Multipartite entanglement in Fusion-Categorical Lattice Models

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ICTP

based on `arXiv:2602.16770` with M. Del Zotto and A. Gadde
$+$ *work in progress*

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Plan

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Setup: lattice models and TQFT

Gapped Hamiltonian lattice model:

ground state on $\mathbb{R}^{d-1}$ (S^{d-1})

$$|\psi\rangle \in \bigotimes_{i \in \text{sites}} \mathcal{H}_i$$

$\longleftrightarrow$

d-dimensional TQFT:

$$Z(M^d) : Z(N_1^{d-1}) \to Z(N_2^{d-1})$$

for a bordism

$$M^d : \quad N_1^{d-1} \to N_2^{d-1}.$$

$$|\psi\rangle \xrightarrow{\textbf{this talk}} Z(M^d) \in \mathbb{C}, \text{ for closed (oriented) manifolds } M^d$$

$\downarrow$ [Blanchet–Habegger–Masbaum–Vogel '95] "universal construction" [McNamara–Wang '26]

Z functor

Remark

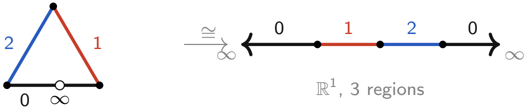
In a sense, this generalizes the **topological entanglement entropy** [Kitaev–Preskill '06; Levin–Wen '06], which recovers $Z(S^3)$.

$|\psi\rangle$ as a $(d+1)$-partite state

Consider $|\psi\rangle$ as a $(d+1)$-**partite state**, corresponding to the splitting of $S^{d-1} = \mathbb{R}^{d-1} \cup \{\infty\}$ into $(d+1)$ regions — the faces of a d-simplex T, via $\partial T \cong S^{d-1}$:

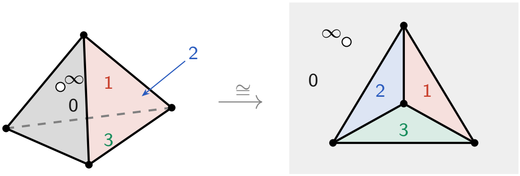
$$|\psi\rangle \in \bigotimes_{j=0}^{d} \mathcal{H}^{(j)}, \qquad \mathcal{H}^{(j)} = \bigotimes_{\substack{i \in \text{sites} \\ \text{in region } j}} \mathcal{H}_i.$$

$d = 2$



$\mathbb{R}^1$, 3 regions

$d = 3$



$\mathbb{R}^2$, 4 regions

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Multi-invariants

Definition

Fix $N \in \mathbb{Z}_{\geq 1}$ (the *replica number*) and permutations $\sigma_j \in S_N$, $j = 0, \ldots, d$. For a $(d+1)$-partite state $|\psi\rangle \in \bigotimes_{j=0}^{d} \mathcal{H}^{(j)}$, one defines a **multi-invariant**

$$\mathcal{Z}\big(|\psi\rangle;\ \sigma_0, \sigma_1, \ldots, \sigma_d\big) \ := \ \langle\psi|^{\otimes N}\, \sigma_0 \otimes \cdots \otimes \sigma_d\, |\psi\rangle^{\otimes N}.$$

[Gadde–Krishna–Sharma '22, '23; Gadde–Jain–Kulkarni '24; Gadde–Jain '25]

It is invariant under local unitary transformations acting on each $\mathcal{H}^{(j)}$, $j = 0, \ldots, d$.

Moreover, for any $g \in S_N$,

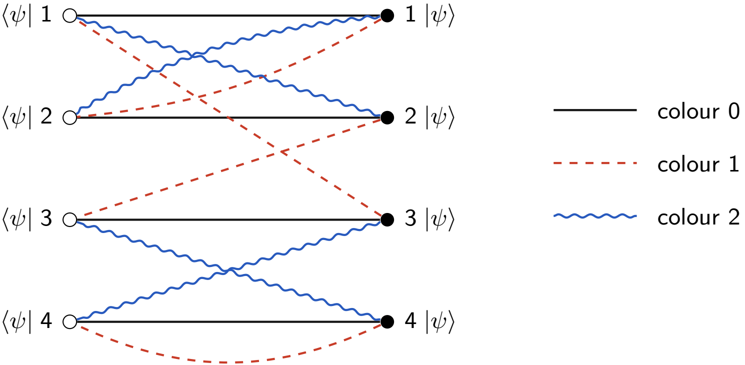
$$\mathcal{Z}\big(|\psi\rangle;\ g\sigma_0, \ldots, g\sigma_d\big) \ = \ \mathcal{Z}\big(|\psi\rangle;\ \sigma_0, \ldots, \sigma_d\big) \ = \ \mathcal{Z}\big(|\psi\rangle;\ \sigma_0 g, \ldots, \sigma_d g\big).$$

The invariant information can be encoded in a **bipartite $(d+1)$-edge-coloured graph with $2N$ vertices**.

Previously used to probe topological order: [Sheffer–Stern–Berg '25; Sheffer–Fan–Stern–Berg–Ryu '25; Sheffer '26]

Example: $d = 2$, $N = 4$

$$\sigma_0 = \mathrm{id}, \qquad \sigma_1 = (123), \qquad \sigma_2 = (12)(34)$$



An edge of colour j joins $\bullet\, i$ to $\circ\, \sigma_j(i)$.

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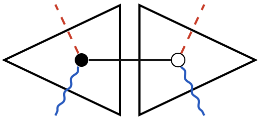
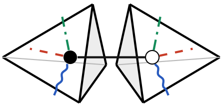
Graphs as manifolds: gems

Such a graph Γ can *also* be used to encode a closed oriented d-manifold, via its triangulation:

- **Vertices** — copies of a d-simplex and of its orientation reversal ($\bullet$ / $\circ$);
- **Edges** — the faces along which they are glued (colour j = the j-th face).



$d = 2$



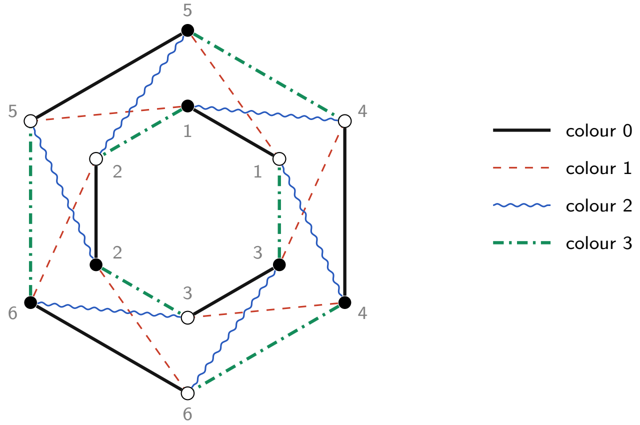
$d = 3$

In this setting such graphs are referred to as d-**gems** (*graph-encoded manifolds*) [Pezzana '74; Ferri '76; Ferri–Gagliardi–Grasselli '86; Lins '95]. Denote by $|\Gamma|$ the (diffeomorphism class of the) corresponding manifold.

- Any smooth manifold can be realized by such a Γ;
- Not all graphs give manifolds: deleting the edges for some of the colours should give a disjoint union of n-gems, $n < d$, realizing n-spheres.

Example: a lens space ($d = 3$, $N = 6$)

$$\sigma_0 = \mathrm{id}, \quad \sigma_1 = (15)(26)(34), \quad \sigma_2 = (14)(25)(36), \quad \sigma_3 = (123)(465)$$



Γ is a 3-gem with $|\Gamma| = L(3,1)$; the two circles are its $\{0,3\}$-coloured cycles.

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The conjecture

Conjecture

Let Γ be a d-gem. Then

$$\mathcal{Z}\big(|\psi\rangle;\Gamma\big) \;=\; Z\big(|\Gamma|\big) \;\times\; \underbrace{U\big(|\psi\rangle,\Gamma\big) \;\times\; \Big(1+O\big(e^{-\text{size}}\big)\Big)}_{\text{non-universal factors}},$$

where U has the form

$$U\big(|\psi\rangle,\Gamma\big) \;=\; \prod_{\gamma} A\big(|\psi\rangle;\gamma\big)^{\#(\gamma,\Gamma)}.$$

Here:

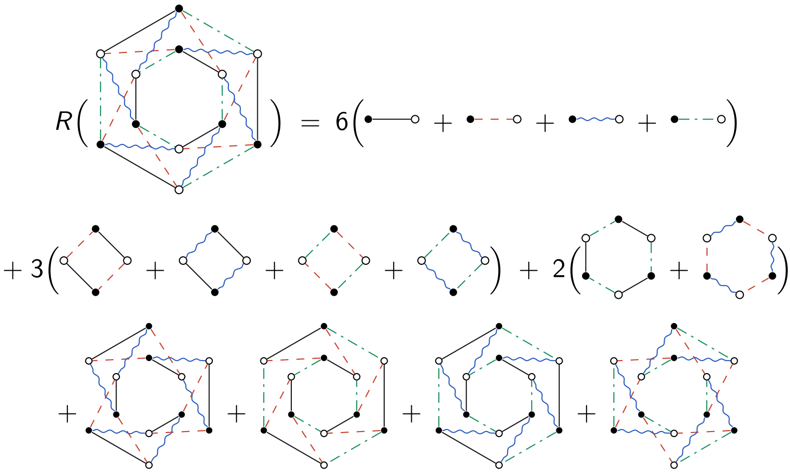
- γ runs over **spherical** $(< d)$**-gems** with edges coloured by *proper* subsets of $\{0,\ldots,d\}$;
- $\#(\gamma,\Gamma) :=$ the number of subgraphs γ in Γ.

Reformulation: the residue homomorphism

Alternatively, define the *residue* homomorphism

$$R:\ \mathbb{Z}[d\text{-gems}] \longrightarrow \mathbb{Z}\left[\begin{matrix}\text{spherical}\\ (<d)\text{-gems}\end{matrix}\right], \qquad \Gamma \longmapsto \sum_{\gamma} \gamma \cdot \#(\gamma, \Gamma).$$

Then $U : \mathbb{Z}[d\text{-gems}] \to \mathbb{C}^*$ **factors through** R.



The last four terms are Γ with one colour deleted — the $\{0,1,2\}$-, $\{0,1,3\}$-, $\{0,2,3\}$- and $\{1,2,3\}$-coloured 2-gems (each connected, each S^2).

Corollary

Corollary

If

$$\frac{R(d\text{-gems})}{R\big(\text{spherical } d\text{-gems}\big)} = 0 \qquad (*)$$

then there exists a universal (non-unique)
$S:\ \mathbb{Z}[d\text{-gems}] \to \mathbb{Z}\big[\text{spherical } d\text{-gems}\big]$ such that $R \circ S = R$, and thus

$$\frac{Z(|\Gamma|)}{Z(S^d)^{\,n_\Gamma}} = \frac{\mathcal{Z}(|\psi\rangle;\Gamma)}{\mathcal{Z}\big(|\psi\rangle; S(\Gamma)\big)},$$

with n_Γ = sum of the coefficients in $S(\Gamma)$.

(extend multiplicatively)

Remark

Remark

There is a **surjective** map

$$\begin{aligned} \Omega_d &\longrightarrow \frac{R(d\text{-gems})}{R\big(\text{spherical } d\text{-gems}\big)}, \\ [\,|\Gamma|\,] &\longmapsto [\Gamma] \end{aligned}$$

where Ω_d is the oriented bordism group.

Consequently,

$$\Omega_d = 0 \qquad \Longrightarrow \qquad \frac{R(d\text{-gems})}{R\big(\text{spherical } d\text{-gems}\big)} \;=\; 0.$$

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Fusion $(d-2)$-categories and their SymTFT

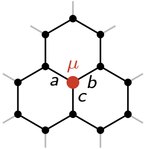
Let $\mathcal{C}$ be a fusion $(d-2)$-category: *topological defects in a $(d-1)$-dimensional QFT*.

Its **SymTFT**, a d-dimensional TQFT, has the following Hamiltonian lattice model — a generalization of Levin–Wen '05 ($d=3$, in which case it is $TV(\mathcal{C}) = RT(\mathcal{Z}(\mathcal{C}))$).

Lattice: a cell decomposition of $\mathbb{R}^{d-1}$ (or S^{d-1}).

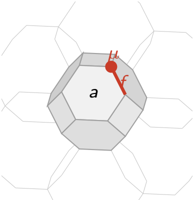
Hilbert space: functions on arbitrary colourings:

$(d-2)$-cells : objects
$(d-3)$-cells : 1-homs
$(d-4)$-cells : 2-homs
$\vdots$
0-cells : $(d-2)$-homs



$d-1=2$: the honeycomb.

a, b, c: 1-cells, μ: 0-cell.



$d-1=3$: the *bitruncated cubic honeycomb* (cells = truncated octahedra).

a: 2-cell, f: 1-cell, μ: 0-cell.

The colouring is *arbitrary* — no compatibility between the labels is imposed here.

The ground state

Ground state: function ψ supported on colourings corresponding to "legal" configurations of topological defects in $(d-1)$ dimensions, invariant under the allowed moves.

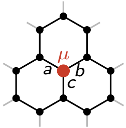
Alternatively, let D be such a configuration of defects. Then

$$\psi(D) \;=\; \mathrm{ev}(D) \cdot \psi(\emptyset),$$

where $\mathrm{ev}(D) \in \mathbb{C}$ is the evaluation of the configuration on S^{d-1}, i.e.

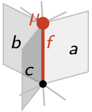
$$D \underset{\text{contraction}}{\sim} \mathrm{ev}(D) \cdot \underbrace{1}_{\text{trivial local op.}}$$

$\Longrightarrow$ the ground state is **unique** up to normalization.



$d = 3$: at each *vertex*,

$\mu \in \mathrm{Hom}(a \otimes b,\ c)$.



$d = 4$: along each *edge*,
$f \in \mathrm{Hom}(a \otimes b,\ c)$;

at each *vertex*, $\mu \in$ 2-$\mathrm{Hom}(f, \cdots)$.

Main theorem

Theorem (conditional on a technical assumption)

The Conjecture holds in this case, with the factors $A(|\psi\rangle;\gamma)$ being constant on 1- and $(d-1)$-gems.

Recall the Conjecture:

$$\mathcal{Z}\big(|\psi\rangle;\Gamma\big) \;=\; Z\big(|\Gamma|\big) \;\times\; \underbrace{\prod_{\gamma\subsetneq\Gamma} A\big(|\psi\rangle;\gamma\big)^{\#(\gamma,\Gamma)} \;\times\; \Big(1+O\big(e^{-\mathrm{size}}\big)\Big)}_{\text{non-universal factors}}.$$

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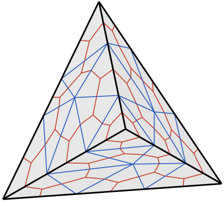
Proof sketch — (i) the state sum

Consider the **state sum model** of the SymTFT by Walker '21 (a generalization of Douglas–Reutter '18, $d = 4$, and Turaev–Viro '92, $d = 3$), for the cell decomposition of $|\Gamma|$ where the d-cells are the gem's d-simplices, but with their faces subdivided into a cell decomposition *dual* to the lattice one — recall that the lattice is what supports the defects.

In the state sum we sum over the colours:

$$\begin{aligned} \text{1-cells} &: \text{objects} \\ \text{2-cells} &: \text{1-homs} \\ \vdots\ &:\ \vdots \\ (d-1)\text{-cells} &: (d-2)\text{-homs} \end{aligned}$$

with certain weights.



A d-simplex of the gem ($d = 3$): the subdivision of its faces (blue) is dual to the lattice (red).

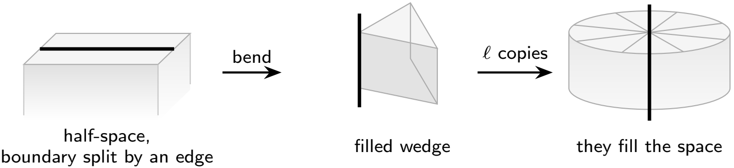
Also, each d-cell contributes the factor $\mathrm{ev}(D)$, with D the corresponding configuration of defects on its boundary.

Proof sketch — (ii) the "wrong" weights

$\mathcal{Z}\big(|\psi\rangle;\Gamma\big)$ is then *almost* the state sum, but with "wrong" weights attached to the 1-, 2-, $\ldots$, $(d-2)$-cells.

Why

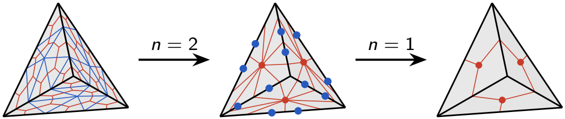
Due to geometrical (but *not* topological) conical defects there, arising in the replica construction.



$d = 3$: the copies are glued around an edge of the gem's simplex; their number ℓ is the number of vertices of the bicoloured loop in Γ dual to that edge (in general $\ell \neq 2N$). Each is intrinsically a half-space — so the geometry there is conical, though the topology is not.

Proof sketch — (iii) correcting the weights

For $n = d - 1$, $d - 2$, $\ldots$ (in that order), in the limit of dense subdivision (i.e. large region sizes), the sum over colourings inside an n-simplex of the gem can be reduced to the sum of the n-simplex *not* being subdivided, and with *correct* weights.



The **cost** is the factor $A(|\psi\rangle; \gamma)$, with γ the spherical $(d - n - 1)$-gem dual to the n-simplex.

Remark

The first step, $n = d - 1$, is "free".

Remark

$A(|\psi\rangle; \gamma)$ depends multiplicatively on the numbers of n-, $(n - 1)$-, $\ldots$ cells in the subdivision of the n-simplex.

The technical assumption for (iii)

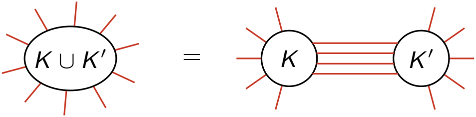
Technical assumption

A solution of a certain bilinear recursion equation (associated with the "gluing" of n-dimensional cell complexes) tends to a certain "projective fixed-point solution" in the limit of large size.

- **Case** $d = 3$**:** a consequence of the Frobenius–Perron theorem.
- **Also holds for** $d = 4$, in the case where $\mathcal{C}$ is a G-crossed braided fusion category (understood as a fusion 2-category), which corresponds to the Cui '16 TQFT state sum (which reduces to Crane–Yetter '93 in the case $G = 1$).

The technical assumption for (iii), schematically

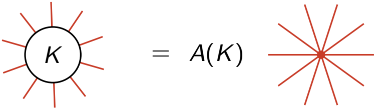
Gluing two n-dimensional cell complexes K, K' along part of their boundaries:



$$X(K \cup K') \;=\; \sum X(K)\, X(K'),$$

where X is a complicated tensor, with its indices suppressed.

The "projective fixed-point solution":



(with *correct* weights)

with $A(K)$ a *scalar*, so that $A(K \cup K') = A(K) \cdot A(K')$.

It is a solution because the sums with correct weights are invariant under changes of cell decomposition.

Summary and outlook

- The multi-invariants of a gapped ground state $|\psi\rangle$ are organized by bipartite $(d+1)$-edge-coloured graphs; those that are d-gems encode closed oriented d-manifolds.
- **Conjecture:** for a d-gem Γ, the multi-invariant $\mathcal{Z}(|\psi\rangle;\Gamma)$ computes $Z(|\Gamma|)$ — so $|\psi\rangle$ determines Z on all closed oriented d-manifolds, up to non-universal factors of a certain form. Provided $(*)$ ($\Leftarrow \Omega_d = 0$) this allows one to extract Z from $\mathcal{Z}$.
- **Theorem:** this holds for the SymTFT lattice models of fusion $(d-2)$-categories, conditionally on the technical assumption.

Outlook:

- the technical assumption in general (known for $d=3$, and for $d=4$ with $\mathcal{C}$ G-crossed braided);
- what can be extracted if $(*)$ does not hold (at least $|Z|^2$ can always be extracted);
- beyond fusion-categorical models — chiral phases;
- manifolds with extra structure (bundle, spin, framing, etc.) *(joint with R. Radhakrishnan)*.

Thank you!