Anomalies and extension of sigma models

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Based on:

- KY and Mayuko Yamashita, 2106.09270
- KY, 2207.13858
- T. Sugeno, H. Wada, KY, To appear

Introduction

In this talk, I will talk about two things:

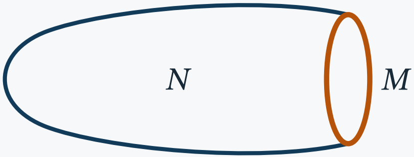
1. The general framework for sigma model anomalies
2. A generalization of sigma models (known as a stack or a gerbe) and its topological structure

General arguments work in any dimension, but in this talk, the main examples will be about string theory.

Introduction

Let us briefly recall basic facts about anomalies.

Anomalous degrees of freedom appear on the boundary of a higher-dimensional bulk on which a nontrivial bulk action exists.



$$M = \partial N$$

- $N\,(\dim N = D)$: Bulk action
- $M\,(\dim M = D-1)$: Anomalous degrees of freedom

Introduction

Example of an action:
The Chern–Simons action for a $\mathrm{U}(1)$ gauge field $A = A_\mu dx^\mu$ in three dimensions is given by

$$\begin{aligned} S(A) &= i\frac{k}{4\pi}\int AdA \qquad (k \in \mathbb{Z}) \\ &= i\frac{k}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \end{aligned}$$

Important remark: This formula is only for the topologically trivial bundle. It is not globally well-defined for topologically nontrivial bundles.

Introduction

Example of an anomaly:
A chiral fermion Ψ in 2-dim. coupled to a U(1) gauge field A:
The partition function is

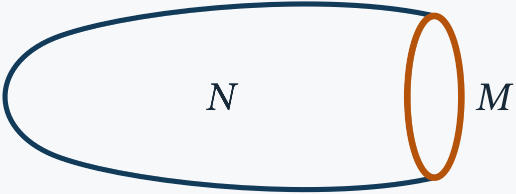
$$Z(A) = \int [\mathcal{D}\Psi]\, e^{-S(\Psi,A)}$$

Under a gauge transformation $A \to A + d\alpha$ (for a topologically trivial $\mathrm{U}(1)$ bundle),

$$Z(A + d\alpha) = Z(A) \exp\left(i \frac{1}{4\pi} \int \alpha dA \right)$$

Introduction

Suppose that the chiral fermion lives on the boundary $M = \partial N$ of a bulk manifold N on which a Chern–Simons action (with $k = 1$) is defined.



$$M = \partial N$$

Introduction

When $\partial N = M$, the Chern-Simons term behaves as

$$S(A + d\alpha) - S(A) = i\frac{1}{4\pi}\int_N d\alpha dA = i\frac{1}{4\pi}\int_M \alpha dA$$
$$= \log(Z(A + d\alpha)/Z(A))$$

Therefore, the combined system of the bulk and boundary is gauge invariant: anomaly inflow.

$$\boxed{\text{Gauge invariant: } Z_{\text{boundary}} \exp(-S_{\text{bulk}})}$$

Callan-Harvey (1985)

It describes an integer quantum Hall system (Chern insulator).

Introduction

I have only described perturbative anomalies for a topologically trivial bundle.

More generally, even for nonperturbative (or global) anomalies, it is believed that

The set of possible anomalies in $(D-1)$ dimensions
= The set of D-dim. actions modulo deformation

I will explain the meaning of ”modulo deformation” in the context of sigma model anomalies.

Contents

Sigma model

A sigma model is described by a scalar field, which is a map

$$\phi : M \to X$$

- M: spacetime manifold (e.g. 2-dim. string worldsheet)
- X: target space (e.g. target space of string theory)

Whenever we consider a bulk N ($\partial N = M$), we always assume that ϕ is also extended to $\phi : N \to X$.

Coupling a sigma model to other fields

A sigma model can be coupled to matter fields, such as fermions.

A class of couplings:
(P, A): a G-bundle P on the sigma model target space X (where G is a group) and a connection A.

The pullback $\phi^*(P, A)$ by $\phi : M \to X$ can be coupled to matter.

There is an interesting concept of sigma model anomalies.

Moore–Nelson (1984)

Anomaly?

Let us consider the example of coupling the 2-dim. chiral fermion Ψ to $\phi^* A$ as in the case of the usual gauge field.

Question:

Does this coupling between the sigma model field ϕ and the fermion Ψ have an anomaly?

Answer:

It depends on the topology of the G-bundle:

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & X \end{array}$$

Topologically trivial case

Example:
Suppose P is a trivial bundle, $P = G \times X$.
This is automatic if, for example, $X = \mathbb{R}^N$ (i.e., for a linear sigma model).

Then A is a globally well-defined 1-form.
The Chern–Simons 3-form

$$\frac{k}{4\pi}\phi^*(AdA)$$

is also a globally well-defined 3-form.

Topologically trivial case

Recall that for the case of $\mathrm{U}(1)$ gauge theory we have

$$\boxed{\text{Gauge invariant: } Z_{\text{boundary}} \exp(-S_{\text{bulk}})}$$

In the current case of a trivial bundle on X,

$$S_{\text{bulk}} = i \int_N \frac{k}{4\pi} \phi^*(AdA)$$

This S_{bulk} is globally well-defined or “gauge invariant”. Therefore, the boundary partition function Z_{boundary} is also well-defined: no anomaly.

Comparison

- In the case of a gauge field, we consider various (P, A) (where P is a bundle and A is a connection). Then, AdA is not always a globally well-defined form.
- In the case of a sigma model, we consider various $\phi : N \to X$ (where N is a bulk spacetime and X is the target space). The bundle $P \to X$ and its connection A are fixed. If P is trivial, AdA is globally well-defined.

More general case

More generally, bulk actions of the form

$$S = i \int_N \phi^* \alpha \qquad \alpha\text{: globally well-defined on } X$$

do not represent anomalies.
In other words, if the bulk action can be cancelled by adding a counterterm

$$S \to S - i \int_N \phi^* \alpha,$$

then the anomaly is cancelled. We call such counterterms as deformation, because their coefficients are not quantized.

Classification of anomalies

The classification of general actions modulo deformation is given by the Anderson dual of bordism, $I\Omega_H^{D+1}(X)$

Theorem
$I\Omega_H^{D+1}(X)$ = {The set of D-dim. actions mod deformation}

Original conjecture : Freed–Hopkins (2016).
More precise statement and proof: Yamashita–KY (2021)

I do not explain $I\Omega_H^{D+1}(X)$ in detail. Let me just discuss some intuition and properties.

de Rham cohomology

First let's recall de Rham cohomology:

$$H^{D+1}_{\mathrm{dR}}(X) = \{\omega \in \ ((D+1)\text{-form}) \mid d\omega = 0\} / \{d\alpha \mid \alpha \in \ (D\text{-form})\}$$

- D-form α corresponds to deformation or counterterm.
- $(D+1)$-form ω corresponds to the anomaly polynomial.

Example:

For a chiral fermion in $(D-1) = 2$ coupled to a $\mathrm{U}(1)$ bundle $P \to X$ with curvature 2-form $F = dA$,

$$\omega = \frac{1}{2}\left(\frac{F}{2\pi}\right)^2 \in H^4_{\mathrm{dR}}(X) \quad : \text{The anomaly polynomial}$$

For a trivial bundle, $F^2 = d(AdA)$.

Cohomology groups

de Rham theorem:

$$H_{\mathrm{dR}}^{D+1}(X) \simeq H^{D+1}(X;\mathbb{R})$$

The cohomology group with integer coefficients contains more refined information:

$$H^{D+1}(X;\mathbb{Z})$$

The more precise cohomology group relevant to anomalies:

$$I\Omega_H^{D+1}(X)$$

Application to heterotic string

To be concrete, let's consider the usual 10-dim. heterotic superstring with $G = E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$.

The anomaly polynomial:

$$\omega = \frac{1}{2}\,\mathrm{tr}\left(\frac{F}{2\pi}\right)^2 - \frac{1}{2}\,\mathrm{tr}\left(\frac{R}{2\pi}\right)^2 \in H^4_{\mathrm{dR}}(X)$$

The condition for the absence of perturbative anomaly is that ω is exact,

$$\omega = dH.$$

Application to heterotic string

Counterterm:

$$S_{\text{bulk}} = -i \int_N H$$

The counterterm H is the field strength 3-form of the B-field.

Roughly, if $H \sim dB$ and $\partial N = M$ where $\dim M = 2$ and $\dim N = 3$,

$$\int_N H \sim \int_M B.$$

The right-hand side is the usual perturbative description of the B-field on the string worldsheet M. A more precise description requires the bulk N $(\dim N = 3)$.

Witten (1999), KY (2022)

Application to heterotic string

Assume that the perturbative anomaly cancels.

It turns out that (as sets)

$$I\Omega^4_{\mathrm{spin}}(X) \sim H^1(X;\mathbb{Z}_2) \times H^2(X;\mathbb{Z}_2) \times \cdots$$

The Majorana–Weyl fermions ψ, which are superpartners of the bosonic field $\phi : M \to X$, turn out to have anomalies given by

$$w_1 \in H^1(X;\mathbb{Z}_2), \quad w_2 \in H^2(X;\mathbb{Z}_2). \quad \text{(Stiefel–Whitney classes)}$$

The absence of anomalies of ψ requires $w_1 = w_2 = 0$:

X must admit an orientation and a spin structure.

KY (2022)

Delgado–Eberhardt–Tomašević (2026)

Contents

Motivating example

The next topic is a generalization of sigma models. Let me first discuss my personal motivation.

Consider the heterotic superstring with $G = \mathrm{Spin}(32)/\mathbb{Z}_2$.

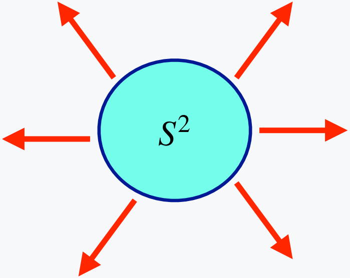
- Lie algebra $so(32)$.
- Cartan subalgebra $u(1)^{16} = so(2)^{16} \subset so(32)$.

Motivating example

Let's consider magnetic fluxes of $u(1)^{16} \subset so(32)$ on S^2. They are parametrized by 16 numbers

$$(q_1, \cdots, q_{16})$$

$$q_i = \int_{S^2} \frac{F_i}{2\pi}$$



$G = \mathrm{Spin}(32)/\mathbb{Z}_2$ allows half-integral fluxes $q_i \in \frac{1}{2} + \mathbb{Z}$. Such a magnetic monopole gives a new brane in heterotic string theory!

Ohmori–Kaidi–Tachikawa–KY (2023)

Motivating example

Worldsheet fermions: 32 Majorana or 16 complex fermions. They are coupled to $\mathrm{U}(1)$ bundles on S^2 with magnetic fluxes q_i.

Q. Are half-integer fluxes inconsistent with Dirac quantization?

A. The fermions are gauged by $\mathbb{Z}_2$. This accounts for the ambiguity around the Dirac string.

Q. OK. But the $\mathrm{U}(1)$-bundle pulled back by $\phi : M \to X (= S^2)$ still seems to be ill-defined. How can we write an explicit Lagrangian for the fermions?

Motivating example

A solution is to modify the sum over topological sectors so that the mapping degrees of $\phi : M \to X$ are multiples of some $n \in \mathbb{Z}$.

$$\int_M \frac{\phi^* F}{2\pi} = \deg(\phi) \int_X \frac{F}{2\pi}, \quad \deg(\phi) \in n\mathbb{Z}.$$

Seiberg (2010)

Taking $n = 2$,

$$\int_X \frac{F}{2\pi} \in \frac{1}{2} + \mathbb{Z}, \quad \text{but} \quad \int_M \frac{\phi^* F}{2\pi} \in \mathbb{Z}.$$

The pullback bundle can be consistently coupled to the fermions.

Questions

- What is the general formulation of coupling a sigma model to matter degrees of freedom in a representation that is “not allowed”? (E.g., fractional charges of $\mathrm{U}(1)$).
- What is the cohomology group that is relevant to the sigma model anomalies?

The setup of the problem

X: sigma model target space.
P: G-bundle

$$\begin{tikzcd} G \arrow[r] & P \arrow[d] \\ & X \end{tikzcd}$$

We consider a group extension

$$1 \to A \to \widetilde{G} \to G \to 1.$$

We want to couple a matter field Ψ that transforms in a representation of $\widetilde{G}$ but not of G.

Example

The previous heterotic string example may be regarded as

$$1 \to \mathbb{Z}_2 \to \mathrm{U}(1) \to \mathrm{U}(1) \to 1.$$

Alternatively, we may consider

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(32) \to \mathrm{Spin}(32)/\mathbb{Z}_2 \to 1.$$

The representation $\mathbf{32}$ carried by the 32 Majorana–Weyl fermions is allowed in $\mathrm{Spin}(32)$.

The formulation

The generalization of sigma models, like Seiberg's modification of the sum over topological sectors, is described by what is known as a stack or a gerbe (with which I am not familiar).

Hellerman–Sharpe (2010)

I will present a very elementary formulation.

The formulation

Consider a sigma model with the target space P (rather than X).

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & X \end{array}$$

We gauge $\widetilde{G}$, which acts on the target space P via $\widetilde{G} \to G$.

$$1 \to A \to \widetilde{G} \to G \to 1.$$

Matter fields can be coupled to the $\widetilde{G}$ gauge field.

Near each point $p \in X$, the fiber G plays the role of Goldstone bosons which spontaneously break the gauge group $\widetilde{G}$ to A.

Example

We consider the extension of $X = S^2 = \mathbb{CP}^1$:

- $P = S^3, \quad X = S^2 = \mathbb{CP}^1, \quad P \to X$: Hopf fibration.
- $1 \to \mathbb{Z}_n \to \mathrm{U}(1) \to \mathrm{U}(1) \to 1$

It can be easily generalized to $\mathbb{CP}^N$.

If $n = 1$ (i.e., the trivial extension), it is the standard description of the $\mathbb{CP}^N$ sigma model by introducing an auxiliary $\mathrm{U}(1)$ gauge field. The case $n > 1$ assigns charge n to the fields.

Pantev–Sharpe (2005)

The data of the field

The field is described by the following data.

- $\widetilde{G}$-bundle $\widetilde{Q} \to M$ and its connection.
- A section of the associated bundle $\widetilde{Q} \times_{\widetilde{G}} P$ (because P is the sigma model target space which is gauged by $\widetilde{G}$ via $\widetilde{G} \to G$).

Topologically (but not geometrically), this is equivalent to:

$$\boxed{\text{Map } \widetilde{\phi} : M \to \widetilde{X}, \text{ where } \widetilde{X} := E\widetilde{G} \times_{\widetilde{G}} P}$$

$E\widetilde{G} \to B\widetilde{G}$ is the universal $\widetilde{G}$-bundle over the classifying space $B\widetilde{G}$.

The new target space

Topologically (but not geometrically), the extended sigma model is described by taking

$$\widetilde{X} := E\widetilde{G} \times_{\widetilde{G}} P$$

as the new target space.

Therefore, anomalies in $(D-1)$ dimensions are expected to be described by the cohomology group

$$I\Omega_H^{D+1}(\widetilde{X})$$

Remark

Do we get a new type of string theory? Basically, No.

- If $\widetilde{G}$ is not coupled to matter fields, we have a 1-form center symmetry by which the theory splits into different "universes", in each of which we get the usual string theory.
- If some matter fields Ψ are coupled, the 1-form center symmetry can be broken. But the theory is equivalent to first gauging Ψ by A, and then coupling it to the original X.

However, the formulation based on $\widetilde{X}$ is convenient for computing topological properties such as anomalies.

Fiber sequence

The space $\widetilde{X} := E\widetilde{G} \times_{\widetilde{G}} P$ fits into a (homotopy) fiber sequence

$$BA \to \widetilde{X} \to X$$

- X is the original target space.
- BA (the classifying space of A bundles) corresponds to the A gauge field.

When the fibration is nontrivial, the anomalies are not just the ones for the group A and the sigma model X separately. We have a nontrivial mixture of them.

Example of anomalies

The motivating example on $X = S^2$ with fractional fluxes:

- $P = S^3, \quad X = S^2 = \mathbb{CP}^1, \quad P \to X$: Hopf fibration.
- $1 \to \mathbb{Z}_n \to \mathrm{U}(1) \to \mathrm{U}(1) \to 1 \qquad (A = \mathbb{Z}_n)$

Denote

$$\tilde{S}^2_n := E\mathrm{U}(1) \times_{\mathrm{U}(1)} S^3 \quad (\mathrm{U}(1) \text{ acts on } S^3 \text{ with charge } n.)$$

The group relevant for anomalies in 2-dim. is

$$I\Omega^4_{\mathrm{spin}}(\tilde{S}^2_n) = \mathbb{Z} \oplus \mathbb{Z}_{2n^2}.$$

$\mathbb{Z}_{2n^2}$ is not obvious from the point of view of $X = S^2$ or $A = \mathbb{Z}_n$.

Example of anomalies

Let us consider chiral (e.g. left-moving) fermions Ψ_i which are coupled to the sigma model with target S^2, such that their charges are

$$\frac{q_i}{n} \quad \text{under} \quad G = \mathrm{U}(1)$$

Such fractional charges are possible in the extended sigma model $\tilde{S}^2_n$. (The charges are q_i under $\widetilde{G} = \mathrm{U}(1)$.)
The anomaly cancellation condition for $\mathbb{Z}_{2n^2} \subset I\Omega^4_{\mathrm{spin}}(\tilde{S}^2_n)$ is:

$$\sum_i q_i^2 = 0 \mod 2n^2$$

Contents

Summary

- When a sigma model with target X is coupled to matter fields, there can be anomalies which are classified by $I\Omega_H^{D+1}(X)$.
- Sigma models have a nontrivial extension which corresponds to a group extension $1 \to A \to \widetilde{G} \to G \to 1$. For a given G-bundle $P \to X$, the extension is topologically described by a target space $\widetilde{X} := E\widetilde{G} \times_{\widetilde{G}} P$. The sigma model X and the A gauge field are mixed in a nontrivial way via $BA \to \widetilde{X} \to X$.