out.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2\kappa h_{\mu\nu}$$

- The action:

$$S[h_{\mu\nu}, \phi] = S_S[\phi] + S_E[h_{\mu\nu}] + S_I[h_{\mu\nu}, \phi]$$

$$S_S = \int \mathrm{d}^4x \sqrt{-\bar{g}} \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]$$

$$S_E + S_{\mathrm{gf}} = \int \mathrm{d}^4x \sqrt{-\bar{g}} \left[\frac{1}{2} h_{\mu\nu} (\Box + 2\Lambda) h^{\mu\nu} - \frac{1}{4} h (\Box + 2\Lambda) h) + \frac{\Lambda}{\kappa^2} \right]$$

(after Fadeev-Popov gauge fixing, in harmonic gauge)

- Interaction

$$S_I = \int \mathrm{d}^4x\sqrt{-\overline{g}}\left(\kappa T^{\mu\nu}h_{\mu\nu} + \kappa^2 U^{\mu\nu\rho\sigma}h_{\mu\nu}h_{\rho\sigma}\right)$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\overline{g}_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \overline{g}_{\mu\nu}V(\phi)$$

- Reduced state: seperate bra + ket d.o.f.

$$\rho_S[\phi,\phi',t] = \int \mathrm{d}h_{\mu\nu}\rho[\phi h_{\mu\nu},\phi' h'_{\mu\nu},t] = \int \mathrm{d}h_{\mu\nu}e^{iS[\phi,h_{\mu\nu}]}\rho[\phi h_{\mu\nu},\phi' h'_{\mu\nu},0]e^{-iS[\phi',h'_{\mu\nu}]}$$

- As usual for decoherence, assume uncorrelated initial state:

$$\rho[\phi h_{\mu\nu},\phi' h'_{\mu\nu},0] = \rho_S[\phi,\phi',0]\rho_E[h_{\mu\nu},h'_{\mu\nu},0]$$

- Now we can use the Feynman-Vernon influence functional to define an effective action for the system alone.

$$
\begin{aligned}
&\rho_S[\phi, \phi', t] \\
&= \int \rho_S[\phi, \phi', 0] e^{i(S_S[\phi] - S_S[\phi'])} \int_{h_{\mu\nu}} \rho_E[h_{\mu\nu}, h'_{\mu\nu}, 0] e^{i(S_E[h_{\mu\nu}] + S_I[h_{\mu\nu}, \phi] - S_E[h'_{\mu\nu}] - S_I[h'_{\mu\nu}, \phi\prime])} \\
&\equiv \int \rho_S[\phi, \phi', 0] e^{i(S_S[\phi] - S_S[\phi'] + S_{IF}[\phi, \phi'])},
\end{aligned}
$$

- Now plug in for a thermal environment.

- Using results of Blencowe 2013 + some textbook nonequilibrium QFT:

$$\begin{aligned}\partial_t\rho(t) = -i[H,\rho(t)] - \int_0^t \mathrm{d}t' \int \mathrm{d}\mathbf{r} \int \mathrm{d}\mathbf{r}' \Big[N(\mathbf{r}-\mathbf{r}',t') \Big(2[T_{\mu\nu}(\mathbf{r}),[T^{\mu\nu}(\mathbf{r}',-t'),\rho(t)]] \\ - [T_\mu{}^\mu(\mathbf{r}),[T_\nu{}^\nu(\mathbf{r}',-t'),\rho(t)]] \Big) \\ - iD(\mathbf{r}-\mathbf{r}',t') \Big(2[T_{\mu\nu}(\mathbf{r}),\{T^{\mu\nu}(\mathbf{r}',-t'),\rho(t)\}] \\ - [T_\mu{}^\mu(\mathbf{r}),\{T_\nu{}^\nu(\mathbf{r}',-t'),\rho(t)\}] \Big) \Big]\end{aligned}$$

- Depends on noise and dissipation kernels in the environment.

- Gravitons obey Bose-Einstein statistics

$$N(\mathbf{r},t) = \frac{\kappa^2}{8}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k}\cos(kt)\left[n(k)+\frac{1}{2}\right]$$

$$D(\mathbf{r},t) = \frac{\kappa^2}{16}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k}\sin(kt),$$

$$n(k) = (e^{\beta k}-1)^{-1}$$

- In the rest frame of the scalar field:

$$T_{\mu\nu} = \Phi \times \mathrm{diag}(1, w, \ldots, w)$$

- The Hilbert space is spanned by the eigenstates of field value. We'll consider a superposition of two field value states and look at whether the graviton bath causes this bath to decohere. This will depend on the equation of state of the field, w.
- This is now a QM problem, involving calculating a 2x2 density matrix. Just need to do the integrals…

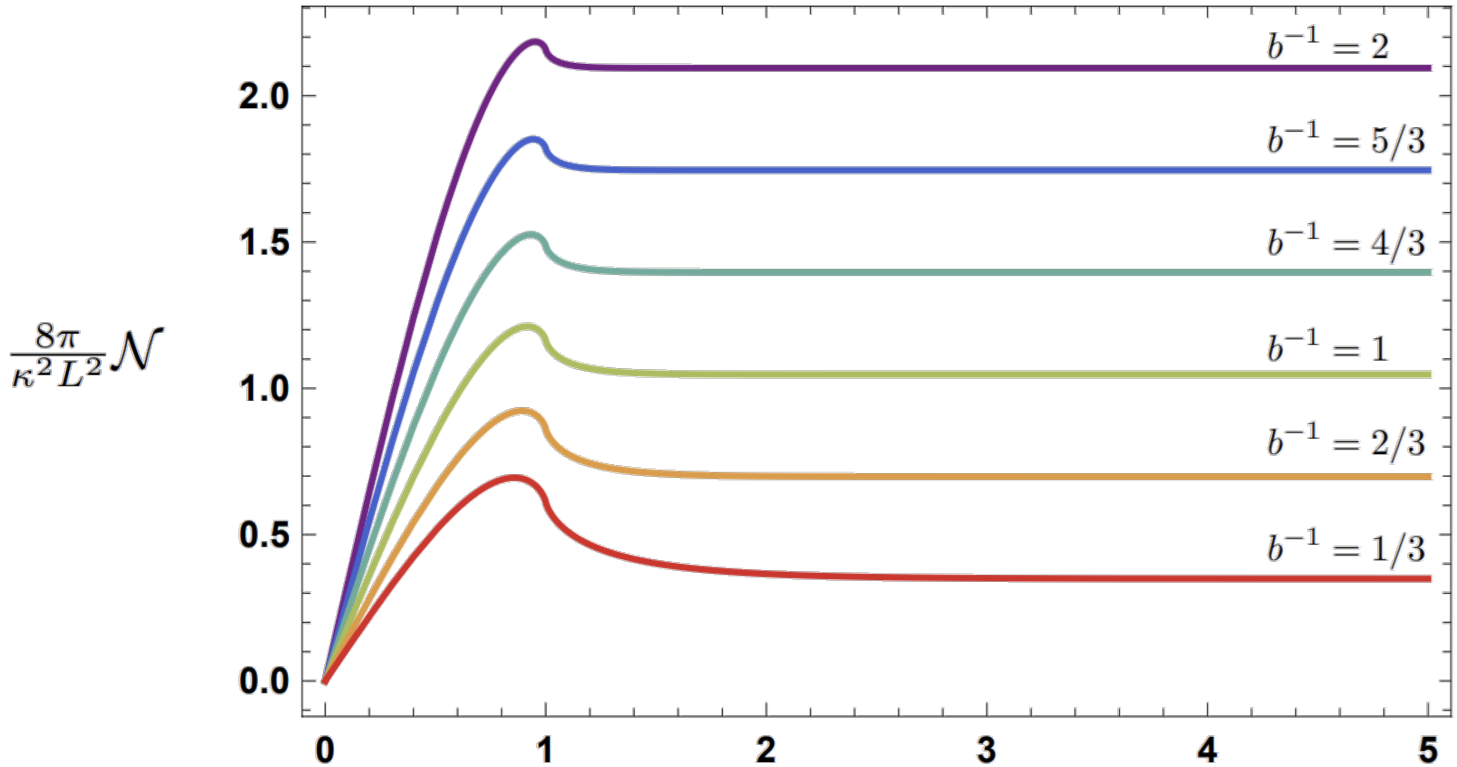
The reduced problem

$$\partial_t \rho(t) = -i\mathcal{V}[\Phi, \rho(t)] - \mathcal{V} f(w)\Big(\mathcal{N}(t)[\Phi, [\Phi, \rho(t)]] - i\mathcal{D}(t)[\Phi, \{\Phi, \rho(t)\}]\Big)$$

$$\mathcal{N}(t) = \int_0^t \mathrm{d}t' \int_{\mathcal{B}} \mathrm{d}\mathbf{r}\, N(\mathbf{r}, t')$$
$$\mathcal{D}(t) = \int_0^t \mathrm{d}t' \int_{\mathcal{B}} \mathrm{d}\mathbf{r}\, D(\mathbf{r}, t')$$

$$f(w) = 1 - 3w(w-2)$$

- The integrals can be done exactly, but we'll only care about the long-time behavior of the noise kernel, as that's what sets the decoherence rate.



- b^{-1} = dimensionless temperature. This is the behavior we expect! Asymptotic decoherence rate is proportional to temperature.

Decoherence analysis

- Initial state

$$\rho = \begin{pmatrix} \rho_1 & \rho_e \\ \rho_e^* & 1-\rho_1 \end{pmatrix}$$

- Asymptotic behavior of off-diagonal element determines the decoherence. If exponential decay: we have decoherence, and the pointer basis is the field value basis.

$$\rho_e(t) = \rho_{e,0} e^{-(\lambda + i\omega)t}$$

$$\lambda = \frac{4\pi\kappa^2 L^6}{72\beta} f(w)(\Phi_1 - \Phi_2)^2$$

$$\omega = \frac{4}{3}\pi L^3 (\Phi_1 - \Phi_2)[1 - 32\kappa^2 L^2 f(w)(\Phi_1 + \Phi_2)]$$

- Recall that $f(w) = 1 - 3w(w-2)$

- So decoherence occurs for some equation of state values:

$$w \in \left(1 - \frac{2}{\sqrt{3}}, 1 + \frac{2}{\sqrt{3}}\right) \approx (-0.155, 2.155)$$

- Outside these values, might decohere, but not in the field value basis.

Example two: finding dynamics from a thermal state

- Say I have a big, complicated many-body quantum system – like a spin chain, or a molecule, or a protein. It's probably intractable to analytically determine its dynamics, e.g. find its spectrum.
- However, nature likes to thermalize things. If I have access to the thermal density matrix $\rho = e^{-\beta H}$ of the system, does this help?

$$\rho = e^{-\beta H}$$

- I could try to do tomography. However, this requires an exponential number of measurements.
- On the other hand, I could restrict myself to local measurements only, and determine the states of subsets of degrees of freedom. Then I could look for a global state which is compatible with all these reduced states.

- This is the "quantum marginal problem".

$$\rho_i = \mathrm{Tr}_{\setminus S_i}\, \rho$$

- If we know our Hamiltonian is 2-local, then finding all the 2-local density matrices will let us find the ground-state energy by solving a polynomially-sized optimization problem, assuming we know how to solve the marginal problem.

$$\min_\rho \mathrm{Tr}\, H\rho = \min_\rho \sum_{i,j} \mathrm{Tr}\, h_{ij}\rho = \min_{\{\rho_{i,j}\text{comp.}\}} \sum_{i,j} \mathrm{Tr}\, h_{i,j}\rho_{i,j}$$

- Unfortunately:

Consistency of Local Density Matrices is QMA-complete

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June 16, 2006

- However, there is a special case in QI in which we can bootstrap our way up from a smaller state to a larger state: when the *quantum Markov property* is satisfied, i.e. the conditional mutual information

$$S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B)$$

vanishes. That is, all of the correlations between A and C are mediated by B.

When the CMI holds, we can reconstruct BC from B exactly using *Petz recovery*. And the reconstruction degrades gracefully.

Thermal reconstruction of chaotic quantum many-body systems

Shozab Qasim[1] and Jason Pollack[2,3]

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[3]Institute for Quantum & Information Sciences, Syracuse University

Theorem 2. *[38] For any density operator ρ_{ABC} on $A \otimes B \otimes C$, where A, B, and C are separable Hilbert spaces, there exists a trace-preserving completely positive map $\mathcal{R}_{B \to BC}$ from the space of operators on B to the space of operators on $B \otimes C$ such that*

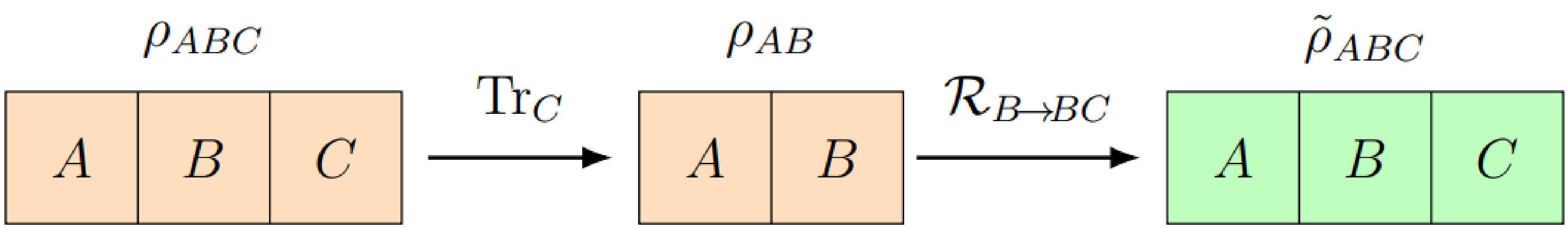
$$2^{-\frac{1}{2} I(A:C|B)_\rho} \leq F(\rho_{ABC}, (\mathcal{I}_A \otimes \mathcal{R}^{(\lambda)}_{B \to BC}(\cdot))\rho_{AB}). \quad (3)$$

Furthermore if A, B and C are finite dimensional then $\mathcal{R}^{(\lambda)}_{B \to BC}(\cdot)$ has the form

$$\mathcal{R}^{(\lambda)}_{B \to BC}(\cdot) = \rho_{BC}^{\frac{1}{2} - \frac{i\lambda}{2}} \rho_B^{-\frac{1}{2} + \frac{i\lambda}{2}} (\cdot) \rho_B^{-\frac{1}{2} - \frac{i\lambda}{2}} \rho_{BC}^{\frac{1}{2} + \frac{i\lambda}{2}}. \quad (4)$$

This most general form is called the Petz recovery map, we will study this in the case where $\lambda = 0$.

- We want to apply this setup to the specific case of *thermal reconstruction*. We'll bound how far apart the reconstructed state is from the true global thermal state, both analytically and numerically. If they're close, I can plausibly learn about global dynamics by locally probing the state.



Analytic results

- Any density matrix is the thermal state of *some* Hamiltonian. If we reconstruct a full state and treat it as thermal, we expect to recover a slightly different Hamiltonian than the actual Hamiltonian with respect to which the actual state is thermal.
- Say we do time evolution with this reconstructed Hamiltonian. How far off will we be from the true Hamiltonian?
- We give a series of lemmas to answer this.

Lemma 3. *Given* $2^{-\frac{1}{2}I(A:C|B)_\rho} \leq F(\rho_{ABC}, \widetilde{\rho}_{ABC})$, *where* $\widetilde{\rho}_{ABC}$ *is the reconstructed state via the Petz map,* $2^{-\frac{1}{2}I(A:C|B)_\rho} \leq F(\mathcal{N}(\rho_{ABC}), \mathcal{N}(\widetilde{\rho}_{ABC}))$ *for some quantum channel* $\mathcal{N}$. *Additionally,* $\|\mathcal{N}(\rho_{ABC}) - \mathcal{N}(\widetilde{\rho}_{ABC})\|_1 \leq \epsilon$ *where* $\epsilon = \|\rho - \widetilde{\rho}\|_1 \leq \sqrt{4(1 - 2^{-\frac{1}{2}I(A:C|B)_\rho})}$.

- For example, we could take the channels to implement time evolution under the true or reconstructed Hamiltonians.

Lemma 5. *Given* $2^{-\frac{1}{2}I(A:C|B)_\rho} \leq F(\rho_{ABC}, \widetilde{\rho}_{ABC})$ *where* $\widetilde{\rho}_{ABC}$ *is the reconstructed state via the Petz map,* $\|\mathcal{O}(\rho - \widetilde{\rho})\|_1 \leq \|\mathcal{O}\|_\infty \sqrt{4(1 - 2^{-\frac{1}{2}I(A:C|B)_\rho})}$ *where* $\epsilon = \|\rho - \widetilde{\rho}\|_1 \leq \sqrt{4(1 - 2^{-\frac{1}{2}I(A:C|B)_\rho})}$.

- How much do expectation values of observables differ?

Lemma 6. *Given thermal states* $\rho = e^{-\beta H}$ *and the Petz reconstructed state* $\widetilde{\rho} = e^{-\beta \widetilde{H}}$ *which satisfy* $2^{-\frac{1}{2}I(A:C|B)_\rho} \leq F(\rho, \widetilde{\rho})$, *a high-temperature expansion (small values of* β*) yields a trace norm difference between the Hamiltonians given by* $\left\| \widetilde{H} - H \right\|_1 \leq \frac{\delta}{\beta}$ *where* $\delta = \sqrt{1 - \frac{1}{4}\|\rho - \widetilde{\rho}\|_1^2}$.

- How much do the reconstructed Hamiltonians differ?

Numerics

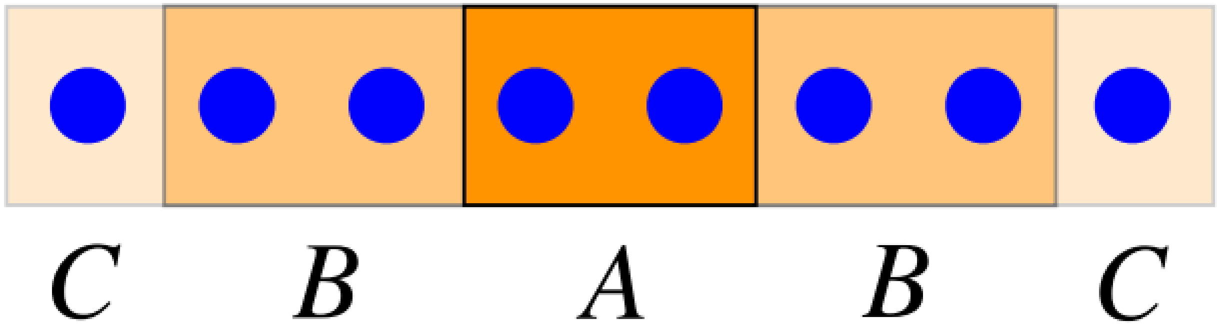
- We work with an Ising-type Hamiltonian

$$H = \alpha \sum_{i}^{L-1} \sigma_i^z \sigma_{i+1}^z + h_z \sum_i^L \sigma_i^z + J_x \sum_i^L \sigma_i^x$$

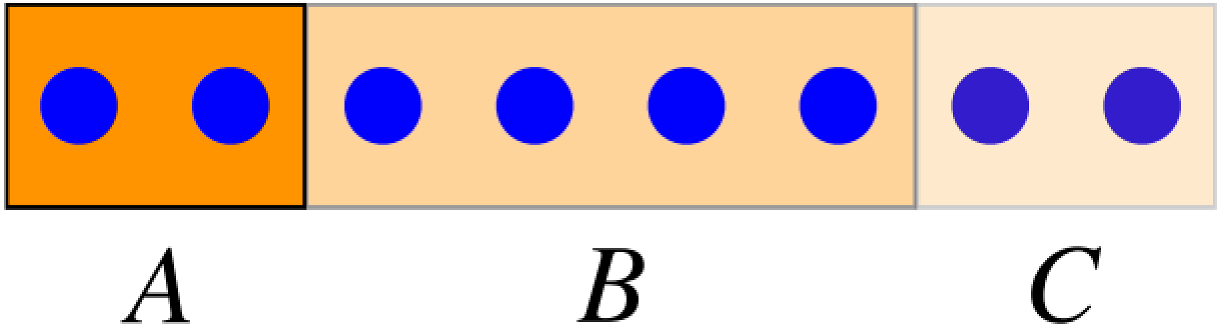
- We'll consider Gibbs states w/r/t H, both at the chaotic point $\alpha = 1.0, J_x = 1.05, h_z = -0.5$

  as well as moving into the integrable phase by

  tuning h_z

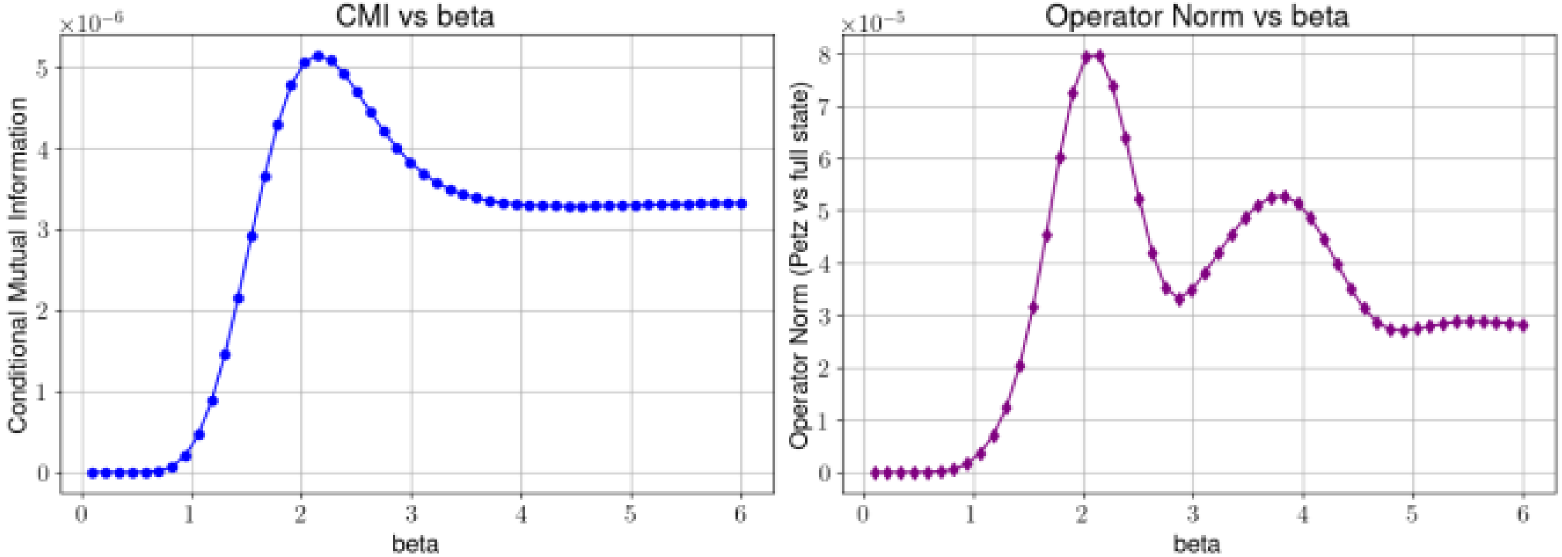


(a) Permuted configuration.



(b) Non-permuted configuration.

Chaotic phase



(a) Reconstruction for thermal states where the sites are not permuted in the chaotic phase.

- Good reconstruction even in the very low-temperature limit
- Special scale: “worst temperature”

What sets the scale?

- One heuristic: characteristic energy in subsystem
  - Large vs small subsystems (compared to temperature scale)
  - Small enough subsystems “don’t feel the temperature”
- Another heuristic: Hawking-Page transition, i.e. large vs small black holes in units of L_AdS
  - Compute CMI using RT: expect transition from hyperbolic to linear geodesics as BH gets larger

Example three: the Redfield equation

- Let's work in a setting more general than Lindblad, but *perturbatively* in the interaction between the system and the environment. It will be straightforward to derive an expression for the *finite*-time evolution map from an initial time t_i to a later time t: we just pick the channel which has $\mathcal{U}(t, t_i)$ as its Stinespring dilation. The challenge will be to try to find an *infinitesimal*-time evolution map which integrates to the finite-time one. This need not be in Lindblad form, and indeed need not even be a channel, but we'd like to try to approximate it as one.

In defense of Redfield: or how I learned to stop worrying and love non-Lindbladian dynamics

Greg Kaplanek[1,2,*] and Jason Pollack[1,2,†]

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Evolution of full state

$$H_{\mathrm{world}}(t) = H_{\mathrm{S}}(t) \otimes \mathbb{I}_{\mathrm{E}} + \mathbb{I}_{\mathrm{S}} \otimes H_{\mathrm{E}}(t) + \lambda H_{\mathrm{int}}(t)$$

- Go to the interaction picture

$$\rho(t) = \mathcal{U}_0^\dagger(t, t_{\mathrm{i}})\widetilde{\rho}(t)\mathcal{U}_0(t, t_{\mathrm{i}})$$

$$H_{\mathrm{int}}^I(t) = \mathcal{U}_0^\dagger(t, t_{\mathrm{i}})H_{\mathrm{int}}(t)\mathcal{U}_0(t, t_{\mathrm{i}})$$

$$\mathcal{U}_0(t, t_{\mathrm{i}}) = \mathcal{T}e^{-i\int_{t_{\mathrm{i}}}^t \mathrm{d}s\ H_{\mathrm{S}}(s)} \otimes \mathcal{T}e^{-i\int_{t_{\mathrm{i}}}^t \mathrm{d}s\ H_{\mathrm{E}}(s)}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i\lambda\left[H_{\mathrm{int}}^I(t), \rho(t)\right]$$

- Start with initial uncorrelated state, integrate and perturb:

$$\begin{aligned}\rho(t) \simeq\ & \varrho_{\mathrm{Si}} \otimes \varrho_{\mathrm{Ei}} \\ & -i\lambda \int_{t_{\mathrm{i}}}^t \mathrm{d}s\ \left[H_{\mathrm{int}}^I(s), \varrho_{\mathrm{Si}} \otimes \varrho_{\mathrm{Ei}}\right] \\ & -\lambda^2 \int_{t_{\mathrm{i}}}^t \mathrm{d}s \int_{t_{\mathrm{i}}}^s \mathrm{d}u \left[H_{\mathrm{int}}^I(s), \left[H_{\mathrm{int}}^I(u), \varrho_{\mathrm{Si}} \otimes \varrho_{\mathrm{Ei}}\right]\right] \\ & +\mathcal{O}(\lambda^3)\ .\end{aligned}$$

Evolution of reduced state

- Now trace out the environment:

$$\varrho_{\mathrm{S}}(t) = \operatorname*{Tr}_{\mathrm{E}}\Big(\mathcal{U}(t,t_{\mathrm{i}})\left(\varrho_{\mathrm{Si}} \otimes \varrho_{\mathrm{Ei}}\right)\mathcal{U}^{\dagger}(t,t_{\mathrm{i}})\Big) = \Phi(t,t_{\mathrm{i}})[\varrho_{\mathrm{Si}}]$$

$$\mathcal{U}(t,t_{\mathrm{i}}) = \mathcal{T}e^{-i\lambda\int_{t_{\mathrm{i}}}^{t}\mathrm{d}s\, H_{\mathrm{int}}(s)}$$

- This is manifestly a Stinespring dilation, so the map $\Phi(t,t_{\mathrm{i}})$ is a channel, and thus completely positive and trace-preserving. However, this may not be respected by the perturbation expansion:

$$\rho_{\mathrm{S}}(t) = \rho_{\mathrm{Si}} + \lambda^2\Phi_2(t)[\rho_{\mathrm{Si}}] + \mathcal{O}(\lambda^4)$$

$$\rho_S(t) = \rho_{Si} + \lambda^2 \Phi_2(t)[\rho_{Si}] + \mathcal{O}(\lambda^4)$$

- We can write the perturbative map in terms of the interaction Hamiltonian:

$$\begin{aligned}\Phi_2(t)[\varrho_{\mathrm{Si}}] \\ = \sum_{\alpha,\beta} \int_{t_\mathrm{i}}^{t} \mathrm{d}s \int_{t_\mathrm{i}}^{s} \mathrm{d}u \Big(& [S_\alpha(u)\varrho_{\mathrm{Si}}, S_\beta(s)]\, \mathcal{E}_{\beta\alpha}(s,u) \\ & + [S_\beta(s), \varrho_{\mathrm{Si}} S_\alpha(u)]\, \mathcal{E}^*_{\beta\alpha}(s,u) \Big)\end{aligned}$$

- We've defined the environmental correlator

$$\mathcal{E}_{\alpha\beta}(t,s) = \operatorname*{Tr}_{\mathrm{E}}\left[E_\alpha(t) E_\beta(s) \rho_{\mathrm{Ei}}\right]$$

A physical assumption

- Let's assume that the environmental kernel admits a positive decomposition:

$$\mathcal{E}_{\alpha\beta}(t,s) = \sum_{\kappa} u_{\kappa\alpha}(t) u^*_{\kappa\beta}(s)$$

- With this decomposition, we can define time-evolved system operators

$$V_\kappa(t) = \sum_\alpha \int_{t_\mathrm{i}}^{t} ds\; u^*_{\kappa\alpha}(s) S_\alpha(s)$$

- Now we can rewrite the perturbative map in a Lindblad-like form:

$$\Phi_2(t)[\varrho_{\mathrm{Si}}] = \sum_\kappa V_\kappa(t)\varrho_{\mathrm{Si}}V_\kappa^\dagger(t) - B(t)\varrho_{\mathrm{Si}} - \varrho_{\mathrm{Si}}B^\dagger(t)$$

$$B(t) = \sum_{\alpha,\beta}\int_{t_{\mathrm{i}}}^{t} ds \int_{t_{\mathrm{i}}}^{s} du\, \mathcal{E}_{\beta\alpha}(s,u)S_\beta(s)S_\alpha(u)$$

- Using this form, we can write the map in an approximate Kraus form:

$$\rho_{\mathrm{S}}(t) = K_0\rho_{\mathrm{Si}}K_0^\dagger + \sum_\kappa K_\kappa\rho_{\mathrm{Si}}K_\kappa^\dagger + \mathcal{O}(\lambda^4)$$

$$K_0 = \mathbb{I}_{\mathrm{S}} - \lambda^2 B \qquad K_\kappa = \lambda V_\kappa$$

$$\rho_{\rm S}(t) = K_0 \rho_{\rm Si} K_0^\dagger + \sum\nolimits_\kappa K_\kappa \rho_{\rm Si} K_\kappa^\dagger + \mathcal{O}(\lambda^4)$$

- So the map is *approximately* CPTP, with corrections given by the higher-order terms in the expansion. Because the terms are time-dependent, this means that there will be a breakdown at late times; in particular

  $K_0 = \mathbb{I}_{\rm S} - \lambda^2 B$ exhibits secular growth.

- In the spirit of field theory, we can try to resum the perturbative map to get better behavior.

Resolvent expansion

$$\rho_{\mathrm{S}}(t) = \rho_{\mathrm{Si}} + \lambda^2 \Phi_2(t)[\rho_{\mathrm{Si}}] + \mathcal{O}(\lambda^4)$$

- This equation says that the initial and time-evolved state only differ by higher-order terms. That means we can rewrite the expression in a cheekier way:

$$\varrho_{\mathrm{S}}(t) = \varrho_{\mathrm{Si}} + \lambda^2 \Phi_2(t)\left[\varrho_{\mathrm{S}}(t)\right] + \mathcal{O}(\lambda^4)$$

- Now we can formally solve for $\varrho_{\mathrm{S}}(t)$:

$$\varrho_{\mathrm{S}}(t) = \left(\mathcal{I} - \lambda^2 \Phi_2(t)\right)^{-1}\left[\varrho_{\mathrm{Si}}\right] + \mathcal{O}(\lambda^4)$$

- The expression in parentheses defines the *resolvent map*.

$$\Phi_{\rm res}(t) := \left(\mathcal{I} - \lambda^2 \Phi_2(t)\right)^{-1}$$

- This agrees with the perturbative map at lowest nontrivial order, but differs at higher orders by the resummation of an infinite number of corrections:

$$\Phi_{\rm res} = \mathcal{I} + \lambda^2 \Phi_2 + \lambda^4 \Phi_2^2 + \cdots$$

- We therefore expect it to have better behavior. Compare to the replacement of bare propagators by dressed (Dyson) propagators in QFT.

The resolvent map is exactly CPTP

- We can actually solve the self-consistency equation $\Phi_{\text{res}}(t) := \left(\mathcal{I} - \lambda^2 \Phi_2(t)\right)^{-1}$. The solution is

$$\Phi_{\text{res}}(t) = \int_0^\infty \mathrm{d}r \; e^{-r} e^{r\lambda^2 \Phi_2(t)}$$

- Recall that

$$\int_0^\infty \mathrm{d}r \;\; e^{-r} \; = \; 1$$

- So this is a positive weighted average of CPTP maps, and since CPTP maps obey convex linearity that means it is itself a CPTP map. However, it might not be very physical…

The Redfield approach

- What I just showed you is fun and (hopefully) interesting, but it can't be interpreted as a master equation: not the exponentiated form of a generating equation.
- The more traditional approach:

$$\frac{\mathrm{d}\varrho_{\mathrm{S}}}{\mathrm{d}t} = \dot{\Phi}(t, t_{\mathrm{i}})\left[\varrho_{\mathrm{Si}}\right] \qquad \varrho_{\mathrm{Si}} = \Phi^{-1}(t, t_{\mathrm{i}})\left[\varrho_{\mathrm{S}}(t)\right]$$

$$\frac{\mathrm{d}\varrho_{\mathrm{S}}}{\mathrm{d}t} = \dot{\Phi}(t, t_{\mathrm{i}})\Phi^{-1}(t, t_{\mathrm{i}})\left[\varrho_{\mathrm{S}}(t)\right]$$

- Now do the weak-coupling expansion and truncate.

$$\Phi^{-1}(t, t_{\mathrm{i}}) \simeq \mathcal{I} - \lambda^2 \Phi_2(t) + \mathcal{O}(\lambda^4)$$
$$\dot{\Phi}(t, t_{\mathrm{i}}) \simeq \lambda^2 \dot{\Phi}_2(t) + \mathcal{O}(\lambda^4)$$

$$\frac{\mathrm{d}\varrho_{\mathrm{S}}}{\mathrm{d}t} \simeq \lambda^2 \dot{\Phi}_2(t) \left[\varrho_{\mathrm{S}}(t)\right]$$

- Explicitly,

$$\frac{\mathrm{d}\varrho_{\mathrm{S}}}{\mathrm{d}t} \simeq \lambda^2 \sum_{\alpha,\beta} \int_{t_{\mathrm{i}}}^{t} \mathrm{d}s \left(\left[S_\alpha(s)\varrho_{\mathrm{S}}(t), S_\beta(t)\right] \mathcal{E}_{\beta\alpha}(t, s) \right.$$
$$\left. + \left[S_\beta(t), \varrho_{\mathrm{S}}(t), S_\alpha(s)\right] \mathcal{E}^*_{\beta\alpha}(t, s) \right)$$

- Formal solution:

$$\Phi_{\mathrm{Red}}(t, t_{\mathrm{i}}) = \mathcal{T} \exp\left(\lambda^2 \int_{t_{\mathrm{i}}}^{t} \mathrm{d}s\, \dot{\Phi}_2(s)\right)$$

- This is now more directly related to local time evolution. The price we pay is that it might not be CPTP.
- To conclude, let's compare all these approaches in two exactly solvable models.

Example one: two qubits

$$H_{\mathrm{S}} = H_{\mathrm{E}} = \frac{\omega}{2}\sigma_3$$

$$H_{\mathrm{int}}(t) = \lambda\,\sigma_1 \otimes \sigma_1$$

$$\rho(0) = \frac{\mathbb{I}}{2} \otimes \left|\downarrow\right\rangle\left\langle\downarrow\right|$$

$$\varrho_{\mathrm{S}}(t) = \operatorname*{Tr}_{\mathrm{E}}\left[e^{-iHt}\varrho_{\mathrm{Si}}\,e^{+iHt}\right] = \begin{bmatrix} p(t) & 0 \\ 0 & 1-p(t) \end{bmatrix}$$

where

$$p(t) := \frac{\lambda^2 \sin^2\left(\sqrt{\lambda^2+\omega^2}\,t\right)}{2\left(\lambda^2+\omega^2\right)} + \frac{1}{2}\cos^2(\lambda t)\ .$$

- That was the exact dynamics. What about the approximations?

$$\mathcal{E}(t,s) = \langle\downarrow|\,\sigma_1(t)\sigma_1(s)\,|\downarrow\rangle = e^{-i\omega(t-s)}$$

First, perturbation theory should be tested. Directly integrating Eq. (27), one finds that

$$\begin{aligned} \varrho_{\mathrm{S}}(t) &\simeq \varrho_{\mathrm{Si}} + \lambda^2\Phi_2(t)[\varrho_{\mathrm{Si}}] \\ &= \frac{\mathbb{I}}{2} + \frac{\lambda^2}{2}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \,\Big([\sigma_1(u),\sigma_1(s)]\,\mathcal{E}(s,u) \\ &\qquad + [\sigma_1(s),\sigma_1(u)]\,\mathcal{E}^*(s,u) \Big) \\ &= \begin{bmatrix} \frac{1}{2} - \frac{\lambda^2 t^2}{2} + \frac{\lambda^2\sin^2(t\omega)}{2\omega^2} & 0 \\ 0 & \frac{1}{2} + \frac{\lambda^2 t^2}{2} - \frac{\lambda^2\sin^2(t\omega)}{2\omega^2} \end{bmatrix}. \end{aligned} \tag{127}$$

Let us next test the resolvant resummation (47), which involves the inversion of the above perturbative series. Explicitly one finds

$$\varrho_{\mathrm{S}}(t) \simeq \Phi_{\mathrm{res}}(t)[\varrho_{\mathrm{Si}}] := \left(\mathcal{I} - \lambda^2\Phi_2(t)\right)^{-1}[\varrho_{\mathrm{Si}}] \tag{128}$$

$$= \begin{bmatrix} p_{\mathrm{res}}(t) & 0 \\ 0 & 1 - p_{\mathrm{res}}(t) \end{bmatrix} \tag{129}$$

where

$$p_{\mathrm{res}}(t) = \frac{\frac{2\lambda^2 \sin^2(t\omega)}{\omega^2} + 1}{2\left(\lambda^2 t^2 + \frac{\lambda^2 \sin^2(t\omega)}{\omega^2} + 1\right)} \tag{130}$$

This agrees with perturbation theory in the limit $\ll 1$ (as it must).

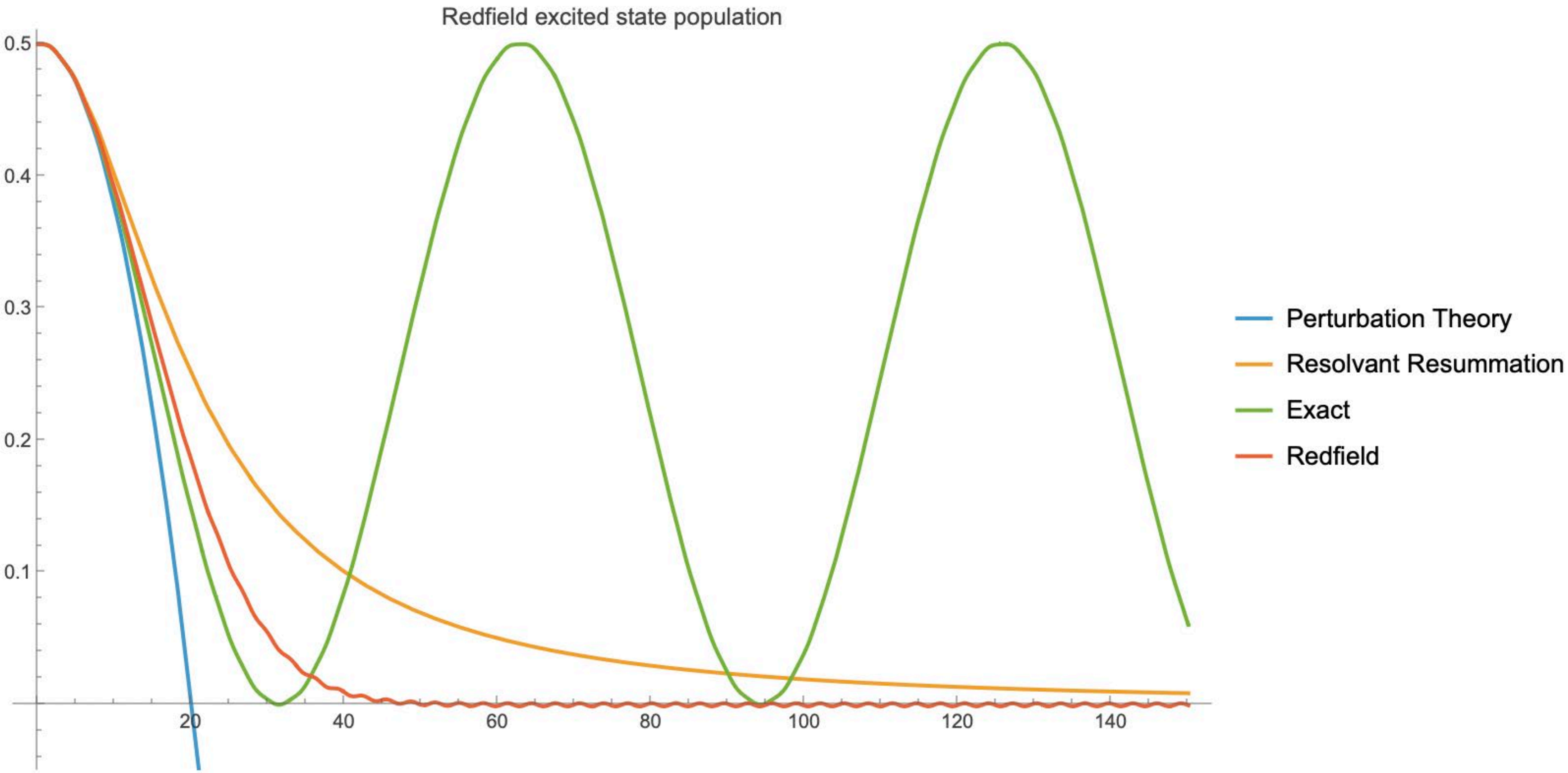
Finally, we integrate the Redfield equation (58) in this context. Explicitly, we solve

$$\frac{\mathrm{d}\varrho_{\mathrm{S}}}{\mathrm{d}t} \simeq \lambda^2 \int_0^t \mathrm{d}s \left([\sigma_1(s)\varrho_{\mathrm{S}}(t), \sigma_1(t)]\, \mathcal{E}(t,s) + [\sigma_1(t), \varrho_{\mathrm{S}}(t)\sigma_1(s)]\, \mathcal{E}^*(t,s) \right), \tag{131}$$

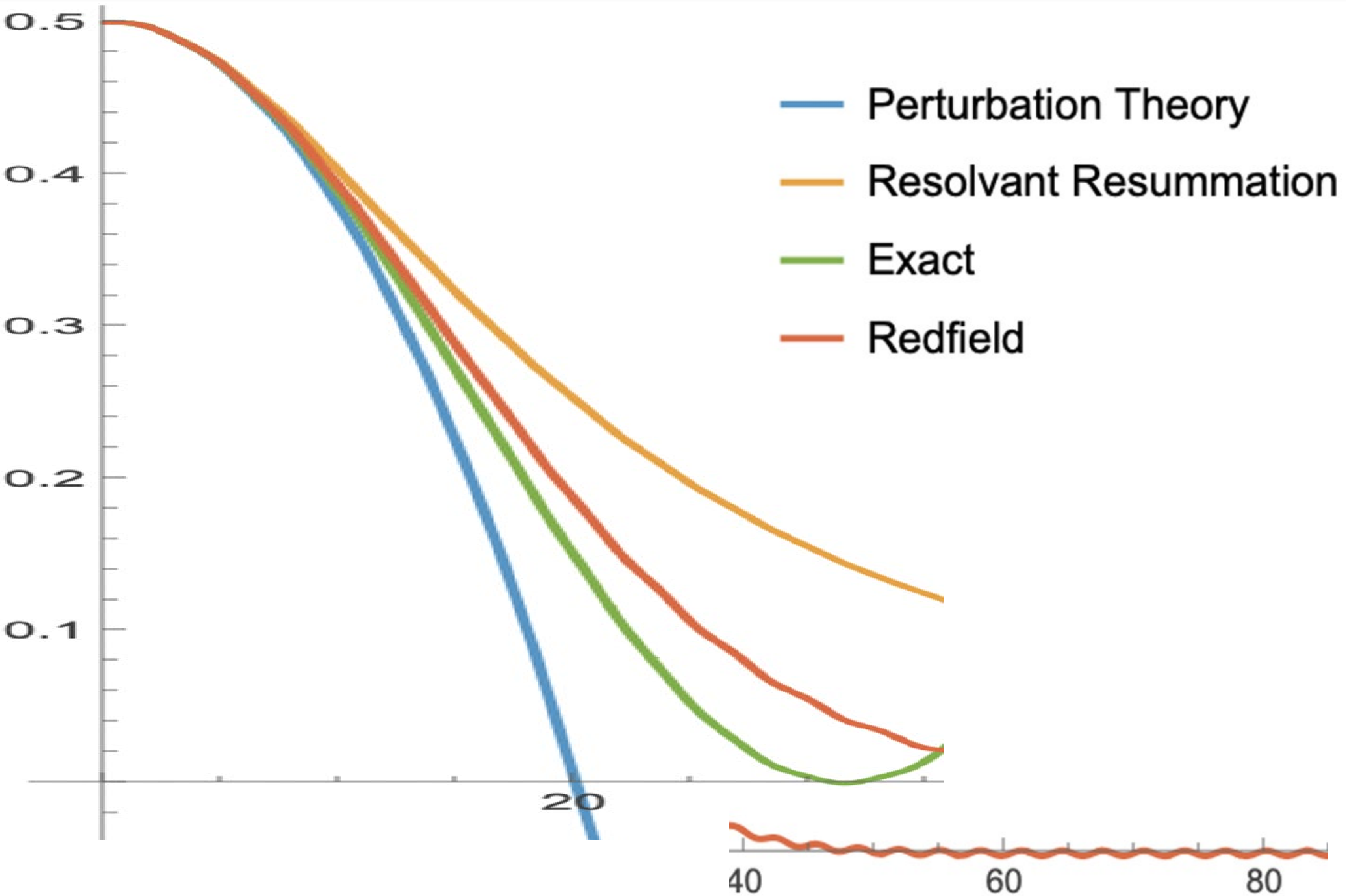
of course subject to the initial condition $\varrho_{\mathrm{Si}} = \mathbb{I}/2$. One finds the diagonal solution

$$\varrho_{\mathrm{S}}(t) \simeq \begin{bmatrix} p_{\mathrm{Red}}(t) & 0 \\ 0 & 1 - p_{\mathrm{Red}}(t) \end{bmatrix} \tag{132}$$

$$p_{\mathrm{Red}}(t) = e^{-\lambda^2 t^2 - \frac{\lambda^2}{\omega^2}\sin(\omega t)} \times \left(\frac{1}{2} + \frac{\lambda^2}{\omega^2} \int_0^t \mathrm{d}s\, e^{\lambda^2 s^2 + \frac{\lambda^2}{\omega^2}\sin(st)} \sin(2\omega s) \right) \tag{134}$$



Comparison done for choice of $\omega = 1$ and $\lambda = 0.05$.



0.5
0.4
0.3
0.2
0.1
20
40
60
80
Perturbation Theory
Resolvant Resummation
Exact
Redfield

Example two: qubit + oscillator

$$H_{\mathrm{S}} = \frac{\omega}{2}\sigma_3 \qquad \text{and} \qquad H_{\mathrm{E}} = \Omega\left(a^\dagger a + \frac{1}{2}\right) \qquad [a, a^\dagger] = \mathbb{I}_{\mathrm{E}}$$

$$H_{\mathrm{int}} = \lambda\,\sigma_3 \otimes (a + a^\dagger)$$

$$[H_S, H_{\mathrm{int}}] = 0,$$

- So this model yields pure dephasing of the qubit system.

Exact solution

$$\rho(0) = \varrho_{\mathrm{Si}} \otimes \frac{e^{-\beta H_{\mathrm{E}}}}{\mathrm{Tr}[e^{-\beta H_{\mathrm{E}}}]}$$

$$\varrho_{\mathrm{S}}(t) = \begin{bmatrix} \varrho_{\uparrow\uparrow}(0) & \varrho_{\uparrow\downarrow}(0)e^{-\lambda^2\Gamma(t)} \\ \varrho_{\downarrow\uparrow}(0)e^{-\lambda^2\Gamma(t)} & \varrho_{\downarrow\downarrow}(0) \end{bmatrix}$$

$$\Gamma(t) = \frac{4}{\Omega^2}\left(1 - \cos(\Omega t)\right) \coth\left(\frac{\beta\Omega}{2}\right)$$

Perturbation theory

$$\mathcal{E}(t,s) = \frac{1}{1-e^{-\beta\Omega}} e^{-i\Omega(t-s)} + \frac{1}{e^{\beta\Omega}-1} e^{+i\Omega(t-s)}$$

$$\mathcal{E}(t,s) = u_+(t)u_+^*(s) + u_-(t)u_-^*(s)$$

$$u_+(t) = \frac{e^{-i\Omega t}}{\sqrt{1-e^{-\beta\Omega}}} \quad \text{and} \quad u_-(t) = \frac{e^{+i\Omega t}}{\sqrt{e^{\beta\Omega}-1}}$$

$$\begin{aligned} \varrho_{\mathrm{S}}(t) &\simeq \varrho_{\mathrm{Si}} + \lambda^2 \Phi_2(t)[\varrho_{\mathrm{Si}}] \qquad (143) \\ &= \varrho_{\mathrm{Si}} + 2\lambda^2 \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \, \left(\sigma_3 \varrho_{\mathrm{Si}} \sigma_3 - \varrho_{\mathrm{Si}}\right) \mathrm{Re}[\mathcal{E}(s,u)] \\ &= \varrho_{\mathrm{Si}} + \frac{\lambda^2 \Gamma(t)}{2} \left(\sigma_3 \varrho_{\mathrm{Si}} \sigma_3 - \varrho_{\mathrm{Si}}\right) \\ &= \begin{bmatrix} \varrho_{\uparrow\uparrow}(0) & \varrho_{\uparrow\downarrow}(0)(1-\lambda^2\Gamma(t)) \\ \varrho_{\downarrow\uparrow}(0)(1-\lambda^2\Gamma(t)) & \varrho_{\downarrow\downarrow}(0) \end{bmatrix} . \end{aligned}$$

Resolvent + redfield

$$\varrho_{\mathrm{S}}(t) \simeq \left(\mathcal{I} - \lambda^2 \Phi_2(t)\right)^{-1}[\varrho_{\mathrm{Si}}] \tag{144}$$
$$= \begin{bmatrix} \varrho_{\uparrow\uparrow}(0) & \varrho_{\uparrow\downarrow}(0)/(1+\lambda^2\Gamma(t)) \\ \varrho_{\downarrow\uparrow}(0)/(1+\lambda^2\Gamma(t)) & \varrho_{\downarrow\downarrow}(0) \end{bmatrix}$$

$$\frac{\partial \varrho_{\mathrm{S}}(t)}{\partial t} = \frac{\lambda^2 \dot{\Gamma}(t)}{2}\left(\sigma_3 \varrho_{\mathrm{Si}} \sigma_3 - \varrho_{\mathrm{Si}}\right)$$

- This is a special case in which the Redfield equation gives the exact solution.

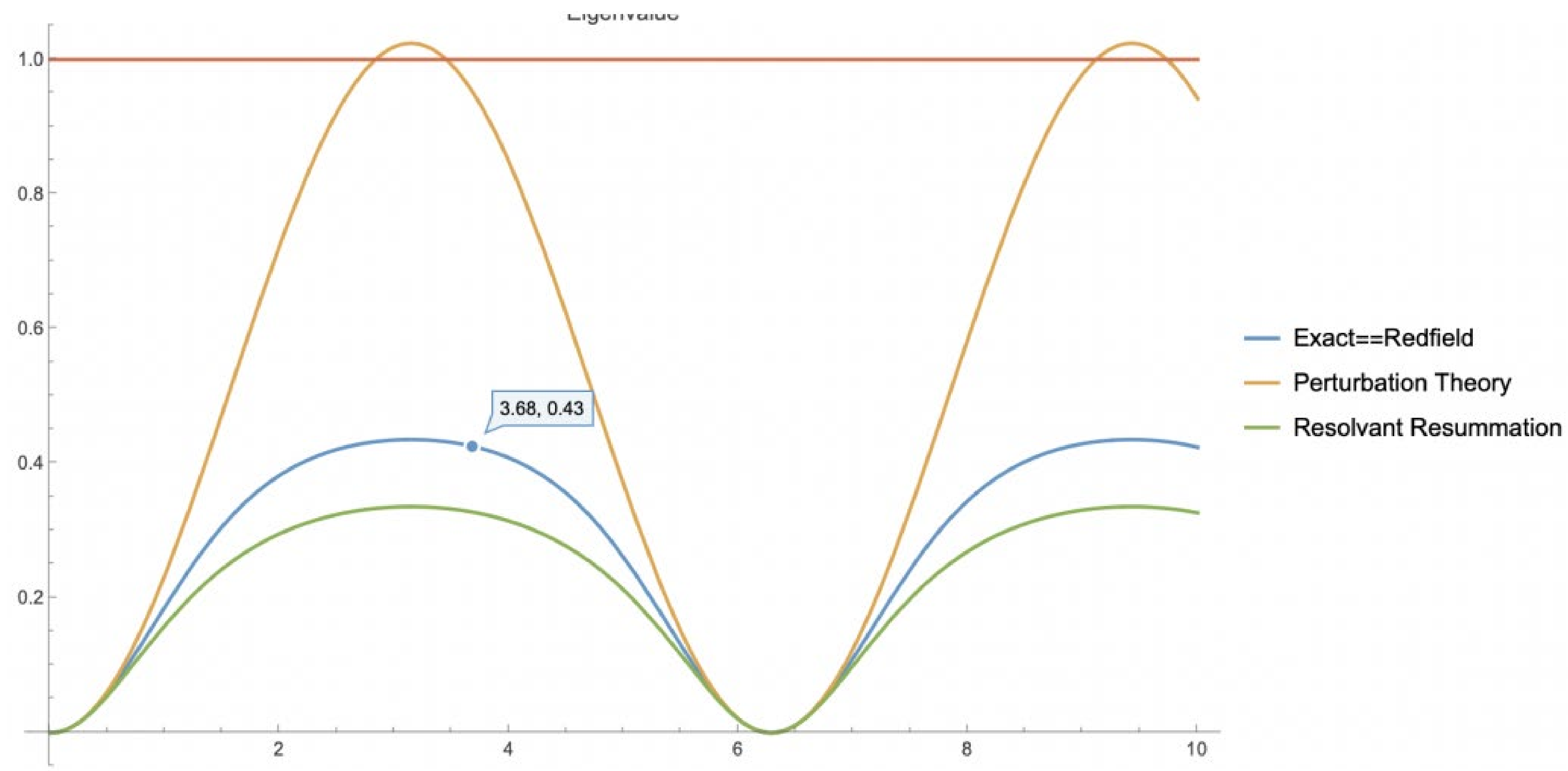


FIG. 3. Comparison done for choice of $\Omega = 1$ and $\lambda = 0.08$ and $\beta = 0.05$. The initial state was taken to be pure where $\varrho_{\mathrm{Si}} = |\uparrow_x\rangle\langle\uparrow_x|$ where $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$. The plot shows

Conclusion

- I sketched some ways in which I think QI intuition helps with physical problems…and some ways in which I think physical intuition helps with QI problems.
- Focused on open systems in this talk. But many other places this might be fruitful. Another example: QI theorists think of implementing algorithms via circuits ala classical EE. But QI experimentalists use physical Hamiltonians to evolve states over time. Pass to a “pulse-based” picture of quantum algorithms/complexity?