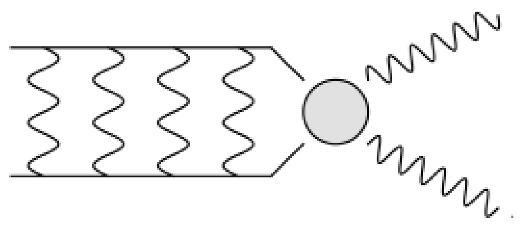
Self-consistent computation of pair production from non-relativistic EFTs in the Schwinger-Keldysh formalism



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Motivation

Usual computation of Sommerfeld-enhanced annihilation via

$$\sigma v = (\sigma v)_0 |\psi_0(\mathbf{r}=0)|^2$$

appears to violate partial-wave unitarity in the presence of nearly zero-energy bound states (e.g., parametrically possible for Yukawa potential, see Fig. 1). Including the short-distance annihilation potential "self-consistently" in the full solution of $|\psi(0)|^2$ unitarizes the annihilation cross section in vacuum [1]. We analyze this system in the Schwinger-Keldysh formalism:

- "Self-consistent" pair annihilation *and* production
- How does the reverse process influence the unitarization?
- What does S-matrix unitarity imply in the in-in formalism?
- Are bound states on shell in the "self-consistent" solution?

Background: Unitarization of Sommerfeld-enhanced annihilation in vacuum

"Self-consistent" solution of Schrödinger equation

$$\left[E + \frac{\Delta_{\mathbf{r}}}{m} - V(r) + ic\delta(\mathbf{r})\right]\psi(\mathbf{r}) = 0\,,$$

unitarizes Sommerfeld-enhanced annihilation [1]:

$$|\psi(\mathbf{r}=0)|^2 = \frac{|\psi_0(0)|^2}{|1 + cG_0(E;0,0)|^2} \approx \frac{|\psi_0(0)|^2}{\left(1 + \frac{(\sigma v)_0 |\psi_0(0)|^2}{(\sigma v)_{\text{uni}}}\right)^2}\,.$$

Respects the inelastic s-wave unitarity bound:

$$\sigma v = (\sigma v)_0 |\psi(\mathbf{r}=0)|^2 \leq \frac{4\pi}{m^2 v} \equiv (\sigma v)_{\text{uni}}\,.$$

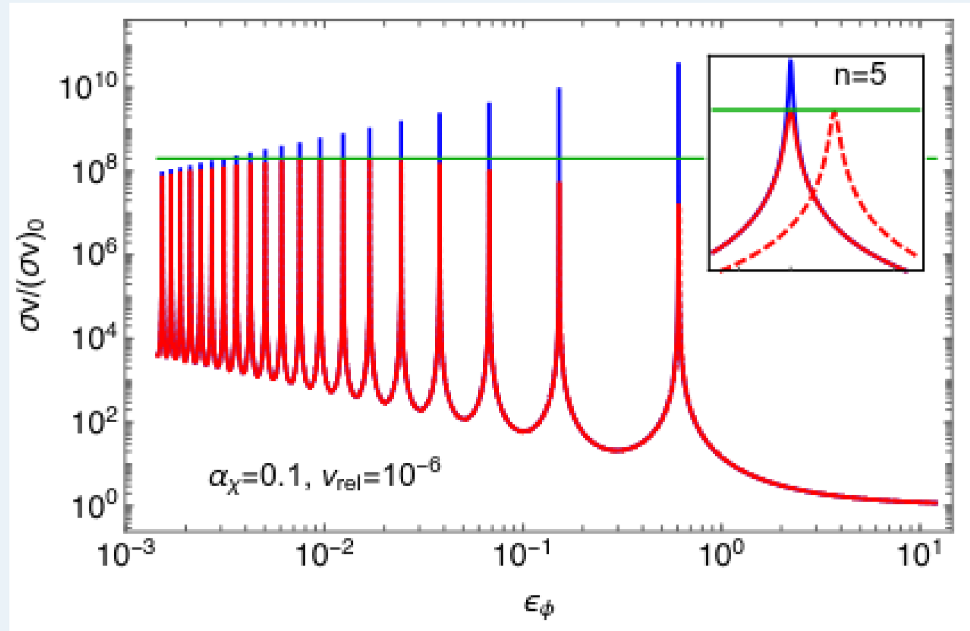

Figure 1: Unitarized (red) and unregulated (blue) Sommerfeld enhancement shown for a Yukawa potential with $\epsilon_\phi = m_\phi/(\alpha m)$, and the s-wave unitarity bound (green). Taken from Ref. [2].

System

- **Environment**: light degrees of freedom which are assumed to remain in equilibrium.
- **Sub-system**: non-relativistic particles and anti-particles which can annihilate into light degrees of freedom and can be pair produced from the environment. Pairs interact via static potential, leading to Sommerfeld effect and bound states.

We integrate out the environment and obtain an **open quantum system** description for the heavy pairs. This resembles the situation of heavy DM pairs in the early Universe or quarkonium inside a quark-gluon plasma.

Simplifications:

- Neglect Hubble expansion
- Neglect ultrasoft contributions

→ analytic solution of the system's dynamics is our goal.

Procedure, using NREFTs in Schwinger-Keldysh

We start from a non-relativistic EFT on the closed-time-path contour $\mathcal{C}$, formulated for scalar particle and antiparticle:

$$S_{\text{NR}} = \int_{x^0\in\mathcal{C}} \mathrm{d}^4x\, \eta^\dagger\left(i\partial_{x^0} + \frac{\Delta}{2m}\right)\eta + \xi^\dagger\left(i\partial_{x^0} + \frac{\Delta}{2m}\right)\xi + \int_{x^0,y^0\in\mathcal{C}} \mathrm{d}^4x\mathrm{d}^4y\, \underbrace{\eta^\dagger(x)\xi(x)i\Gamma(x,y)\xi^\dagger(y)\eta(y)}_{\text{"annihilation and production"}} + \text{static potential}\,.$$

1. From S_{NR}, derive the kinetic equation for the particle number density $n \equiv \langle\eta^\dagger(x)\eta(x)\rangle$. Due to the Martin-Schwinger hierarchy, it depends on the higher **4-point correlation function**:
$$G_{\eta\xi}(x,y,z,w) \equiv \langle T_{\mathcal{C}}\eta(x)\xi^\dagger(y)\xi(w)\eta^\dagger(z)\rangle.$$
2. Compute the 4-point function **"self-consistently"** in i) NREFT and ii) *potential non-relativistic EFT* (pNREFT).

[*Technical remark*: In the pNREFT approach, we make the replacement (see [3] for statistical component):

$$G_{\eta\xi}(t,\mathbf{x_1},\mathbf{x_2};t',\mathbf{x_3},\mathbf{x_4}) \to G(x,y;\mathbf{r},\mathbf{r}') \equiv \langle T_{\mathcal{C}}S(x,\mathbf{r})S^\dagger(y,\mathbf{r}')\rangle\,.$$

S is the two-body field in CM and relative coordinates. The EoM of G follow from pNREFT on the closed-time path:

$$S_{\text{pNR}} = \int_{x^0\in\mathcal{C}} \mathrm{d}^4x\mathrm{d}^3r\, S^\dagger(x,\mathbf{r})[i\partial_{x^0} - h]S(x,\mathbf{r}) + i\int_{x^0,y^0\in\mathcal{C}} \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^3r\, S^\dagger(x,\mathbf{r})\delta(\mathbf{r})\Gamma(x,y)S(y,\mathbf{r})\,.\,]$$

Result 1: pair annihilation only via gradient method

Following the procedure, we obtain for the particle number density equation (see also [4]):

$$\dot{n} = -(\sigma v)_0 G^{++--}_{\eta\xi}(x,x,x,x)\,,\ \text{for s-wave annihilation only, where}\ \Gamma(x,y) = \delta^4(x-y)\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\underbrace{(\sigma v)_0/2}_{=:c}\,.$$

Let G_0 be the solution without annihilation. Then, at LO gradient expansion we obtain the coupled eqs.:

$$G^{+-}(T,P;\mathbf{r},\mathbf{r}') = G_0^{+-}(T,P;\mathbf{r},\mathbf{r}') - c\left[G_0^R(P;\mathbf{r},0)G^{+-}(T,P;0,\mathbf{r}') - G_0^{+-}(T,P;\mathbf{r},0)G^A(P;0,\mathbf{r}')\right].$$

The solution of this coupled system gives the unitarization of s-wave Sommerfeld-enhanced annihilation:

$$G^{+-}(P;0,0) = \frac{G_0^{+-}(P;0,0)}{\left|1 + cG_0^R(P;0,0)\right|^2}\,,\ \textbf{consistent with}\ [1].$$

For bound states, the time gradients must be retained. For $E<0$, the solution is **on shell** and given by:

$$G^{+-}(T,P;0,0) = \sum_n e^{-\Gamma_n^{\text{dec}}T} f_n(T_0, 2m+\omega)(2\pi)\delta(\omega - \mathbf{P}^2/(4m) - E_n)|\psi_n(0)|^2\,.$$

Result 2: pair annihilation and production as initial value problem

Assume that the environment is translationally invariant, implying $\Gamma = \Gamma(x-y)$. At LO in gradient exp.:

$$\dot{n} = \int \frac{d^4P}{(2\pi)^4}\Gamma^{+-}(P)G^{-+}(T,P;0,0) - \Gamma^{-+}(P)G^{+-}(T,P;0,0)\,.$$

4-point funcs. obtained by adapting method of Ref. [5]. Decomposing $G = G^F + \text{sign}_{\mathcal{C}}(x^0,y^0)G^\rho/2$ and writing $G^F = G^F_h + G^F_m$, where G^F_h contains the initial condition, the homogeneous solution reads:

$$G^F_h(x^0,y^0,\mathbf{P};\mathbf{r},\mathbf{r}') = \sum\!\!\!\!\!\!\int_\lambda e^{\Im(E_\lambda)(x^0+y^0)}e^{-i\Re(E_{\mathbf{P},\lambda})(x^0-y^0)}\psi_\lambda(\mathbf{r})\psi^\star_\lambda(\mathbf{r}')\left[\frac{1}{2} + f_\lambda(\mathbf{P})\right].$$

- **Bound states**: $\Im(E_n) = -\Gamma_n^{\text{dec}}/2$. On-shell decay, consistent with gradient method.
- **Scattering states**: $\Im(E_\lambda) = 0$ → time-translation invariant, consistent with gradient method.

In the large time limit, we obtain for the statistical correlator:

$$G^{+-}(T\to\infty,P;0,0) = \frac{G_0^\rho(E;0,0)}{\left|1 + \Gamma^R(P)G_0^R(E;0,0)\right|^2}\left[f_{\mathbf{P}}(\mathbf{P}) - f_B^{\text{eq}}(2m+\omega)\right] + f_B^{\text{eq}}(2m+\omega)G^\rho(P;0,0)\,.$$

- **Bound states**: equilibrate, satisfy KMS relation in large time limit.
- **Scattering states**: unitarization temperature dependent due to Γ^R. Satisfy KMS relation only for equilibrium f, which requires the solution of two-point function, or in kinetic eq. the solution of $\dot{n}$.

Conclusion

- Confirmed insertion of "self-consistent" computation of s-wave Sommerfeld enhancement into Boltzmann equation for dark matter freeze-out studies from thermal field theory directly.
- **Unitarization of Sommerfeld-enhanced annihilation becomes temperature dependent** when the reverse process is included in the in-in formalism; however, finite-T corrections are exponentially suppressed for non-relativistic, dilute particles.
- **Scattering states:** stationary for diagonal density matrix in momentum space, and require the two-point function for reaching thermal equilibrium and to satisfy the KMS relation at large times.
- **Bound states:** decay on shell out of equilibrium, thermalize at late times, and satisfy the KMS relation. Their expected Boltzmann equation emerges dynamically from the 4-point function under semiclassical approximations.
- **Outlook**: Including ultrasoft may unitarize bound-state formation, an interesting topic on its own [6]. The origin of the stationary scattering-state solution requires further investigation (cf., [3]).

References

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