Frame invariant analysis of first order hydrodynamics as IR effective theory

Atsuhisa Ota

Chongqing U.

Based on Phys. Rev. D 114, L011902 (2026)
and Phys. Rev. D 114, 016001 (2026)

August 6, 2026

Roadmap

Stability puzzle → Reframe as IR EFT → Hydro EFT construction → Frame-invariant dispersion → Lorentz boost → Conclusions & outlook

1. Motivation: the long-standing stability puzzle in first order hydrodynamics
2. Reframing hydrodynamics as an IR effective theory
3. Effective theory for Hydrodynamic modes
4. Mode analysis: a frame-invariant dispersion relation
5. Boosted frames and Lorentz covariance
6. Conclusions and outlook

Motivation: the stability puzzle

First, the textbook problem: first-order theory looks unstable.

What is "first-order" hydrodynamics?

Zeroth order: ideal fluid

$$T^{\mu\nu}_{(0)} = \rho\, u^\mu u^\nu + P\,\gamma^{\mu\nu},$$
$$J^\mu_{(0)} = n\, u^\mu.$$

- No derivatives of β, u^μ, μ
- Reversible: no entropy production
- The relativistic Euler equations

First order: dissipative fluid

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)},$$
$$J^\mu = J^\mu_{(0)} + J^\mu_{(1)}.$$

- Exactly *one* gradient of β, u^μ, μ per term
- Irreversible: viscosity, heat/charge diffusion
- A relativistic Navier–Stokes

"First order" means first order in ∇_μ.

[Landau, Lifshitz (1987)]

A practical goal

Everyone wants to evolve a dissipative relativistic fluid as a closed system of PDEs built from conservation laws.

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu J^\mu = 0.$$

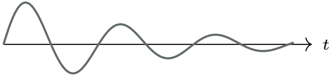
The problem: naive first-order hydrodynamics is unstable; simulations fail.

Step 1– Plane-wave expansion of perturbations around equilibrium:

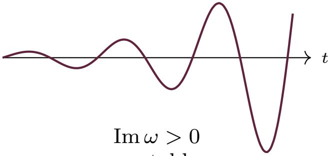
$$\delta\Phi \propto e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Step 2– Solve the dispersion relation and find

$$\omega = \omega(k).$$



$\mathrm{Im}\,\omega < 0$
stable



$\mathrm{Im}\,\omega > 0$
unstable

[Hiscock, Lindblom (1985)]

Two kinds of “frame”: (1) hydrodynamic frame

Hydrodynamic frame – the freedom in *redefining* the hydrodynamic variables:

$$\beta \to \beta + \delta\beta, \qquad u^\mu \to u^\mu + \delta u^\mu, \qquad \mu \to \mu + \delta\mu.$$

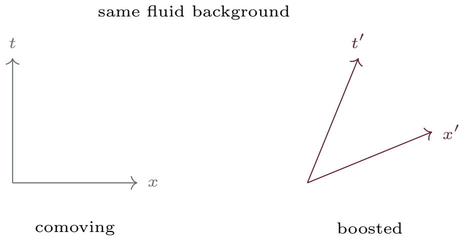
- Some first order coefficients can be absorbed in $\delta\beta$, δu^μ and $\delta\mu$.
- Landau frame: velocity defined by energy flow
- Eckart frame: velocity defined by particle flow

[Landau, Lifshitz (1987); Eckart (1940)]

Two kinds of “frame”: (2) inertial frame

Inertial frame – which *coordinate* describes the same fluid background:

- Comoving frame: moves together with the fluid
- Non-comoving frame: boosted relative to the fluid



The classical mode analysis: verdict

Only a special combination is stable:

Landau frame + comoving inertial frame

Shifting either frame then appears to produce unstable modes. [Hiscock, Lindblom (1985)]

> The Landau frame was regarded as "a peculiar singular limit".

But a physical IR pole should not depend on how we choose to label the fluid nor coordinates.

Stability puzzle → **Reframe as IR EFT** → Hydro EFT construction → Frame-invariant dispersion → Lorentz boost → Conclusions & outlook

Reframing hydrodynamics as an IR effective theory

The fix: stop treating hydrodynamics as exact — treat it as an IR EFT.

Hydrodynamics is not a fundamental theory

Hydrodynamics is an IR effective theory describing the low-frequency, long-wavelength regime – it is not fundamental.

Analogy: chiral perturbation theory. Truncating the derivative expansion at finite order and then solving *exactly* can generate spurious UV poles. But the expansion is only trusted for

$$p \ll \Lambda_{\mathrm{EFT}},$$

and UV poles produced this way are not treated as physical degrees of freedom.

[Weinberg (1979)]

Is first-order hydrodynamics any different?

- A truncated first-order equation should *not* be treated as an exact PDE valid at all frequencies and wavenumbers
- Pathological poles appearing in the UV should not disqualify the theory as a valid IR effective theory
- Nonlinear response and loop corrections only require IR poles that are controlled order-by-order in the derivative expansion

Question: Can first-order hydrodynamics be treated consistently as an IR effective theory?

[AO (2026)]

Stability puzzle → Reframe as IR EFT → **Hydro EFT construction** → Frame-invariant dispersion → Lorentz boost → Conclusions & outlook

Effective theory for hydrodynamic modes

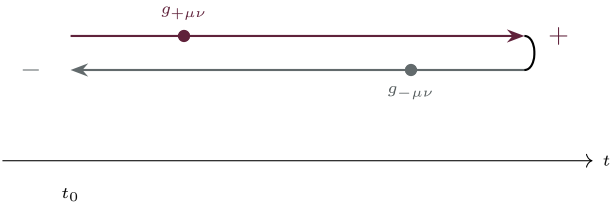
Build the EFT action explicitly, with hydrodynamic fluctuations as NG modes.

Schwinger–Keldysh: doubling the metric

Hydrodynamics is realized by integrating fast modes (χ) on the closed-time-path:

$$e^{iI[g_{+\mu\nu},g_{-\mu\nu}]} = \oint_{\varrho} \mathscr{D}\chi_+ \mathscr{D}\chi_- e^{iS[\chi_+,\chi_-;g_{+\mu\nu},g_{-\mu\nu}]}$$

The closed time path doubles every external source:



We will develop $I[g_{+\mu\nu}, g_{-\mu\nu}]$ as the effective action for the hydrodynamic modes.

[Crossley, Glorioso, Liu (2017); Glorioso, Liu (2018)]

Strong diffeomorphism symmetry

The generating functional is invariant under $\text{Diff}_+ \times \text{Diff}_-$.

$$
\begin{aligned}
e^{iI[g_{+\mu\nu},g_{-\mu\nu}]} &= \oint_{\varrho} \mathscr{D}\chi_+ \mathscr{D}\chi_- e^{iS[\chi_+,\chi_-;g_{+\mu\nu},g_{-\mu\nu}]} \\
&= \oint_{\varrho} \mathscr{D}\chi'_+ \mathscr{D}\chi'_- e^{iS[\chi'_+,\chi'_-;g'_{+\mu\nu},g'_{-\mu\nu}]} \\
&= e^{iI[g'_{+\mu\nu},g'_{-\mu\nu}]}
\end{aligned}
$$

But, a realized fluid select a preferred rest frame: $\text{Diff}_+ \times \text{Diff}_-$ is *fixed*.

Hydro EFT: Non linear realization of $\text{Diff}_+ \times \text{Diff}_-$.

Symmetry structure

The Keldysh basis:

$$g_{\mu\nu} \equiv \frac{g_{+\mu\nu} + g_{-\mu\nu}}{2}, \qquad g_{A\mu\nu} \equiv g_{+\mu\nu} - g_{-\mu\nu},$$

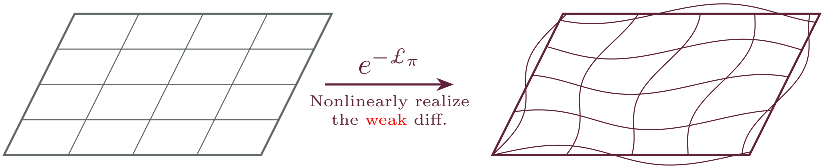
The symmetry structure:

Symmetry	**Generators**	**Acts on**
Strong	$\mathrm{Diff}_+ \times \mathrm{Diff}_-$	$g_{+\mu\nu}, g_{-\mu\nu}$ independently
Weak (physical)	diagonal, $\xi_+ = \xi_- \equiv \xi$	$g_{\mu\nu}$ and $g_{A\mu\nu}$
Noise	remaining part	$g_{A\mu\nu}$ only

[Crossley, Glorioso, Liu (2017); Glorioso, Liu (2018)]

Weak diffeomorphisms

A fluid emerges with a rest frame.
$\rightarrow$ a local hydrodynamic configuration is generated by a diff. Stückelberg field π^μ



$\beta_c^\mu = \beta_c \delta_0^\mu$
Globally uniform

$\beta^\mu(x) = \beta_c(\delta_0^\mu + \dot{\pi}^\mu)$
locally deformed

π^μ contains the NG fields associated with spontaneously broken boosts.

[Dubovsky, Hui, Nicolis, Son, (2011), Crossley, Glorioso, Liu (2017)]

Noise diffeomorphisms

Physical and noise coordinates:

$$x^\mu \equiv \frac{x_+^\mu - x_-^\mu}{2}, \qquad x_A^\mu \equiv x_+^\mu - x_-^\mu$$

Under the noise diffeomorphism

$$T^{\mu\nu} \to T^{\mu\nu},$$
$$g_{A\mu\nu} \to g_{A\mu\nu} + \nabla_\mu \xi_{A\nu} + \nabla_\nu \xi_{A\mu}$$

Introduce a Stückelberg field $\xi_{A\nu} \to -\pi_{A\nu}$ to nonlinearly realize the noise diff:

$$G_{A\mu\nu} = g_{A\mu\nu} - \nabla_\mu \pi_{A\nu} - \nabla_\nu \pi_{A\mu} \qquad \Rightarrow \qquad G_{A\mu\nu} \to G_{A\mu\nu}.$$

[Crossley, Glorioso, Liu (2017)]

Noise Stückelberg fields: defining $G_{A\mu\nu}$

The stress tensor couples to the noise metric in the SK path integral:

$$e^{iI[g,g_A]} \propto \exp\left[\frac{i}{2}\int d^4x\sqrt{-g}\,T^{\mu\nu}g_{A\mu\nu}\right].$$

Replacing $g_A \to G_A$, we have

$$I = \frac{1}{2}\int d^4x\sqrt{-g}\,\pi_{A\nu}\,\nabla_\mu T^{\mu\nu} + \cdots .$$

An "equation of motion" in $\pi_{A\mu}$ is the conservation law:

$$\frac{\delta I[g, G_A]}{\delta \pi_{A\nu}} = 0 \qquad \Longrightarrow \qquad \nabla_\mu T^{\mu\nu} = 0.$$

[Crossley, Glorioso, Liu (2017)]

Schwinger–Keldysh effective action

The effective action for the hydrodynamic modes:

$$I = \int d^4x \sqrt{-\det g} \sum_{n=1} \mathcal{L}_n,$$

with ($C_{A\mu}$ for the U(1) source)

$$\begin{aligned}
\mathcal{L}_1 &= \tfrac{1}{2} T^{\mu\nu} G_{A\mu\nu} + J^\mu C_{A\mu}, \\
\mathcal{L}_2 &= \tfrac{i}{4} W^{\mu\nu\rho\sigma} G_{A\mu\nu} G_{A\rho\sigma} + i\, X^{\mu\nu\rho} G_{A\mu\nu} C_{A\rho} + i\, Y^{\mu\rho} C_{A\mu} C_{A\rho}.
\end{aligned}$$

Constraints inherited from the UV generating functional:

- Conjugation condition
- Unitarity
- Local KMS symmetry

[Crossley, Glorioso, Liu (2017)]

KMS and unitarity fix the structure

Condition	**Implies**
Unitarity + KMS	local second law
KMS at 0th order	thermodynamic relations
KMS at 1st order	fluctuation–dissipation relations

$$3 \text{ thermodynamic} + 18 \text{ dissipative} + 10 \text{ noise coefficients}$$

$$\Downarrow \text{KMS} \; + \; \text{unitarity}$$

$$\mathbf{12} \text{ independent parameters}$$

[Crossley, Glorioso, Liu (2017)]

The general first-order constitutive relations

Most general U(1) first order hydrodynamics:

$$T^{\mu\nu} = \left(\rho + \rho^{(1)}\right)u^\mu u^\nu + \left(P + P^{(1)}\right)\gamma^{\mu\nu} + q^\sigma u^{(\mu}\gamma^{\nu)}{}_\sigma - \Sigma^{\mu\nu},$$
$$J^\mu = \left(n + n^{(1)}\right)u^\mu + \gamma^{\mu\nu} j_\nu.$$

The first-order pieces $\rho^{(1)}, P^{(1)}, n^{(1)}, q_\mu, j_\mu, \Sigma^{\mu\nu}$ split into a scalar sector and a vector/tensor sector – shown next.

[Kovtun (2019)]

Structure of the derivative expansion: scalar sector

Possible building blocks:

$$u^\rho \nabla_\rho \beta, \; u^\rho \nabla_\rho (\beta\mu), \; \nabla_\rho u^\rho .$$

The first order scalars:

$$\begin{aligned}
\rho^{(1)} &= \beta^{-1} \epsilon \, u^\rho \nabla_\rho \beta - \lambda_1 \nabla_\rho u^\rho - \nu_4 \, u^\rho \nabla_\rho (\beta\mu), \\
P^{(1)} &= \beta^{-1} \lambda_2 \, u^\rho \nabla_\rho \beta - \zeta \nabla_\rho u^\rho - \nu_5 \, u^\rho \nabla_\rho (\beta\mu), \\
n^{(1)} &= \nu_1 \, u^\rho \nabla_\rho \beta - \beta \nu_2 \nabla_\rho u^\rho - \beta \nu_3 \, u^\rho \nabla_\rho (\beta\mu).
\end{aligned}$$

$\epsilon, \lambda_1, \lambda_2, \zeta, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5$: scalar transport coefficients, functions of β, μ [Kovtun (2019)]

Structure of the derivative expansion: vector & tensor sector

The heat flux q_μ, charge flux j_μ, and shear stress $\Sigma^{\mu\nu}$ follow the same logic:

$$
\begin{aligned}
q_\mu &= \beta^{-1}\kappa_1 \nabla_\mu \beta - \kappa_2\, u^\rho \nabla_\rho u_\mu - \tau_3 \nabla_\mu(\beta\mu) - \beta\tau_4\, u^\rho F_{\rho\mu}, \\
j_\mu &= \tau_1 \nabla_\mu \beta - \beta\tau_2\, u^\rho \nabla_\rho u_\mu - \beta\chi_1 \nabla_\mu(\beta\mu) - \beta^2\chi_2\, u^\rho F_{\rho\mu}, \\
\Sigma^{\mu\nu} &= \eta\, \gamma^{\mu\alpha}\gamma^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3}\gamma_{\alpha\beta}\nabla_\gamma u^\gamma\right).
\end{aligned}
$$

- $\kappa_1, \kappa_2, \tau_1, \ldots, \tau_4, \chi_1, \chi_2$: vector transport coefficients.
- η: shear viscosity.

[Kovtun (2019)]

Hydrodynamic-frame invariants: energy sector

Physical predictions must be built from frame-invariant combinations, e.g.

$$\tilde{\lambda} = \lambda_2 - \left(\frac{\partial P}{\partial \rho}\right)_n \epsilon - \left(\frac{\partial P}{\partial n}\right)_\rho \nu_1,$$

$$\tilde{\zeta} = \zeta - \left(\frac{\partial P}{\partial \rho}\right)_n \lambda_1 - \left(\frac{\partial P}{\partial n}\right)_\rho \nu_2,$$

$$\tilde{\nu} = \nu_5 - \left(\frac{\partial P}{\partial \rho}\right)_n \nu_4 - \left(\frac{\partial P}{\partial n}\right)_\rho \nu_3,$$

$$\tilde{\tau}_1 = \tau_1 - \frac{n}{\beta(\rho + P)}\kappa_1,$$

$$\tilde{\tau}_2 = \tau_2 - \frac{n}{\beta(\rho + P)}\kappa_2,$$

$$\tilde{\chi}_1 = \chi_1 - \frac{n}{\beta(\rho + P)}\tau_3.$$

[Kovtun (2019)]

Hydrodynamic-frame invariants: charge sector & positivity

KMS and unitarity of the UV theory imply

$$\tilde{\tau}_1 = \tilde{\tau}_2, \quad \eta \geq 0, \quad \tilde{\chi}_1 - \frac{n}{\beta(\rho + P)}\tilde{\tau}_2 \geq 0, \quad \tilde{\zeta} + \left(\frac{\partial P}{\partial \rho}\right)_n \tilde{\lambda} + \left(\frac{\partial P}{\partial n}\right)_\rho \tilde{\nu} \geq 0.$$

These are exactly the ingredients that will guarantee stability later.

[Crossley, Glorioso, Liu (2017); AO (2026)]

Stability puzzle → Reframe as IR EFT → Hydro EFT construction → **Frame-invariant dispersion** → Lorentz boost → Conclusions & outlook

Mode analysis: a frame-invariant dispersion relation

Solve the mode equations and show the dispersion relation is frame invariant.

From the effective action to a dispersion relation

Two steps turn the effective action into the mode equation.

Step 1 – constraints. Solve

$$\frac{\delta I}{\delta \pi_{A0}} = \frac{\delta I}{\delta \theta_A} = \frac{\delta I}{\delta \pi_0} = \frac{\delta I}{\delta \theta} = 0$$

to eliminate the non-dynamical fields $\pi_0,\ \pi_{A0},\ \theta,\ \theta_A$ in favor of π_i and π_{Ai}.

Step 2 – equation of motion. Substituting back and varying with respect to the remaining field,

$$\frac{\delta I}{\delta \pi_{Ai}} = 0,$$

gives the dispersion relations. [AO (2026)]

Eliminating redundant modes

A formally sixth-order dispersion relation:

$$\sum_{n=0}^{6} d_n \, \omega^n = 0.$$

- Solving this exactly reproduces the known gapped, unstable mode with $\mathrm{Im}\, \omega \propto \kappa_2^{-1}$
- But this over-interprets a truncated first-order theory

> We must instead solve *perturbatively* in the derivative-counting parameter ε:
>
> $$\omega = \omega_0 + \varepsilon \, \omega_1 + O(\varepsilon^2).$$

[Hiscock, Lindblom (1985); AO (2026)]

Leading order: the sound speed

The leading-order solution is fixed purely by zeroth-order thermodynamics:

$$\omega_0 = \pm c_{s/n}\, k, \qquad c_{s/n}^2 \equiv \left(\frac{\partial P}{\partial \rho}\right)_{s/n}.$$

- The adiabatic sound speed is completely insensitive to any first-order (dissipative) coefficient
- Zeroth-order stability requires $c_{s/n}^2 \geq 0$

[AO (2026)]

First-order correction: a messy start...

A direct evaluation from the general constitutive relations at first order looks hopeless – a long expression in many frame-dependent coefficients:

$$\omega_1 = -ik^2\, F\big(\epsilon, \lambda_{1,2}, \zeta, \nu_{1..5}, \kappa_{1,2}, \tau_{1..4}, \chi_{1,2}, \eta\big).$$

[AO (2026)]

Remarkably, everything reorganizes into

$$\omega_1 = -i\,\Gamma_L\,k^2,$$

with

$$\Gamma_L = \frac{1}{2(\rho+P)}\Bigg[\frac{4}{3}\underbrace{\eta}_{\geq 0} + \underbrace{\tilde{\zeta} + \left(\frac{\partial P}{\partial \rho}\right)_n \tilde{\lambda} + \left(\frac{\partial P}{\partial n}\right)_\rho \tilde{\nu}}_{\geq 0}$$

$$+ f^2\left(\left(\frac{\partial P}{\partial \rho}\right)_{s/n}\underbrace{\left(\tilde{\chi}_1 - \frac{n}{\beta(\rho+P)}\tilde{\tau}_2\right)}_{\geq 0} + \beta\left(\frac{\partial P}{\partial n}\right)_\rho \underbrace{(\tilde{\tau}_1 - \tilde{\tau}_2)}_{=0}\right)\Bigg].$$

This is hydrodynamic-frame invariant. [AO (2026)]

The main result: stability from positivity

$$\boxed{\rho + P > 0 \implies \Gamma_L \geq 0 \implies \operatorname{Im}\omega \leq 0}$$

- This holds for any choice of hydrodynamic frame – Landau, Eckart, or otherwise
- The transverse mode gives the same conclusion, with

$$\Gamma_T = \frac{\eta}{\rho + P} \geq 0$$

[AO (2026)]

Stability puzzle → Reframe as IR EFT → Hydro EFT construction → Frame-invariant dispersion → **Lorentz boost** → Conclusions & outlook

Lorentz boosted frames

Check the same conclusion holds in any inertial frame, not just comoving.

Non comoving frame

Lorentz boost between comoving ($\tilde{k}_\mu$) and non comoving (k_μ) frames:

$$k_\mu \to \tilde{k}_\mu = \Lambda_\mu{}^\nu k_\nu .$$

For a boost in t-x plain ($\gamma \equiv 1/\sqrt{1-v^2}$),

$$\begin{aligned}
\tilde{\omega} &= \gamma\left(\omega - v k_x\right), \\
\tilde{k}_x &= \gamma\left(k_x - v\omega\right), \\
\tilde{k}_y &= k_y, \\
\tilde{k}_z &= k_z .
\end{aligned}$$

In comoving frame, we found

$$\tilde{\omega} = \pm c_{s/n}\tilde{k} - i\Gamma_L \tilde{k}^2$$

Then what about $\omega = \omega_0 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2)$?

First order dispersion

The leading order dispersion depends on the relative direction:

$$\omega_0 = \frac{v\left(1 - c_{s/n}^2\right)k_x \ \pm \ c_{s/n}\sqrt{1-v^2}\sqrt{\left(1-v^2\right)k_x^2 + \left(1 - c_{s/n}^2 v^2\right)k_\perp^2}}{1 - c_{s/n}^2 v^2}.$$

If $\vec{k} \propto \vec{x}$ and then $k_\perp = 0$,

$$\omega_0 = \frac{v \pm c_{s/n}}{1 \pm c_{s/n} v} k_x.$$

First order

The first order dispersion relation:

$$\omega_1 = -\frac{i\,\Gamma_{\mathrm{L}}\,\tilde{k}^2}{\gamma\left(1 \pm\, c_{s/n}\, v\cos\tilde{\theta}\right)},$$

with

$$\cos\tilde{\theta} \equiv \frac{\gamma\,(k_x - v\omega_0)}{\left|\tilde{k}\right|},$$

$$\tilde{k}^2\Big|_{\omega_0} \equiv \gamma^2\,(k_x - v\omega_0)^2 + k_\perp^2.$$

This is stable.

Stability puzzle → Reframe as IR EFT → Hydro EFT construction → Frame-invariant dispersion → Lorentz boost → Conclusions & outlook

Conclusions and outlook

Put it together: first-order hydrodynamics is a healthy IR effective theory.

First-order hydrodynamics is a healthy IR effective theory

- The physical IR pole does not depend on the hydrodynamic frame
- Under KMS + unitarity, $\rho + P > 0 \Rightarrow \mathrm{Im}\,\omega \leq 0$
- Lorentz boosts maintain the stability of the IR spectrum.

> As an IR effective theory, no instability issue in first-order hydrodynamics.

[AO (2026)]

Outlook: curved and time-dependent backgrounds

- Coupling hydrodynamic modes to gravitational perturbations
- Extension to FLRW, Einstein static universe, black hole backgrounds
- NG modes and stability under different spacetime symmetry breaking
- Effects of background curvature and time dependence on the IR pole

Outlook: nonlinear fluctuations and loop corrections

- Loop corrections sourced by the hydrodynamic modes themselves
- IR renormalization of transport coefficients; long-time tails
- A hydrodynamic-frame-invariant nonlinear effective action

Next goal: a frame-independent, nonlinear, quantum effective theory of relativistic fluids.

Thank you for your attention.

References

- A. Ota, *Frame-independent analysis of relativistic hydrodynamics via effective theory*, Phys. Rev. D 114, L011902 (2026).
- A. Ota, *Mode analysis of Nambu-Goldstone modes in $U(1)$ charged first-order relativistic viscous hydrodynamics*, Phys. Rev. D 114, 016001 (2026).

Contact: aota@cqu.edu.cn