

Resurgence of the Thermal Transition between Bounce and Sphaleron

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Physics and Mathematics of Generalized Symmetry in QFT

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■ INTRODUCTION

Metastability is everywhere

A metastable state is not truly stable — it is only **trapped**. The structure is always the same



Supersaturated sodium acetate — a single seed crystal triggers runaway crystallization.

Supersaturated solution → crystallization seed

Superheated liquid → critical vapor bubble

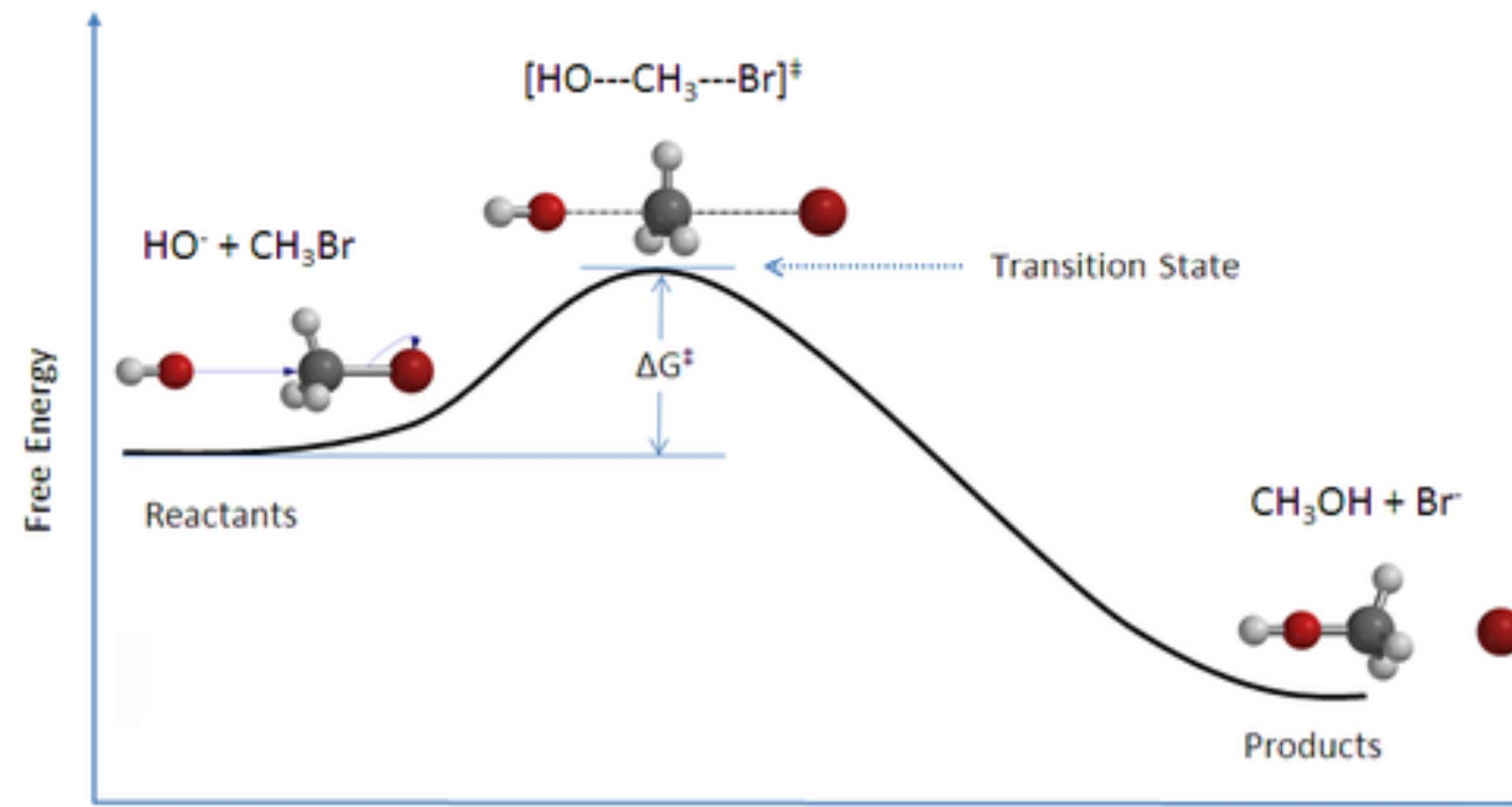
Supercooled liquid → ice nucleus

local basin → barrier → rare escape

■ INTRODUCTION

Chemical reaction

The same picture — a free-energy barrier $\Delta G^\ddagger$ sets the rate.



EYRING EQUATION

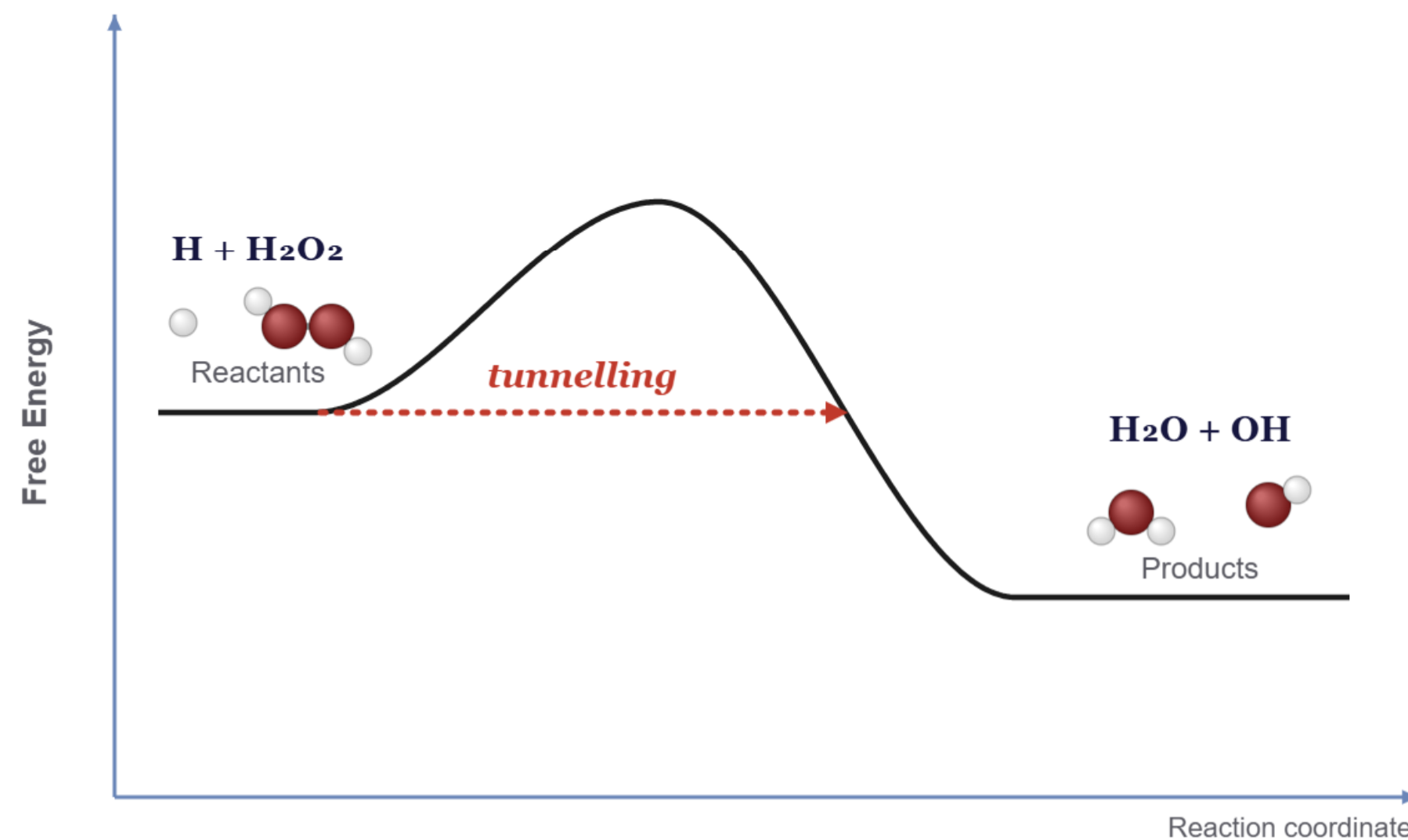
$$k_{\text{thermal}} \sim A e^{-\Delta G^\ddagger / k_B T}$$

- k — the thermal rate constant.
- $\Delta G^\ddagger$ — the free-energy barrier.

■ INTRODUCTION

Chemical reaction

A light atom can cross by going straight through the barrier.

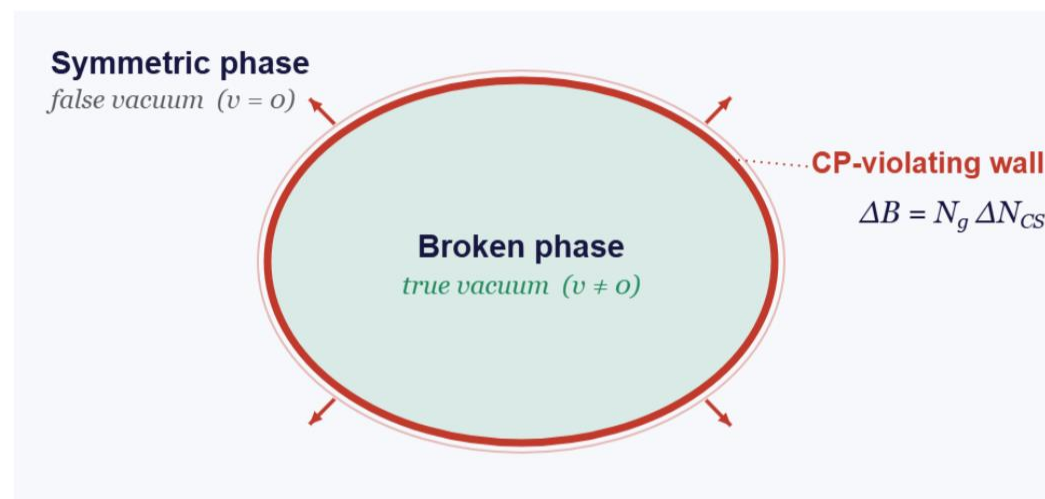
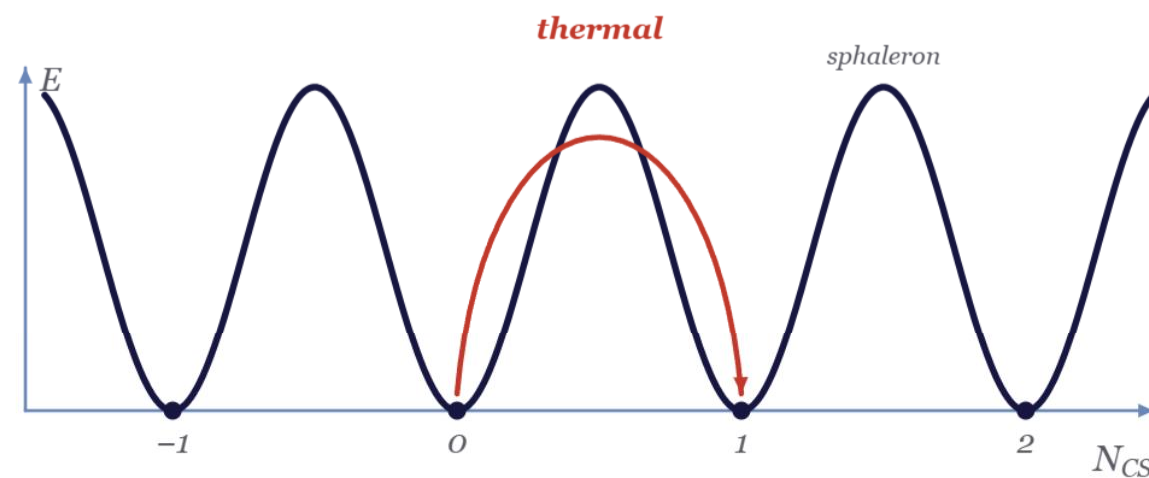


- A light hydrogen atom passes **through** the barrier, not over it.
- **No transition state** at the barrier top to consider.
- Quantum tunnelling — the regime the rest of this talk computes.

■ INTRODUCTION

Electroweak baryogenesis as false-vacuum decay

A CP-violating bubble wall and active sphalerons write the baryon asymmetry.

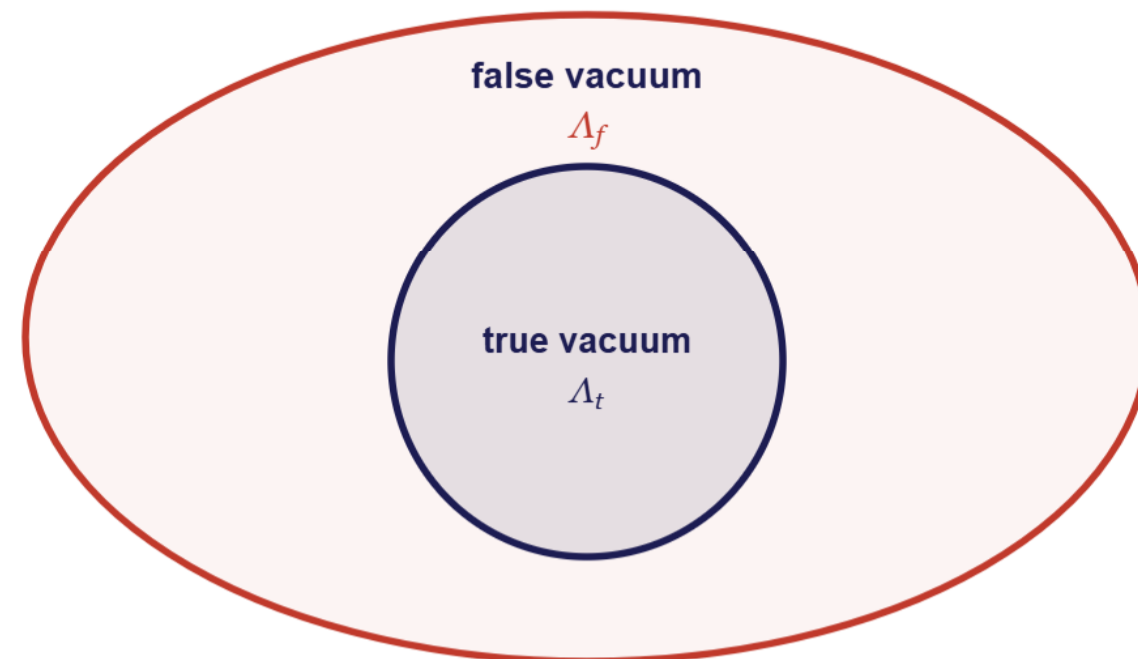


- 1 A **first-order** electroweak transition — bubbles of broken vacuum nucleate and expand.
- 2 **CP violation** at the moving wall builds a chiral asymmetry ahead of it.
- 3 Active **sphalerons** in the symmetric phase convert it into baryon number.
- 4 The wall sweeps it into the broken phase — sphalerons **freeze out**, the asymmetry survives.

False-vacuum decay → the matter excess of the universe.

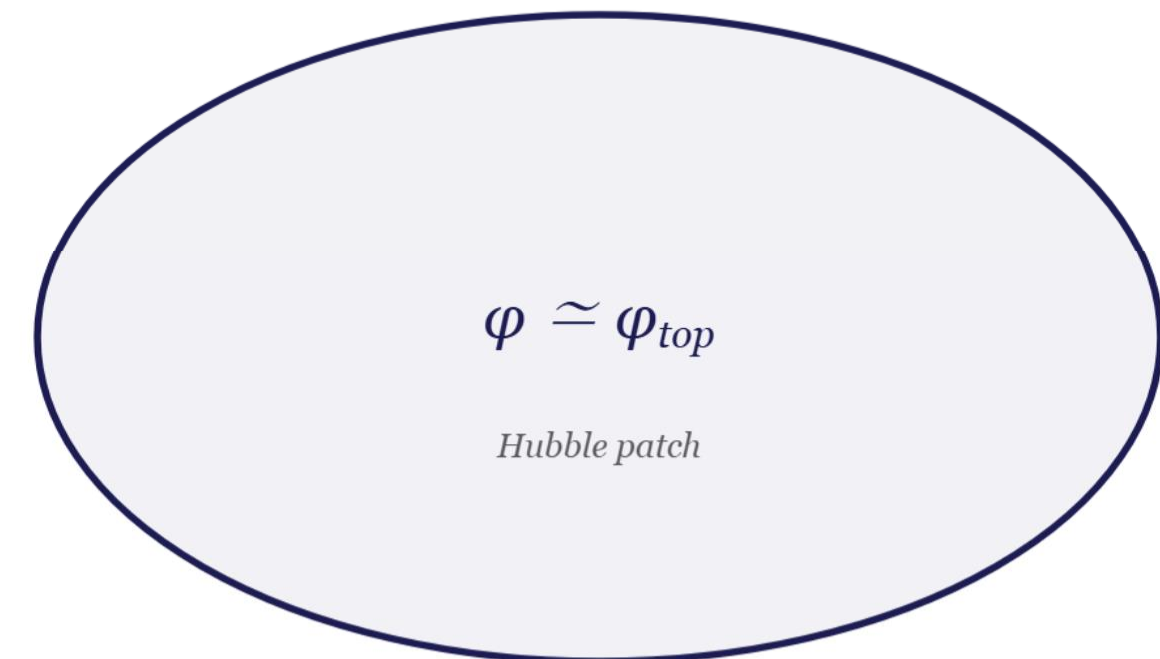
False-vacuum decay in cosmology

Coleman–De Luccia



- A gravitating bubble of true vacuum.
- Interior and exterior: universes of different cosmological constant Λ .

Hawking–Moss



- A Hubble patch transitions homogeneously to $\varphi \simeq \varphi_{top}$.
- An inflationary graceful exit.

■ MOTIVATION

A historical lineage

Reaction-rate theory, instantons, and thermal field theory converge on a single finite-temperature picture of metastability.



Thermal escape

Kramers • Langer

The statistical theory of metastability and thermally activated escape over a barrier.



The bounce

Coleman • Callan–Coleman

False-vacuum decay via the bounce instanton, computed at one loop.



Finite temperature

Affleck • Linde

Finite-temperature decay — read off from $\text{Im } F$ in thermal field theory.

■ MOTIVATION

Affleck's proposal

The false-vacuum decay rate at $T = 1/\beta$ is encoded in the **imaginary part** of the thermal free energy $F(\beta)$:

LOW TEMPERATURE

$$\Gamma = -\frac{2}{\hbar} \operatorname{Im} F(\beta)$$

HIGH TEMPERATURE

$$\Gamma = -\frac{\beta |\omega_c|}{\pi} \operatorname{Im} F(\beta)$$

- The false vacuum is only **metastable** $\rightarrow F(\beta)$ acquires an imaginary part.

How to justify the proposal by Affleck?

■ SETUP

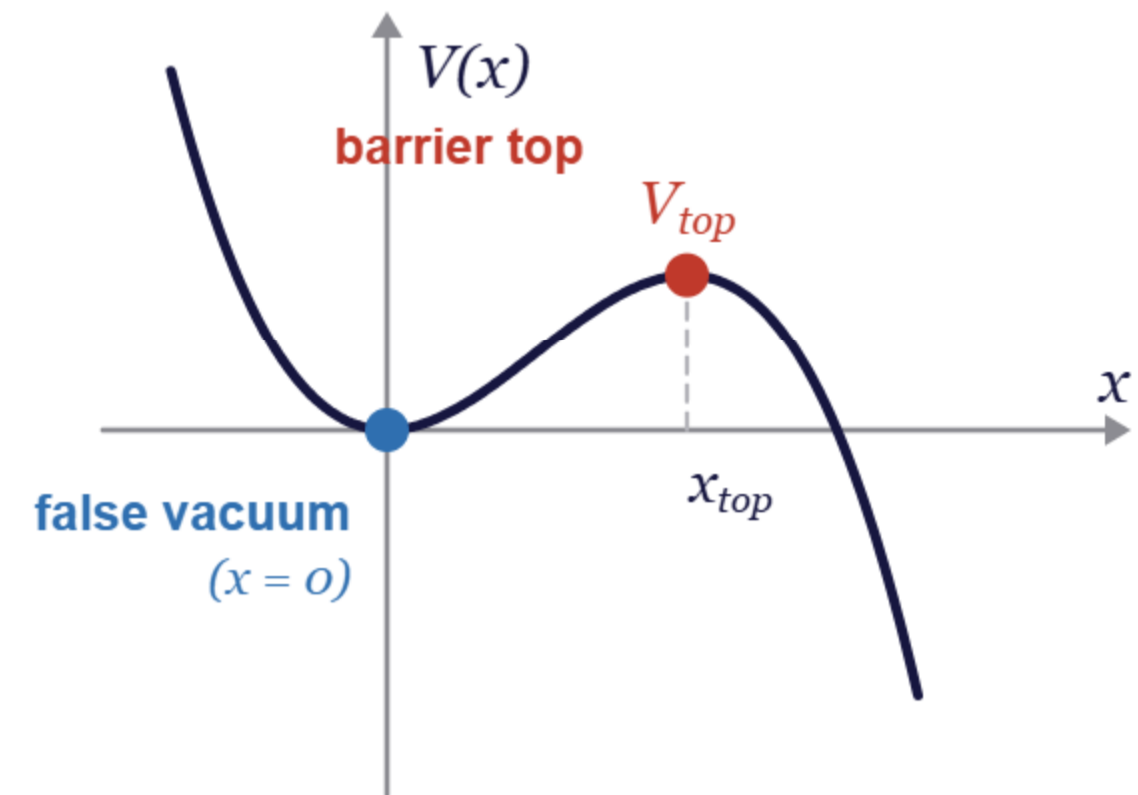
Thermal free energy: the Euclidean path integral

Simplest setting — quantum mechanics in a metastable potential V .

$$e^{-\beta F(\beta)} = \text{tr}_{\mathcal{H}} [e^{-\beta H}] = \int \mathcal{D}x(\tau) \exp \left[-\frac{1}{\hbar} S_E[x(\tau)] \right]$$

$$S_E = \int_0^{\beta \hbar} d\tau \left(\frac{1}{2} \dot{x}^2 + V(x) \right)$$

$$\underbrace{x(\tau + \beta \hbar) = x(\tau)}_{\text{periodic boundary condition}}$$



■ SETUP

Semiclassical limit: two competing saddles

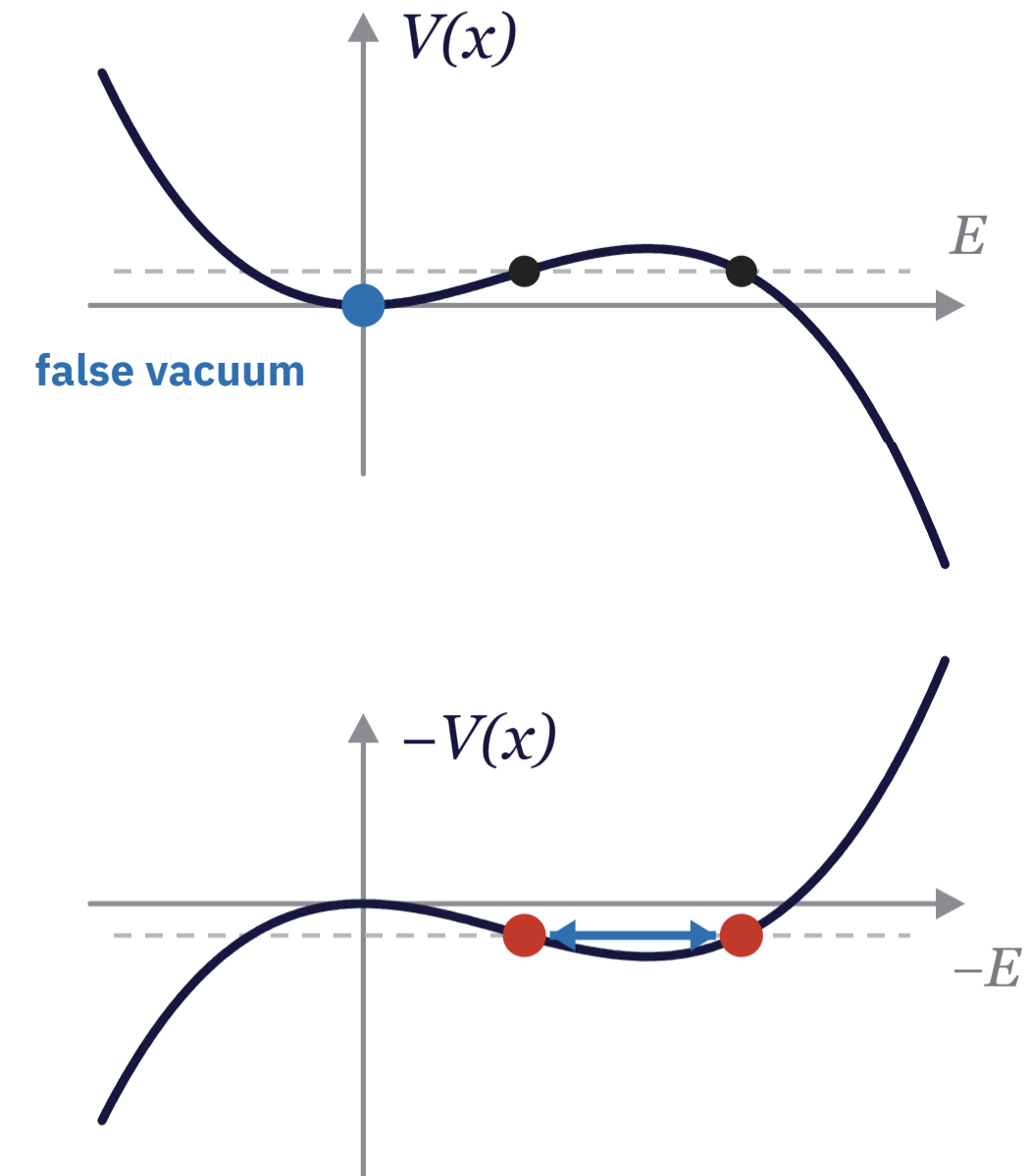
As $\hbar \rightarrow 0$, the path integral is dominated by saddle points:

$$-\frac{d^2 x}{d\tau^2} + V'(x) = 0, \quad x = \bar{x}(\tau)$$

$$\frac{1}{2} \left(\frac{d\bar{x}}{d\tau} \right)^2 - V(\bar{x}) = -E$$

A 1D particle of energy $-E$ moving in the inverted potential $-V(x)$

Two saddles compete.



First saddle: the bounce

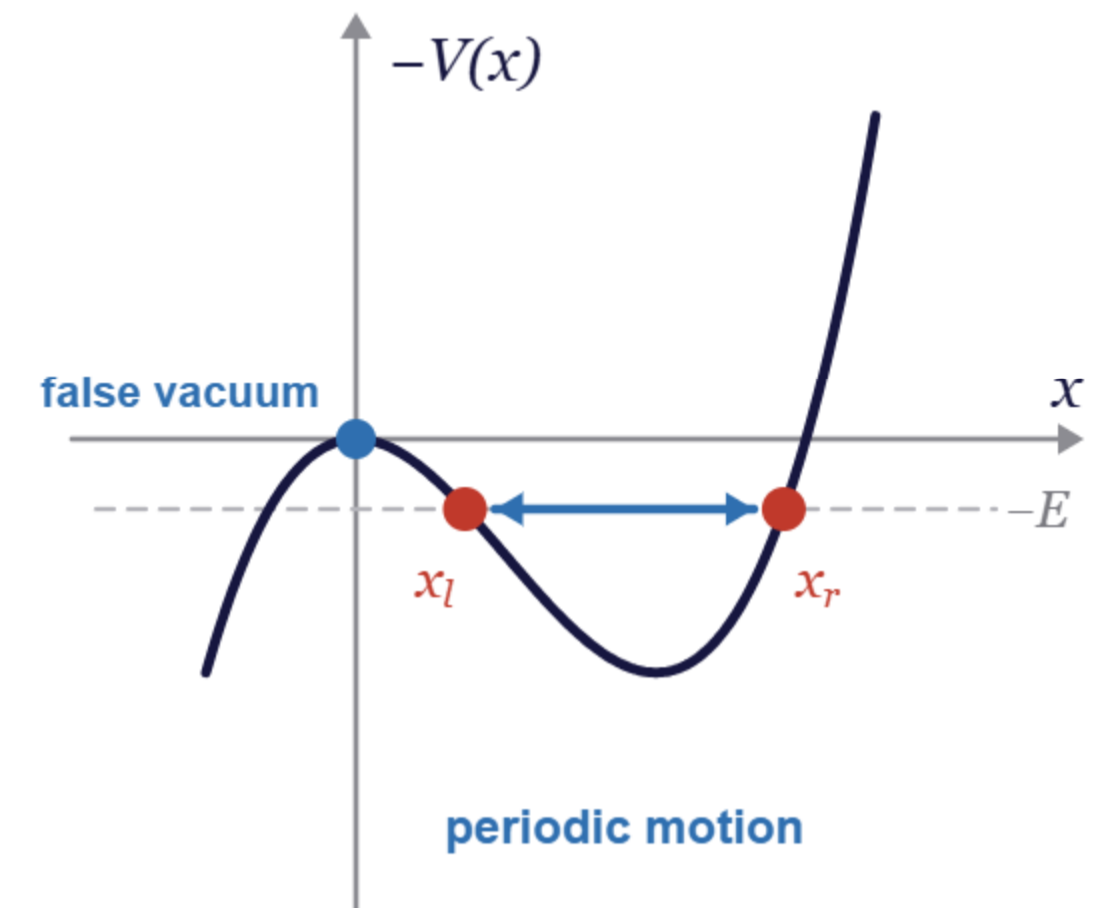
A non-static, periodic solution $\bar{x}(\tau)$.

$$\frac{1}{2} \left(\frac{d\bar{x}}{d\tau} \right)^2 - V(\bar{x}) = -E \quad E < V_{top}$$

The period map fixes the orbit energy $E(\beta)$:

$$\beta = \frac{2}{\hbar} \int_{x_l}^{x_r} \frac{dx}{\sqrt{2(V(x) - E(\beta))}}$$

Temperature fixes the bounce energy.



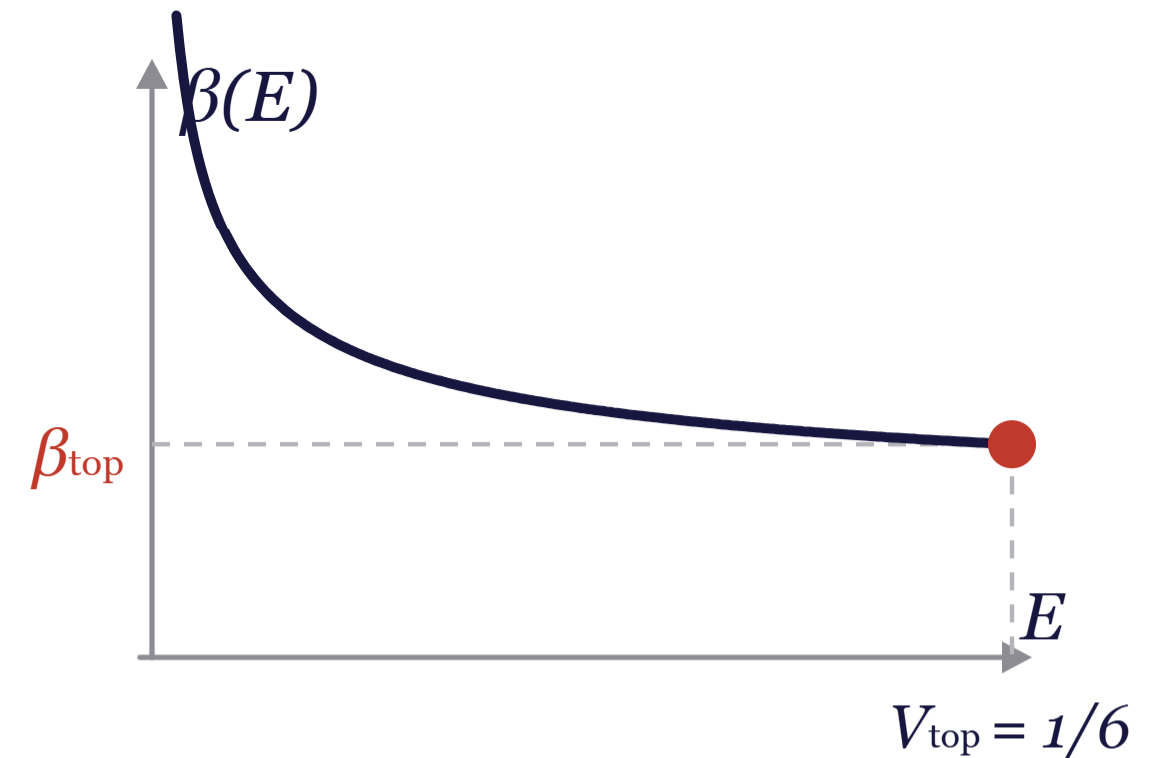
First saddle: the bounce

An example — the cubic model $V(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$.

- The period map $\beta(E)$ is monotonic.
- A bounce solution exists only for $\beta < \beta_{\text{top}}$.

Harmonic limit at the barrier top:

$$\beta_{\text{top}} = \left(\frac{\sqrt{|V''(x_{\text{top}})|}}{2\pi} \right)^{-1}$$



The false-vacuum decay rate

The exponential — the classical action factor ($\hbar \rightarrow 0$).

$$\Gamma_b = \left(\frac{B(\beta)}{2\pi\hbar} \right)^{1/2} \left| \frac{\det[-\partial_\tau^2 + \omega_0^2]}{\det'[-\partial_\tau^2 + V''(\bar{x})]} \right|^{1/2} e^{-S_b(\beta)/\hbar}$$

with the Euclidean on-shell action $S_b(\beta)$:

$$S_b(\beta) = 2 \int_{x_l}^{x_r} dx \sqrt{2(V(x) - E(\beta))} + \beta\hbar E(\beta) \equiv B(\beta) + \beta\hbar E(\beta)$$

The false-vacuum decay rate

The prefactor — the translational zero mode.

$$\Gamma_b = \left(\frac{B(\beta)}{2\pi\hbar} \right)^{1/2} \left| \frac{\det[-\partial_\tau^2 + \omega_0^2]}{\det'[-\partial_\tau^2 + V''(\bar{x})]} \right|^{1/2} e^{-S_b(\beta)/\hbar}$$

- A translational **zero mode** $d\bar{x}/d\tau$ — Euclidean-time translation.
- Integrating over its collective coordinate yields $(B(\beta)/2\pi\hbar)^{1/2}$; the power counts the $\hbar^{-1/2}$ zero modes.

The false-vacuum decay rate

The determinant — the single negative mode.

$$\Gamma_b = \left(\frac{B(\beta)}{2\pi\hbar} \right)^{1/2} \left| \frac{\det[-\partial_\tau^2 + \omega_0^2]}{\det'[-\partial_\tau^2 + V''(\bar{x})]} \right|^{1/2} e^{-S_b(\beta)/\hbar}$$

- The determinant ratio is the **one-loop fluctuation determinant**.
- The bounce has a single **negative mode** → a factor of i in F , hence , i.e., $\text{Im } F(\beta) \neq 0$
false-vacuum decay.

The false-vacuum decay rate

The determinant — Gelfand–Yaglom in closed form.

Evaluate the determinant with the **Gelfand–Yaglom theorem** (1D quantum mechanics):

$$\left| \frac{\det[-\partial_\tau^2 + \omega_0^2]}{\det'[-\partial_\tau^2 + V''(\bar{x})]} \right|^{1/2} = \sqrt{\frac{1}{W(\beta) B(\beta)}}$$

$$W(\beta) = 2 \int_{x_l}^{x_r} \frac{dx}{[2(V - E(\beta))]^{3/2}}$$

Statistical-mechanical interpretation

Reading the rate as a thermal average over tunnelling.

$$\Gamma(T) \simeq \frac{1}{Z_{\text{fv}}} \int dE \, e^{-\beta E} T(E)$$

Below the barrier (WKB):

$$T(E) \sim e^{-B(E)/\hbar} \quad (E < V_{\text{top}})$$

$$B(E) = 2 \int_{x_l}^{x_r} dx \sqrt{2(V(x) - E)}$$

■ $e^{-\beta E}$ — the **Boltzmann weight** (thermal population)

■ $T(E)$ — the **transmission probability** (quantum tunneling)

Statistical-mechanical interpretation

The saddle condition — Boltzmann versus tunnelling.

$$\beta = \frac{2}{\hbar} \int_{x_l}^{x_r} \frac{dx}{\sqrt{2(V(x) - E(\beta))}}$$

$$\frac{d}{dE} [B(E) + \beta \hbar E] = 0 \Rightarrow -\frac{dB}{dE} = \beta \hbar \Rightarrow E = E(\beta)$$

THERMAL FACTOR

favours lower energy.

TUNNELLING FACTOR

favours higher energy.

COMPETITION

selects $E(\beta)$: the bounce period
equals the inverse temperature

Statistical-mechanical interpretation

The result — evaluating at the saddle $E(\beta)$.

$$\Gamma(T) = \left(\frac{1}{2\pi\hbar W(\beta)} \right)^{1/2} \frac{1}{Z_{\text{fv}}} e^{-\beta E(\beta) - B(\beta)/\hbar}$$

→ this reproduces **Affleck's bounce rate** exactly.

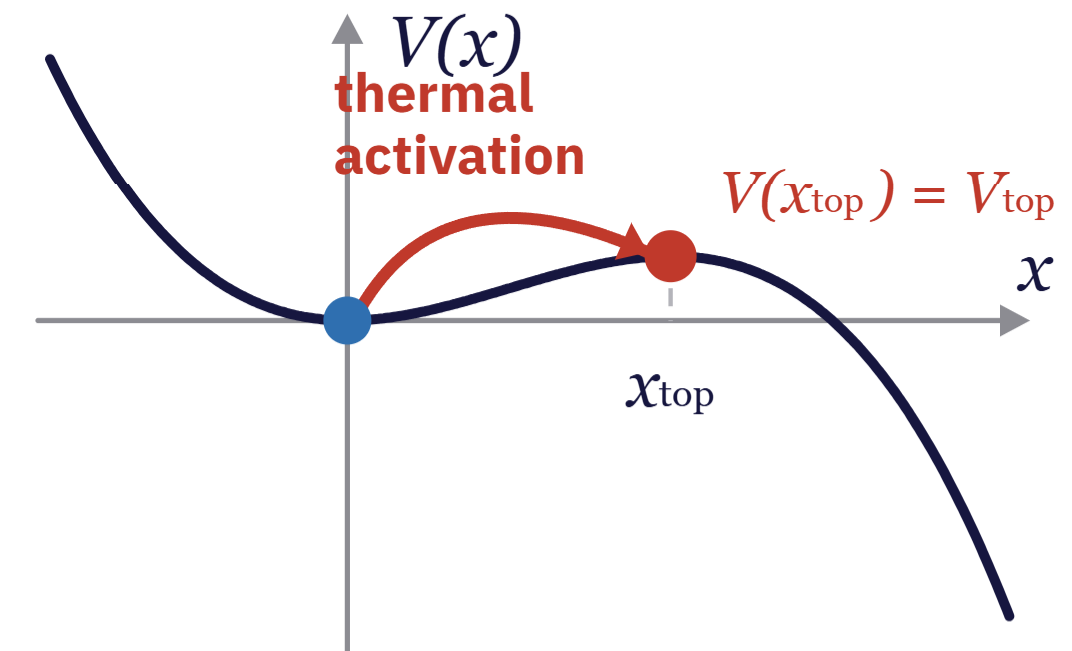
Second saddle: the sphaleron

A static saddle at the barrier top — thermal activation.

Static Euclidean saddle:

$$x(\tau) = x_{\text{top}}, \quad V(x_{\text{top}}) = V_{\text{top}}$$

- **No tunnelling** — a purely thermal jump to the barrier top.
- **Bottleneck:** the Boltzmann cost of reaching V_{top} .
- Dominant at high temperature.



Second saddle: the sphaleron

The one-loop decay rate of the static saddle.

$$\Gamma_{\text{sph}} = \frac{\omega_{\text{top}}}{2\pi} \left[\frac{\sinh(\beta\hbar\omega_0/2)}{\sin(\beta\hbar\omega_{\text{top}}/2)} \right] e^{-S_{\text{sph}}/\hbar}$$

$$S_{\text{sph}} = \beta\hbar V_{\text{top}}$$

with the frequencies

$$\omega_0 = \sqrt{V''(0)}, \quad \omega_{\text{top}} = \sqrt{|V''(x_{\text{top}})|}$$

Sphaleron action and the negative mode

- **No zero mode** (unlike the bounce): no extra $\hbar^{-1/2}$ in the prefactor.

Negative mode?

$$T_{\text{top}} = \frac{\omega_{\text{top}}}{2\pi}$$

$$T > T_{\text{top}}$$

a single negative mode $\rightarrow \text{Im } F \neq 0$ — the sphaleron contributes.

$$T < T_{\text{top}}$$

no negative mode $\rightarrow$ the sphaleron does not contribute to the decay.

Statistical-mechanical interpretation

High-energy states ($E > V_{\text{top}}$) — an inverted oscillator near the top.

$$\Gamma(T) \simeq \frac{1}{Z_{\text{fv}}} \int dE \, e^{-\beta E} T(E)$$

Near the top — an inverted oscillator:

$$V(x) \simeq V_{\text{top}} - \frac{1}{2} \omega_{\text{top}}^2 (x - x_{\text{top}})^2$$

Exact transmission coefficient:

$$T(E) = \frac{1}{1 + \exp\left[-\frac{2\pi}{\hbar\omega_{\text{top}}} (E - V_{\text{top}})\right]}$$

■ $T(E) \approx 1$ for $E \gg V_{\text{top}}$.

■ Dominated by E just above the barrier.

Thermal integral

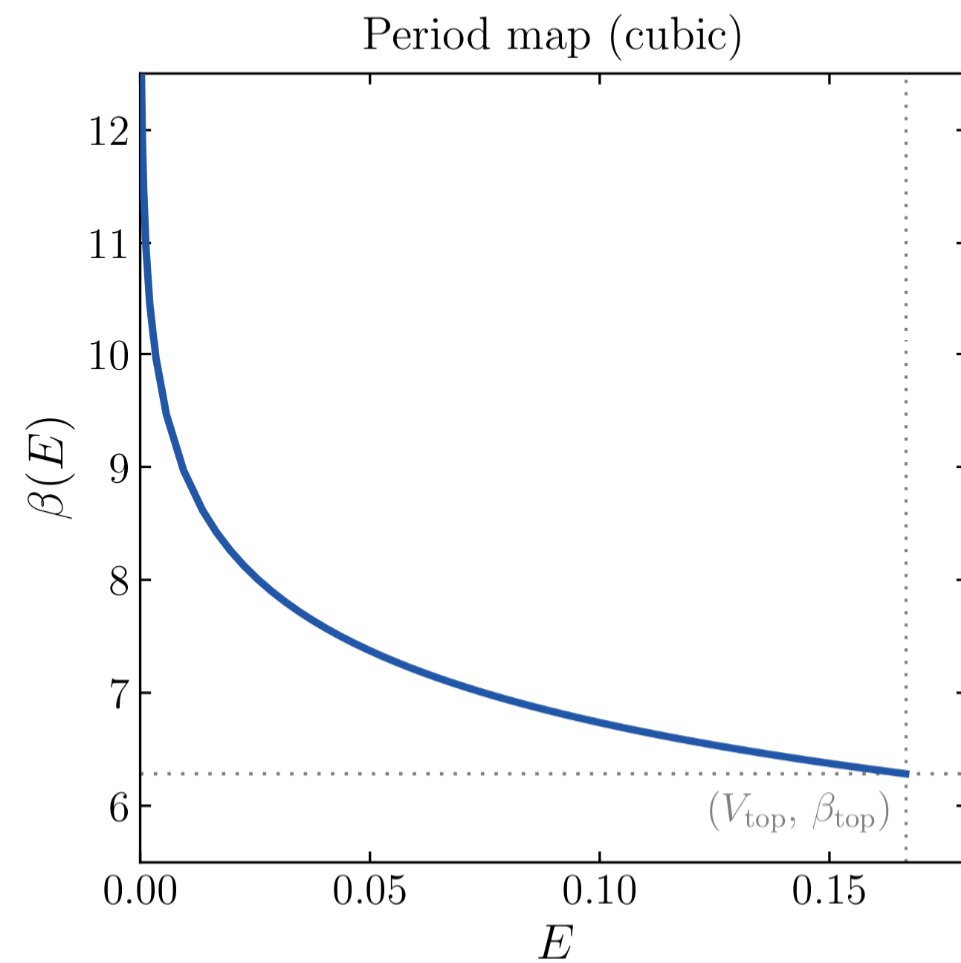
The thermal average with the parabolic-barrier transmission.

$$\begin{aligned}\Gamma_{\text{sph}} &= \frac{1}{Z_{\text{fv}}} \int dE \, e^{-\beta E} T(E) \\ &= \frac{\omega_{\text{top}}}{2\pi} \left[\frac{\sinh(\beta \hbar \omega_0 / 2)}{\sin(\beta \hbar \omega_{\text{top}} / 2)} \right] e^{-S_{\text{sph}} / \hbar}\end{aligned}$$

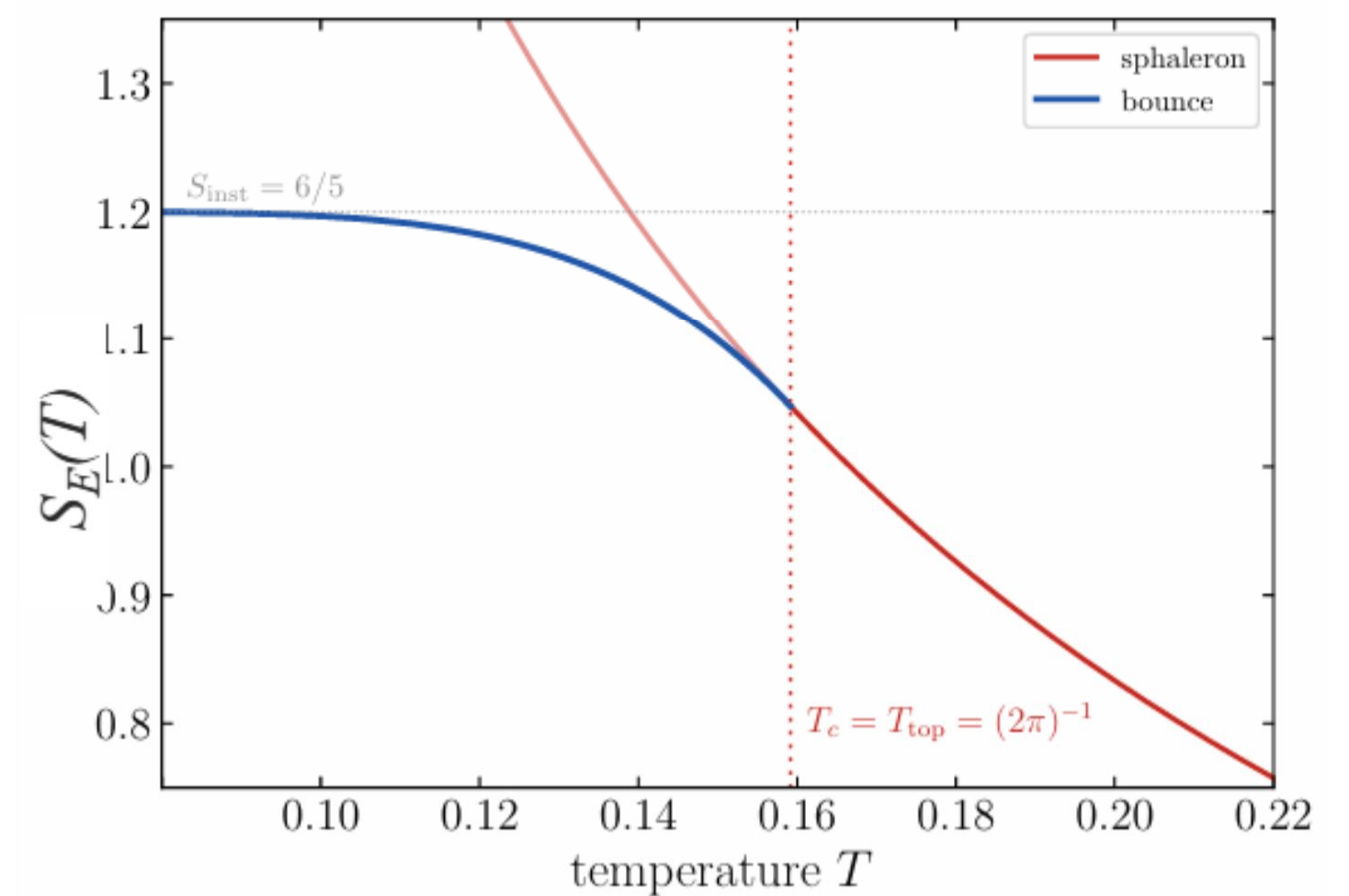
→ exactly **Affleck's result** at high temperature.

A second-order transition

A monotone period map — the bounce joins the sphaleron tangentially.



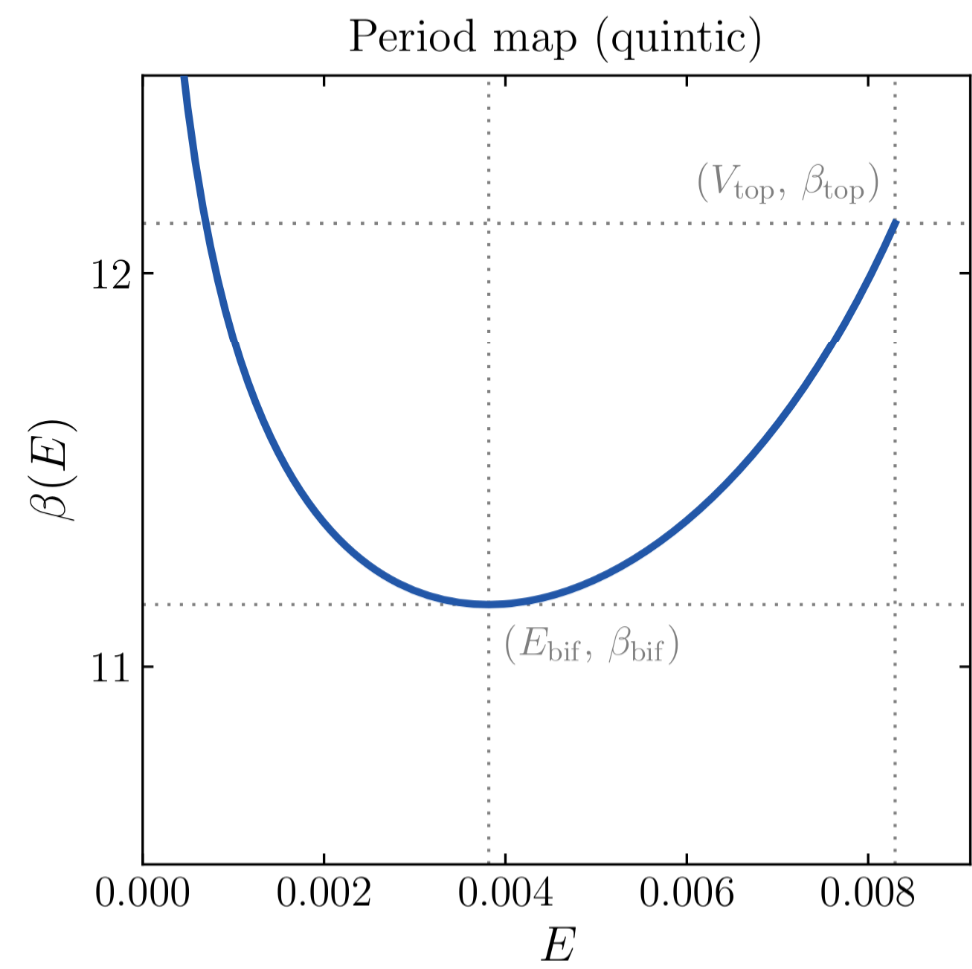
Monotone $\beta(E)$ — a single bounce branch.



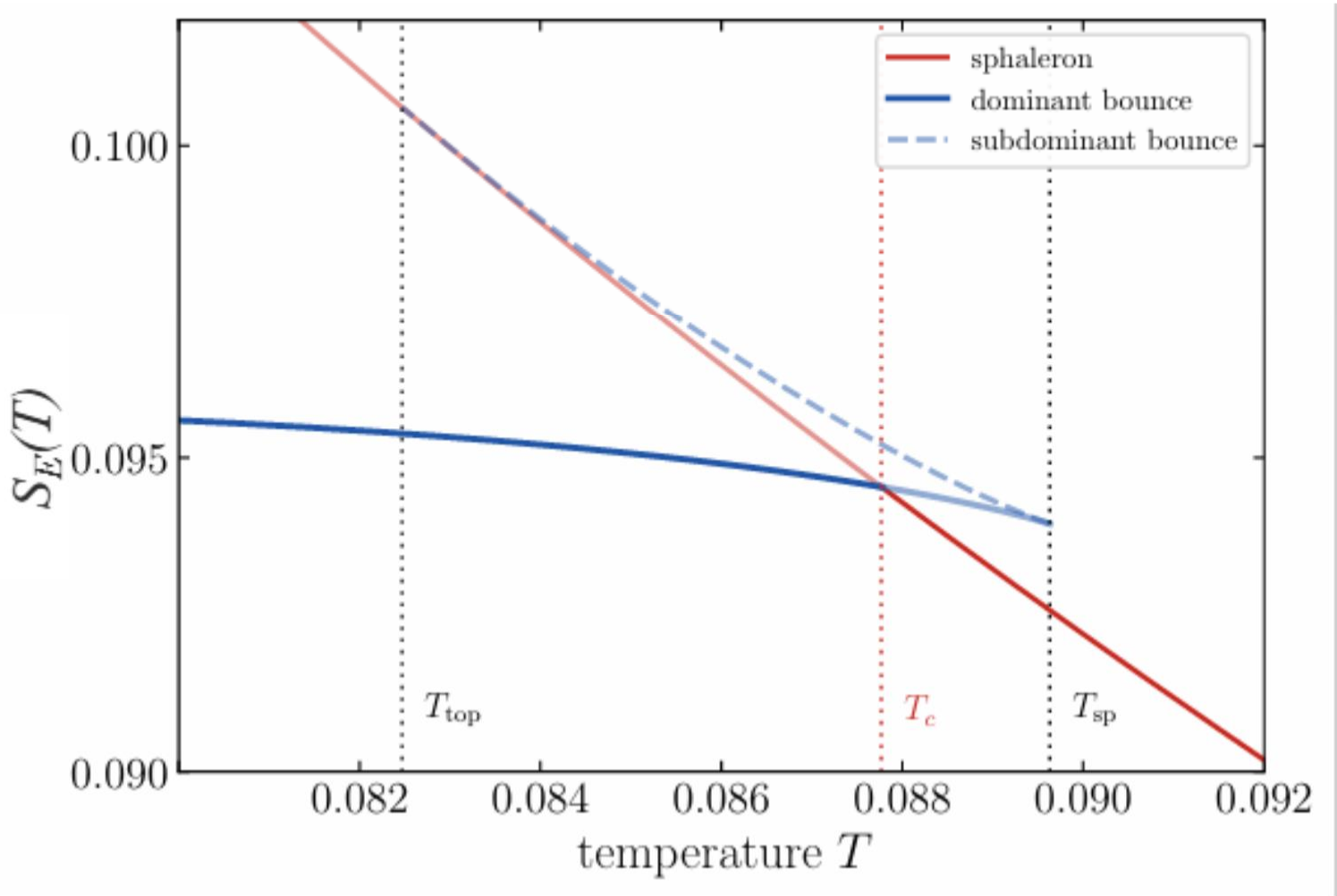
The bounce meets the sphaleron **tangentially** at T_c .

A first-order transition

A folded period map — two bounce branches whose actions cross.



Folded $\beta(E)$ — coexisting bounce branches.



Dominant bounce and sphaleron **cross** at T_c — a kink.

■ THE DETERMINANT PROBLEM

Beyond one dimension

Decay rates need a fluctuation determinant around the saddle.

$$\left[\frac{\det \mathcal{M}_{\text{false}}}{\det' \mathcal{M}_{\text{saddle}}} \right]^{1/2}, \quad \mathcal{M}_{\text{saddle}} = -\partial^2 + V''(\phi_{\text{saddle}})$$

MULTI-DIMENSIONAL QM

- Need the non-perturbative saddle and its spectrum — Gelfand–Yaglom is far less powerful than in 1D.
- each saddle only has a single negative mode

SCALAR QFT — HARDER

- Which saddle? Thin-wall vs thick-wall.
- UV divergences and renormalization; numerical saddle profiles are often needed for the spectral analysis.
- a single negative mode ? - no proof yet at finite temperature

■ THE DETERMINANT PROBLEM

Gauge theories are worse

- **Gauge modes & ghosts** enter the determinant.
- **Goldstone / gauge mixing** complicates the spectrum.
- **Intermediate quantities** are gauge dependent.
- **The physical rate** must be gauge independent — a nontrivial check.

Example The electroweak / Yang–Mills–Higgs sphaleron determinant.

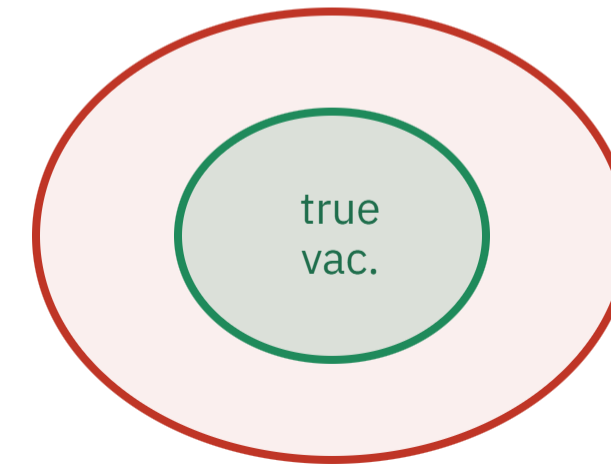
■ THE DETERMINANT PROBLEM

Gravity is even worse

- Scalar–metric fluctuations mix.
- Diffeomorphism constraints.
- **Conformal-factor problem**; boundary conditions.
- CDL / HM negative-mode counting is delicate.

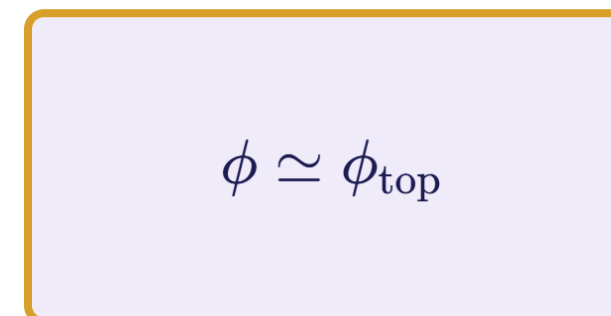
Some formulations give one negative mode, others infinitely many (kinetic-sign / conformal-factor issues) — a known, unresolved subtlety.

Coleman–De Luccia



localized bubble (with gravity)

Hawking–Moss

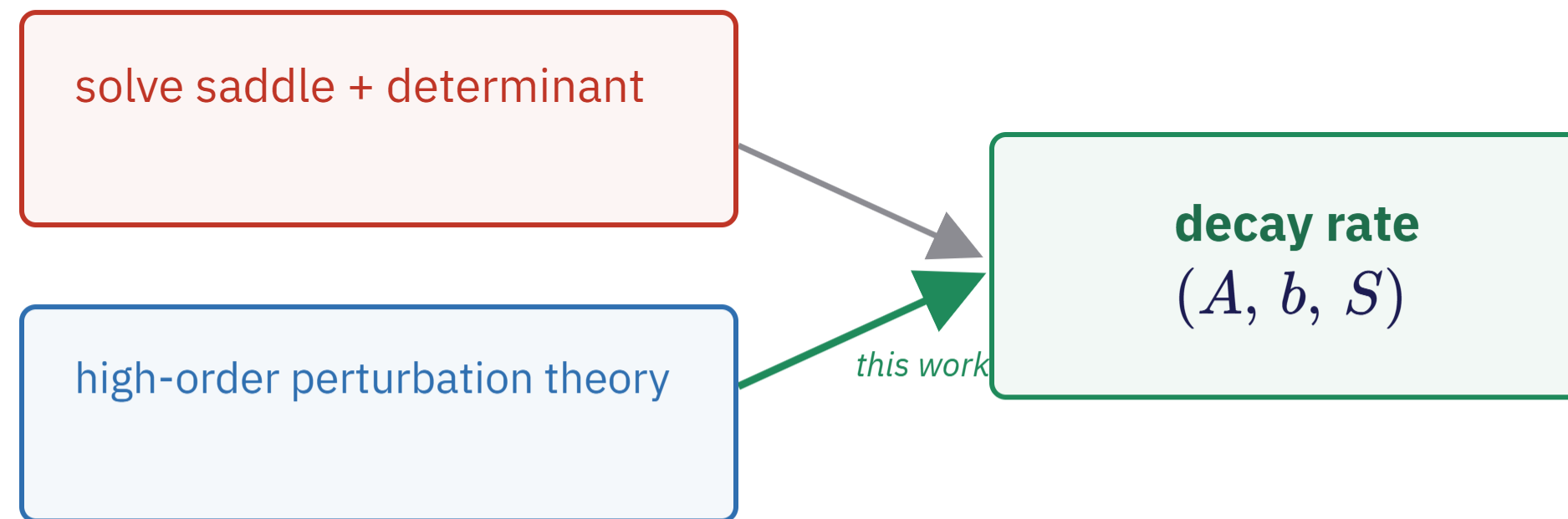


homogeneous Hubble patch

■ MOTIVATION

A way around the determinant

Two routes to the same decay rate.



The perturbative expansion

A physical observable as a formal power series.

$$\Phi(\lambda) \sim \sum_{n=0}^{\infty} a_n \lambda^n \quad (\lambda \rightarrow 0)$$

λ — coupling constant (g^2 , $\hbar$, ...)

- Coefficients grow factorially; zero radius of convergence $\Rightarrow$ an **asymptotic series**.
- Divergence is not a flaw — it **encodes the non-perturbative physics**. [Bender–Wu]

Large-order behaviour

The coefficients share a universal growth.

$$a_n \sim \frac{S}{2\pi} \frac{\Gamma(n+b)}{A^{n+b}} \quad (n \rightarrow \infty)$$

A

location of the leading Borel singularity.

b

characteristic exponent.

S

Stokes constant.

Borel resummation

Remove the factorial growth — the **Borel transform** of the formal series:

$$\widehat{\mathcal{B}}(t) = \sum_{n \geq 0} \frac{a_n}{\Gamma(n+b)} t^n \quad (|t| < A)$$

Reconstruct $\mathcal{S}\Phi(\lambda)$ by a Laplace-type integral, provided $\widehat{\mathcal{B}}(t)$ is regular on the positive axis:

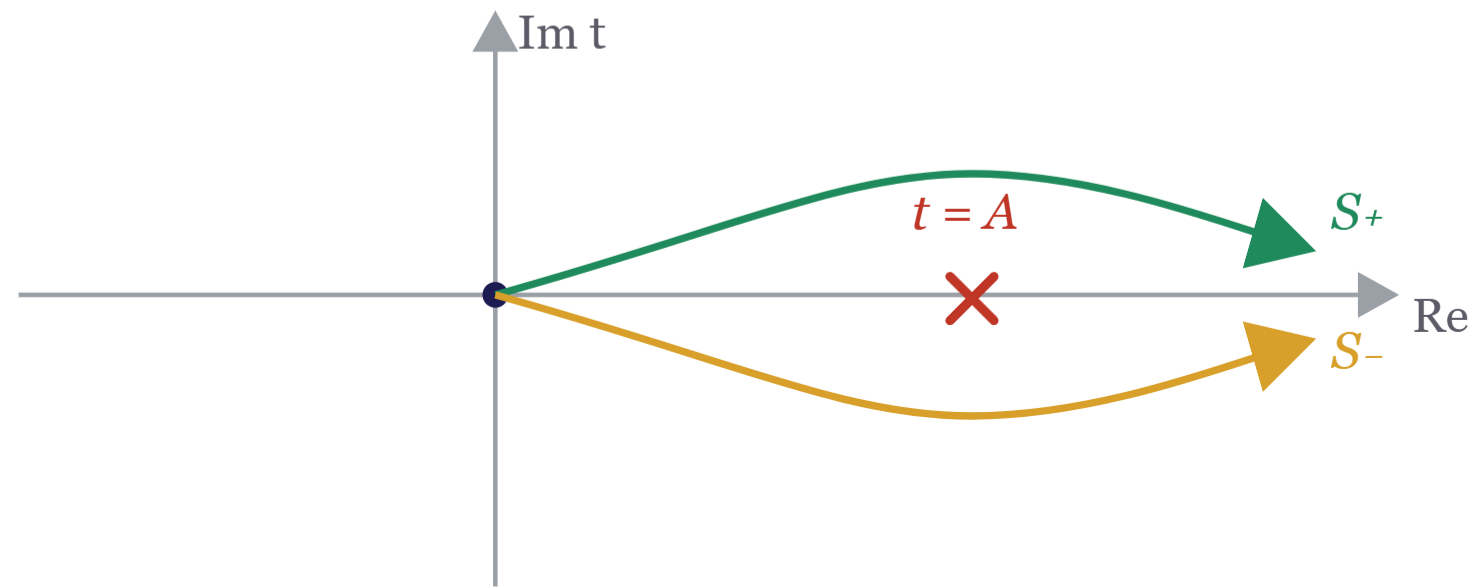
$$\begin{aligned} \mathcal{S}\Phi(\lambda) &= \int_0^\infty dt e^{-t} t^{b-1} \widehat{\mathcal{B}}(\lambda t) \\ &\stackrel{?}{=} \sum_{n \geq 0} \frac{a_n}{\Gamma(n+b)} \lambda^n \int_0^\infty dt e^{-t} t^{n+b-1} = \sum_{n \geq 0} a_n \lambda^n \end{aligned}$$

Same-sign coefficients $\implies$ a singularity on the positive real axis:

$$\widehat{\mathcal{B}}(t) \xrightarrow{t \rightarrow A} \frac{S}{2\pi A^b} \frac{1}{1 - t/A}$$

Lateral Borel resummation

The pole sits on the contour — deform it just above / below:



$$\mathcal{S}_+ \Phi(\lambda) - \mathcal{S}_- \Phi(\lambda) = -iS \frac{e^{-A/\lambda}}{\lambda^b}$$

$$\mathcal{S}_\pm \Phi(\lambda) = \mathcal{P}\Phi(\lambda) \mp \frac{i}{2} S \frac{e^{-A/\lambda}}{\lambda^b}$$

$$|\operatorname{Im} \mathcal{S}_\pm \Phi(\lambda)| = \frac{1}{2} S \frac{e^{-A/\lambda}}{\lambda^b}$$

Remark

The non-perturbative piece appears as an imaginary part — exactly $\operatorname{Im} F$, the decay rate.

Trans-series

The perturbative series alone is *incomplete*; its consistent non-perturbative completion is a trans-series.

$$\Phi(\lambda) = \sum_{k \geq 0} \sigma^k e^{-kA/\lambda} \Phi_k(\lambda)$$

[Écalle '81 · Dunne-Ünsal · Aniceto, Başar & Schiappa '19]

- σ — the trans-series parameter, labelling the non-perturbative sectors.
- $k = 0$ is the perturbative series; $k \geq 1$ are non-perturbative, weighted by $e^{-kA/\lambda}$.
- **Resurgence:** each sector's Borel data fix all the others — the $k = 1$ ambiguity cancels that of $k = 0$.

The Borel data are physical

$$\mathcal{S}_{\pm}\Phi(\lambda) = \mathcal{P}\Phi(\lambda) \mp \frac{i}{2} S \frac{e^{-A/\lambda}}{\lambda^b}$$

A

Euclidean saddle action

b

one-loop spectrum: number of zero modes

S

one-loop prefactor (Stokes constant)

All three follow from the coefficients alone — e.g. the singularity from a ratio:

$$n \frac{a_{n-1}}{a_n} \xrightarrow{n \rightarrow \infty} A$$

■ THE GOAL

RESURGENCE SO FAR

Mostly zero-temperature spectral problems in QM, field theory on S^1 with twisted boundary conditions — where fractional instantons and bions make the structure accessible semiclassically — and Stokes phenomena in the space of coupling constant.

A direct resurgent account of the thermal (**bounce-to-sphaleron**) transition remains largely unexplored.

Questions

Does the perturbative thermal free energy $F(\beta, \lambda)$ know about the thermal transition?

Can one read off T_c , and distinguish first- from second-order, from the large-order behaviour alone?

Two models

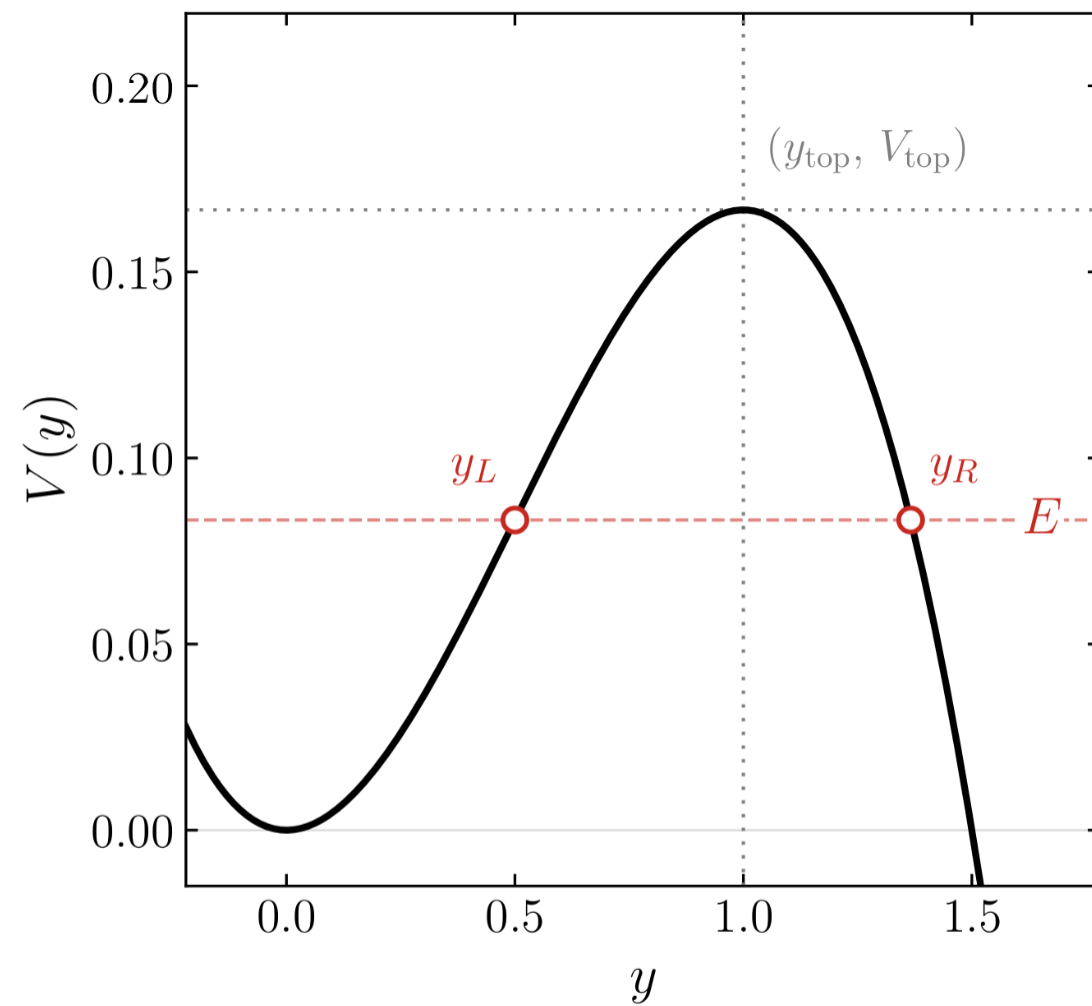
$$S_E = \int_0^\beta d\tau \left(\frac{1}{2} \dot{x}^2 + \frac{1}{g^2} V(gx) \right) \quad \lambda = g^2 = \hbar_{\text{eff}}$$

Cubic

second order · $T_c \approx 0.159$

$$V(y) = \frac{1}{2}y^2 - \frac{1}{3}y^3$$

Potential (cubic)

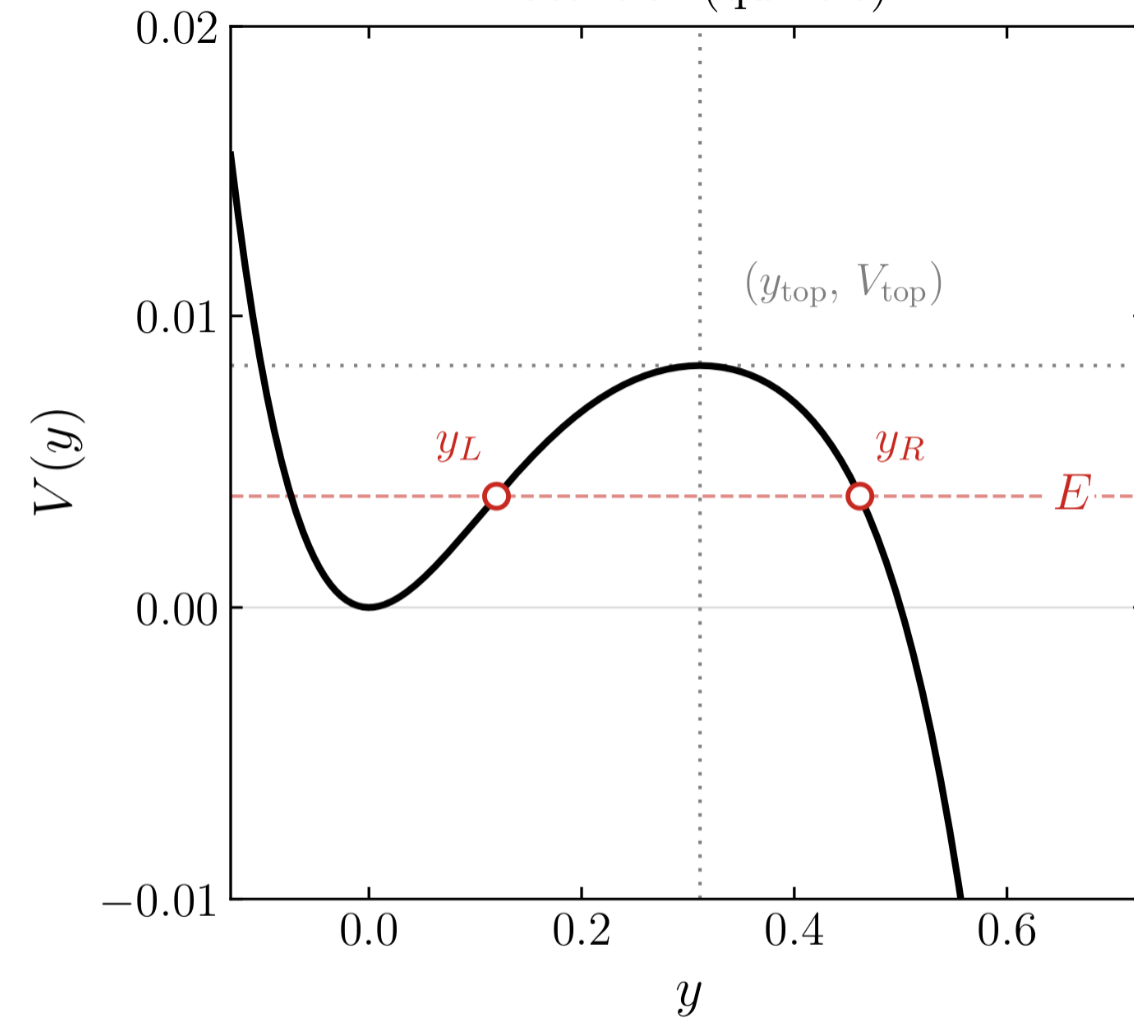


Quintic

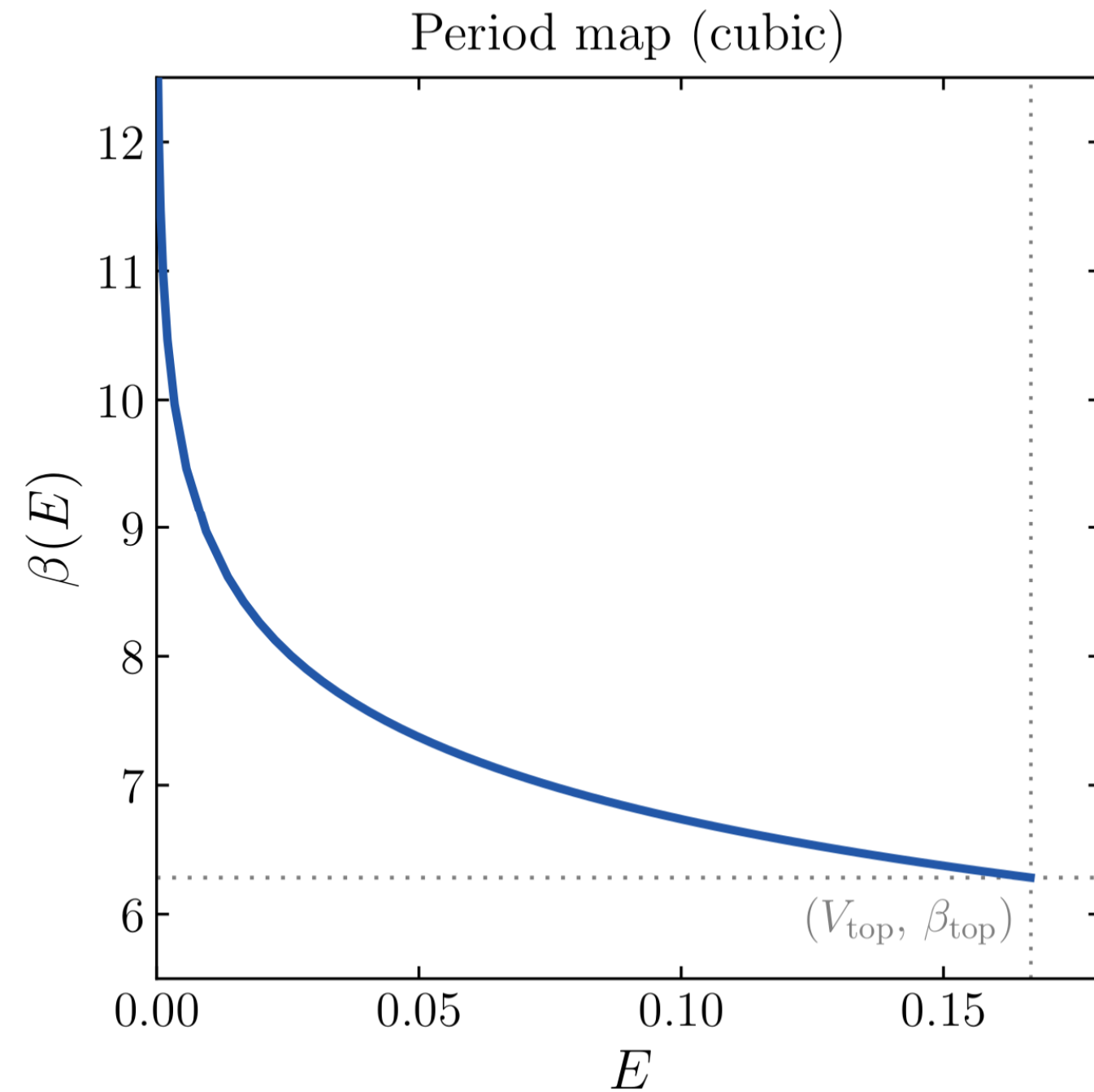
first order · $T_c \approx 0.0878$

$$V(y) = \frac{1}{2}y^2 - \frac{5}{2}y^3 + 5y^4 - 4y^5$$

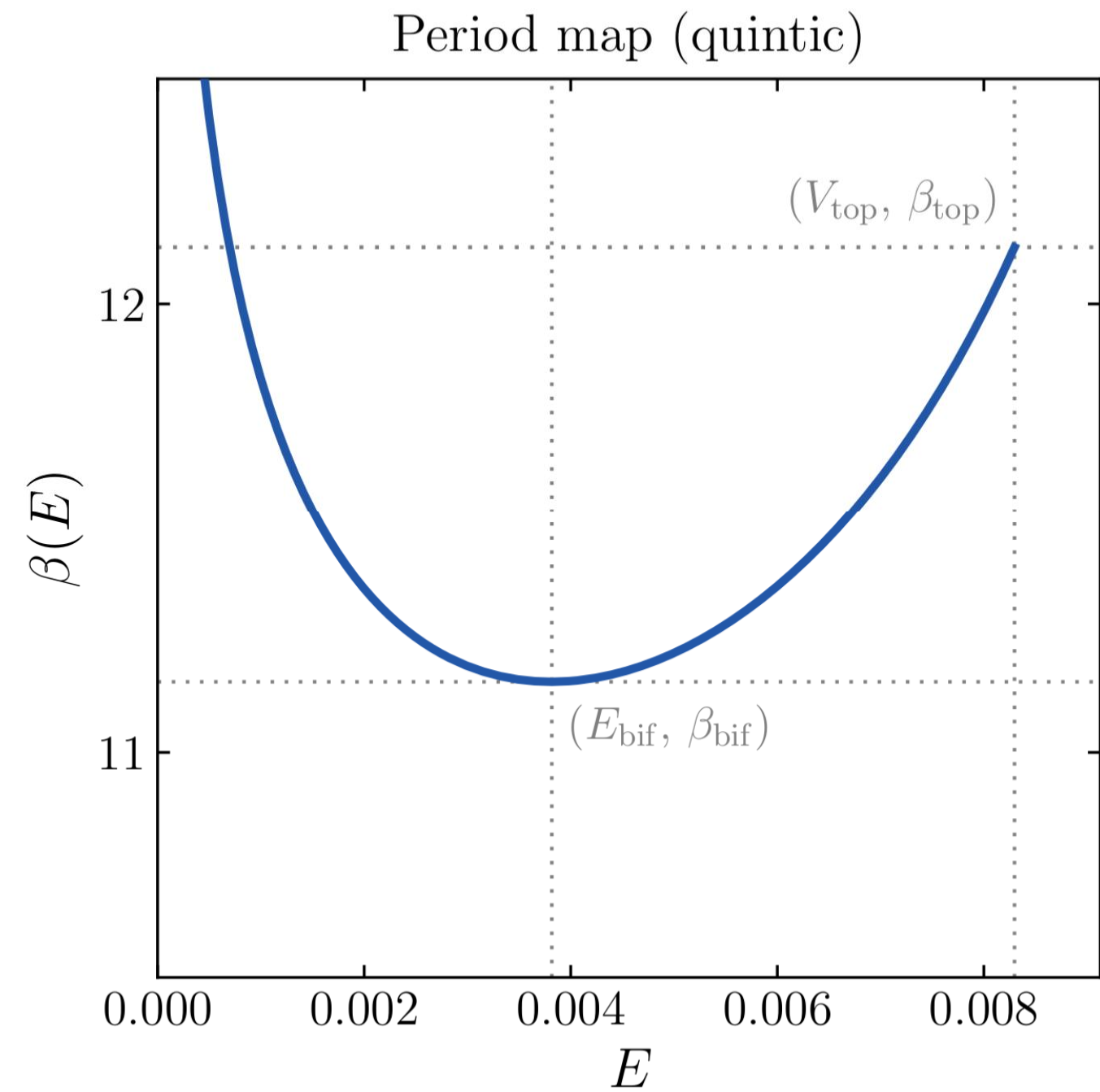
Potential (quintic)



Period maps — monotone vs fold



Monotone: bounce joins the sphaleron tangentially — second order.



Folded: two bounces coexist; the actions cross — first order (a kink).

Free energy around the false vacuum

Expand about the false vacuum (a harmonic well):

$$H = H_0 + H_I, \quad H_0 = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 x^2$$

$\lambda = g^2 = \hbar_{\text{eff}}$ – perturbation in g^2 is the semiclassical expansion.

Each level by perturbation theory (Bender–Wu):

$$E_\nu(\lambda) = \left(\nu + \frac{1}{2}\right) + \sum_{n \geq 1} \varepsilon_{\nu,n} \lambda^n$$

Assemble the thermal free energy as a power series in $\lambda = g^2$:

$$Z(\beta, \lambda) = \sum_{\nu} e^{-\beta E_\nu(\lambda)}$$

$$F(\beta, \lambda) = -\frac{1}{\beta} \log Z = \sum_n F_n(\beta) \lambda^n$$

The coefficients $F_n(\beta)$ – around the false vacuum – are all we feed to the large-order analysis.

Temperature-dependent Borel data

The observable is the perturbative thermal free energy $F(\beta, \lambda)$.

$$F_n(\beta) \sim -\frac{S(\beta)}{2\pi} \frac{\Gamma(n + b(\beta))}{A(\beta)^{n+b(\beta)}} \quad (n \rightarrow \infty)$$

- The coefficients $F_n(\beta)$ depend on temperature.
- The Borel data become functions of β : $A(\beta)$, $b(\beta)$, $S(\beta)$.

Transmutation of Borel data

Do the perturbative data **reorganize themselves across** T_c and detect the thermal transition ?

Extracting (A, b, S)



- **Borel-Padé** → location $A(\beta)$ (smallest stable positive pole).
- **Dlog-Padé** → exponent $b(\beta)$.
- **Residue of the Borel transform** → Stokes constant $S(\beta)$; compare $S/S_{\text{th}} \rightarrow 1$.

Production runs: median over diagonal orders $m = 100 \dots 125$ · $n_{\text{max}} = 250$ · $v_{\text{max}} = 100$

■ THE GOAL

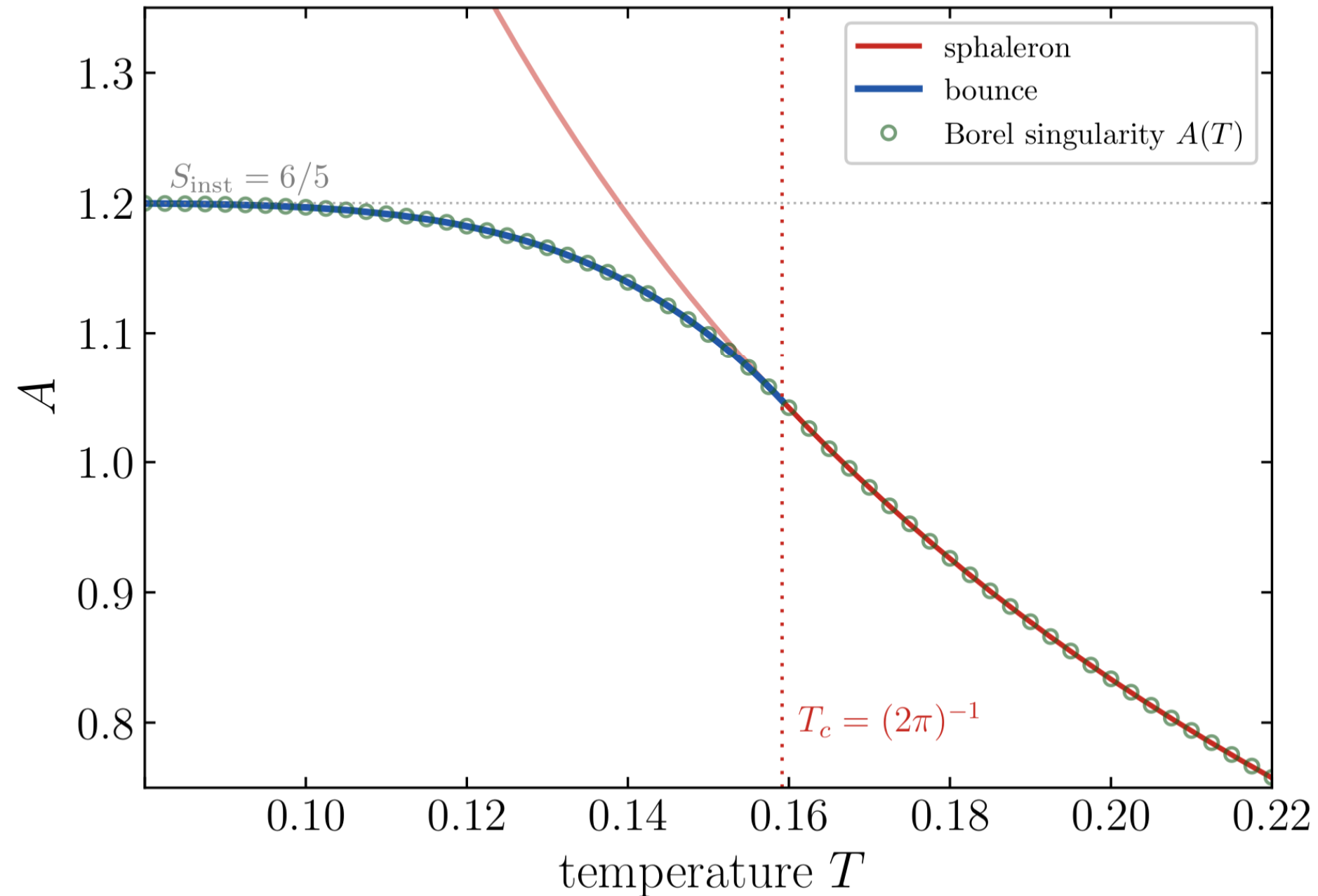
The question

From the large-order behaviour of $F_n(\beta)$ **alone**, can we determine —

- ? which saddle drives the decay at each temperature — bounce or sphaleron; $A(\beta) = \min\{ S_b(\beta), \beta V_{\text{top}} \}$
- ? the number of zero modes of the dominant saddle — the exponent b ; $b = \frac{1}{2}$ (bounce), $b = 0$ (sphaleron)
- ? the one-loop (Affleck) prefactor — the Stokes constant S ;
- ? the transition temperature T_c , and the order of the thermal transition?

Our goal — check each of these **numerically** from $F_n(\beta)$.

A smooth, second-order transition



- $A(T)$ follows the bounce action S_b for $T < T_c$, the sphaleron βV_{top} for $T > T_c$.
- A smooth, tangential join at $T_c = (2\pi)^{-1} \approx 0.159$.
- Agreement $\Delta \lesssim 0.005\%$ – no semiclassical input.

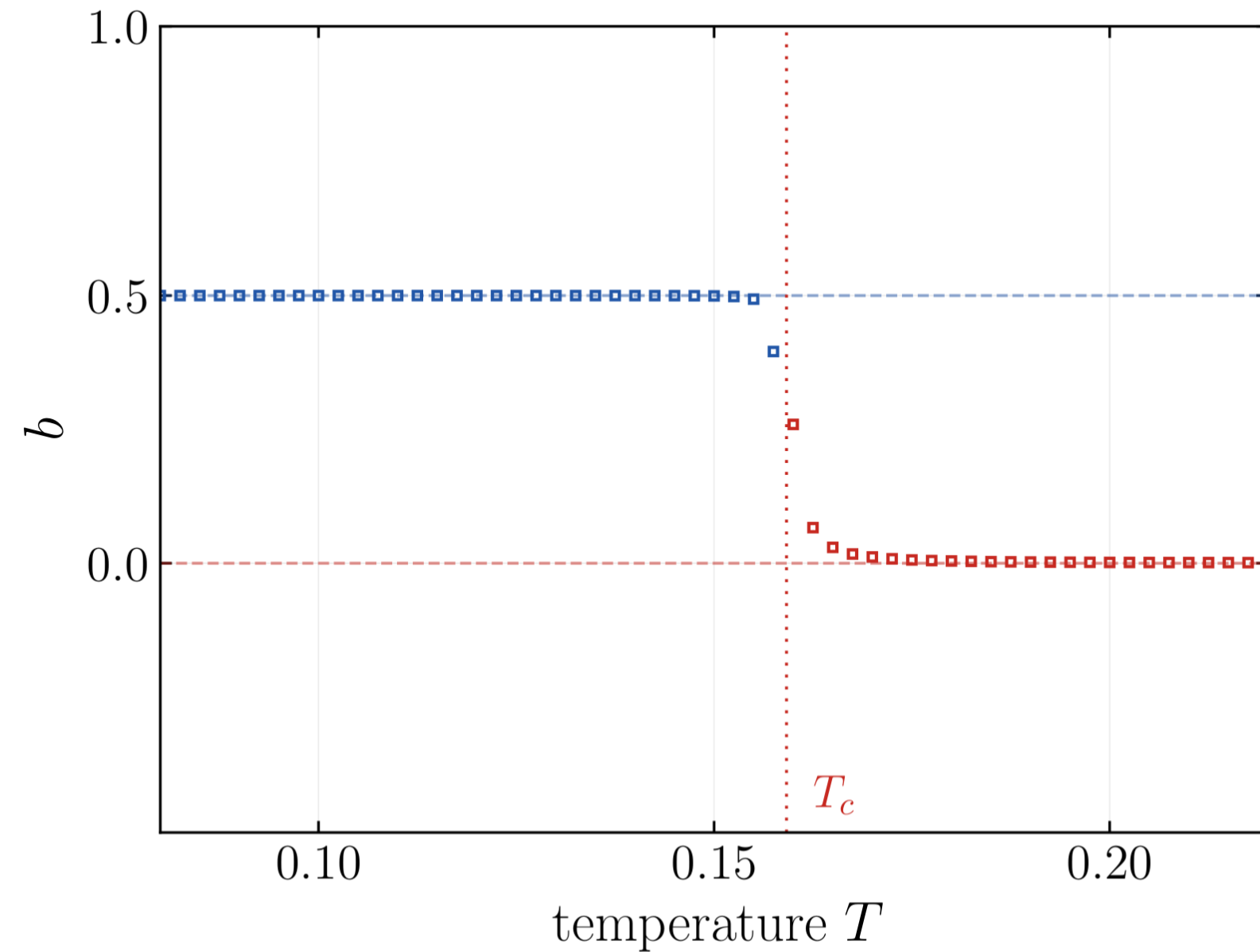
■ RESULTS • CUBIC MODEL

BOUNCE • COLD $T < T_c$					
T	A_{th}	A	Δ %	b	S/S_{th}
0.0800	1.199731	1.199791	0.0050	0.500	1.000
0.0850	1.199439	1.199499	0.0050	0.500	1.000
0.0900	1.198921	1.198983	0.0052	0.500	1.000
0.0950	1.198060	1.198121	0.0051	0.500	1.000
0.1000	1.196707	1.196765	0.0049	0.500	1.000
0.1050	1.194680	1.194738	0.0049	0.500	1.000
0.1100	1.191761	1.191819	0.0048	0.500	1.000
0.1150	1.187698	1.187757	0.0049	0.500	1.000
0.1200	1.182204	1.182264	0.0050	0.500	1.000
0.1250	1.174961	1.175017	0.0048	0.500	1.000
0.1300	1.165614	1.165679	0.0056	0.500	1.000
0.1350	1.153779	1.153846	0.0058	0.500	1.000
0.1400	1.139039	1.139112	0.0064	0.500	1.000
0.1450	1.120945	1.121024	0.0070	0.500	1.000
0.1500	1.099016	1.099094	0.0071	0.499	1.000
0.1550	1.072736	1.072817	0.0075	0.493	1.000
0.1575	1.057796	1.057873	0.0073	0.396	0.978

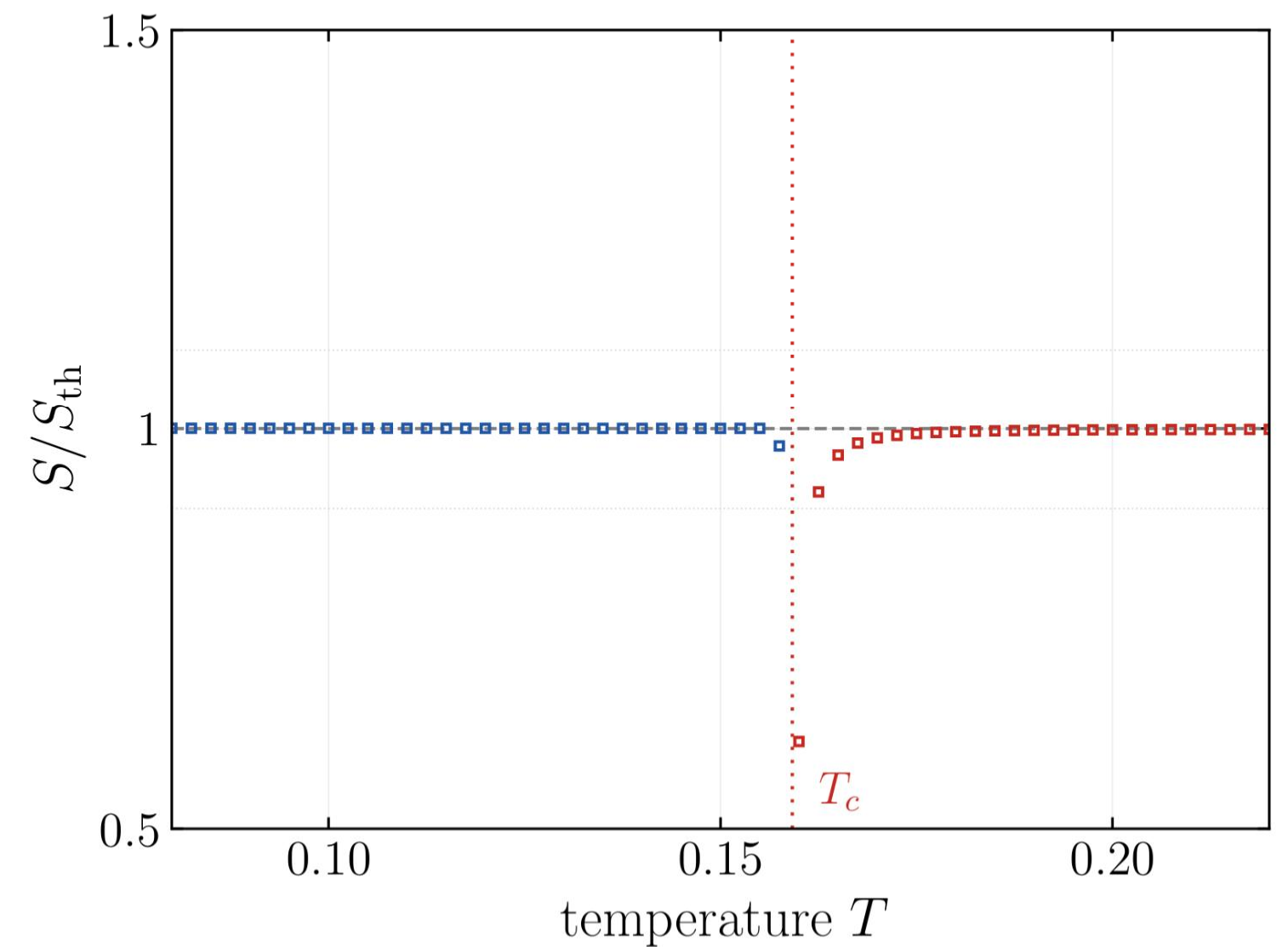
SPHALERON • HOT $T > T_c = (2\pi)^{-1}$					
T	A_{th}	A	Δ %	b	S/S_{th}
0.1600	1.041667	1.041807	0.0135	0.258	0.609
0.1650	1.010101	1.010301	0.0198	0.030	0.967
0.1700	0.980392	0.980606	0.0218	0.012	0.988
0.1750	0.952381	0.952614	0.0245	0.006	0.994
0.1800	0.925926	0.926155	0.0248	0.004	0.996
0.1850	0.900901	0.901144	0.0270	0.003	0.997
0.1900	0.877193	0.877455	0.0299	0.002	0.998
0.1950	0.854701	0.854972	0.0318	0.002	0.998
0.2000	0.833333	0.833624	0.0349	0.002	0.998
0.2050	0.813008	0.813309	0.0370	0.001	0.998
0.2100	0.793651	0.793959	0.0388	0.001	0.999
0.2150	0.775194	0.775518	0.0418	0.001	0.999
0.2200	0.757576	0.757924	0.0460	0.001	0.999
0.2250	0.740741	0.741097	0.0481	0.001	0.999
0.2300	0.724638	0.725013	0.0518	0.001	0.999

$\Delta = |A - A_{th}| / A_{th}$ (%); A, b, S are Borel–Padé values from the perturbative series alone. Every second grid point shown.

Exponent and Stokes ratio

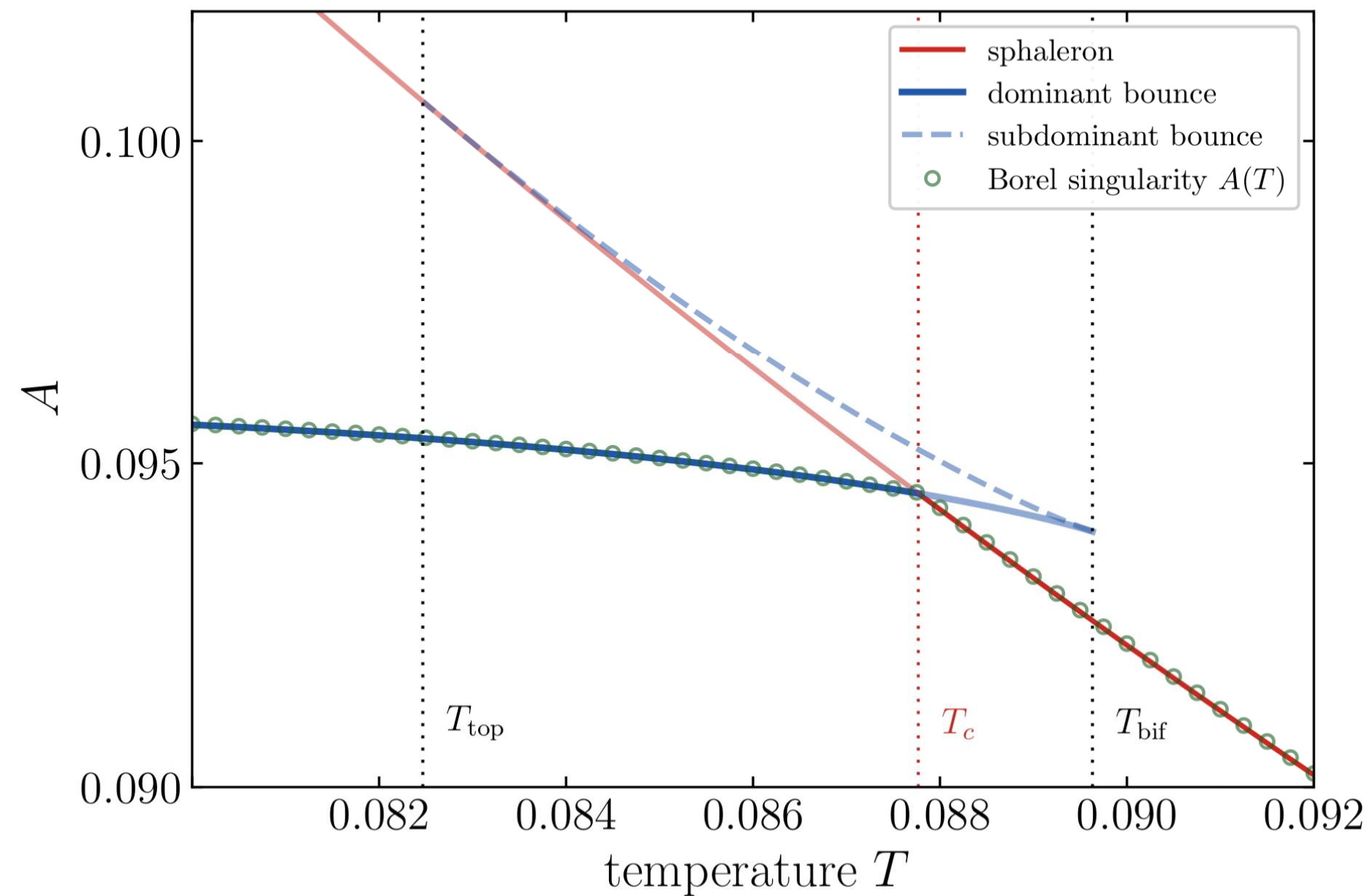


$b(T)$: $\frac{1}{2}$ (bounce) $\leftrightarrow$ 0 (sphaleron) — the zero-mode count.



$S/S_{\text{th}} \approx 1$ away from T_c — the Affleck prefactor reproduced.

A kink — first-order transition



- Same extraction, different shape: $A(T)$ develops a kink at $T_c \approx 0.0878$.
 - Two bounce branches (solid / dashed) — a transversal crossing, not tangent.
 - Agreement $\Delta \lesssim 0.03\%$ on each branch.
- *The kink is the fingerprint of a first-order transition on the Borel plane.*

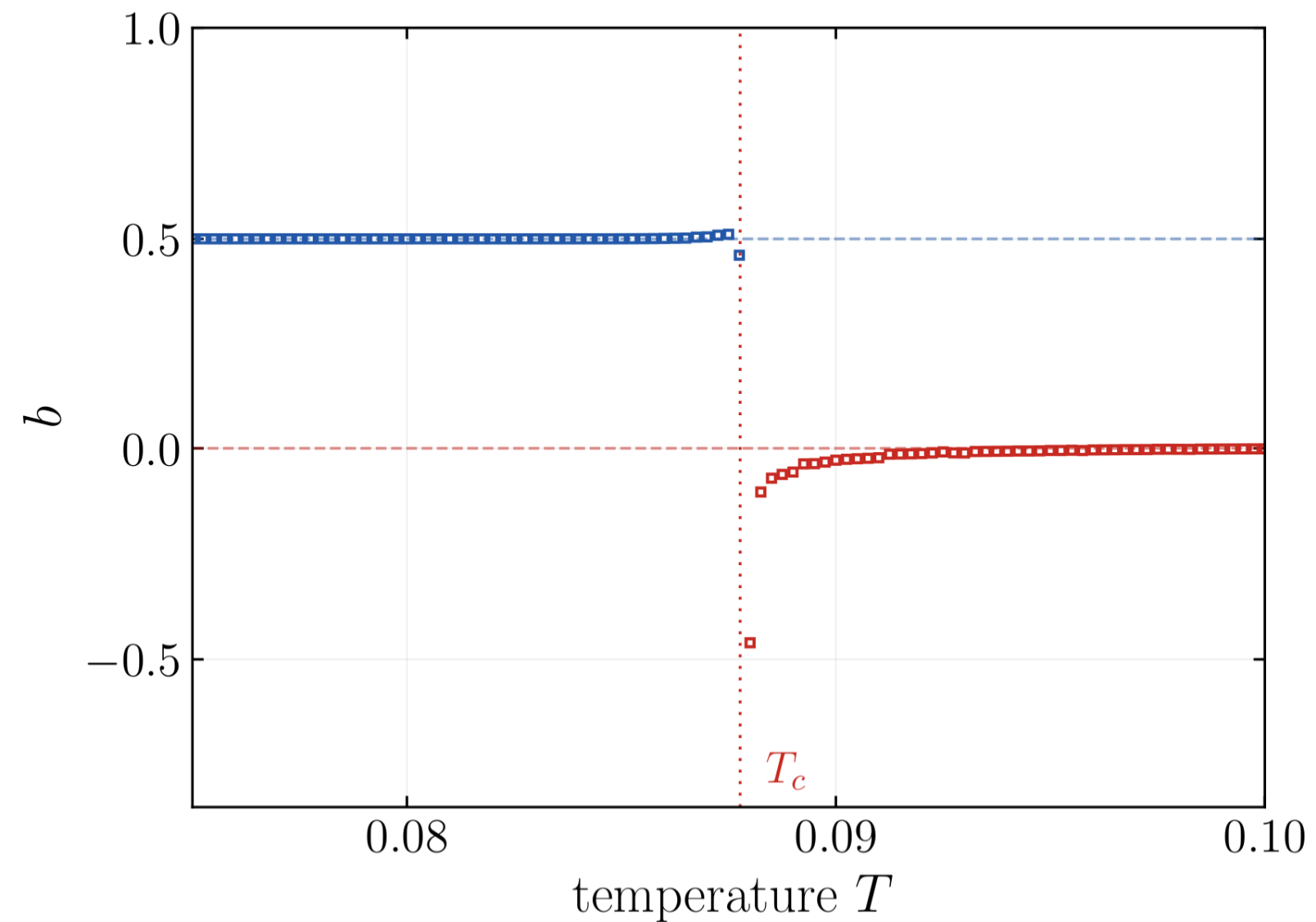
■ RESULTS · QUINTIC MODEL

BOUNCE · COLD $T < T_c$					
T	A_{th}	A	Δ %	b	S/S_{th}
0.07500	0.095845	0.095858	0.0131	+0.500	0.999
0.07600	0.095809	0.095823	0.0146	+0.500	0.999
0.07700	0.095768	0.095781	0.0134	+0.500	0.999
0.07800	0.095719	0.095732	0.0132	+0.500	0.999
0.07900	0.095663	0.095674	0.0120	+0.500	0.999
0.08000	0.095597	0.095610	0.0131	+0.500	0.999
0.08100	0.095521	0.095533	0.0133	+0.500	1.000
0.08200	0.095432	0.095444	0.0127	+0.500	1.000
0.08300	0.095329	0.095341	0.0132	+0.500	1.000
0.08400	0.095209	0.095220	0.0119	+0.500	1.001
0.08500	0.095069	0.095081	0.0129	+0.500	1.002
0.08600	0.094905	0.094917	0.0130	+0.501	1.004
0.08700	0.094710	0.094722	0.0125	+0.505	1.007
0.08775	0.094539	0.094552	0.0132	+0.461	1.057

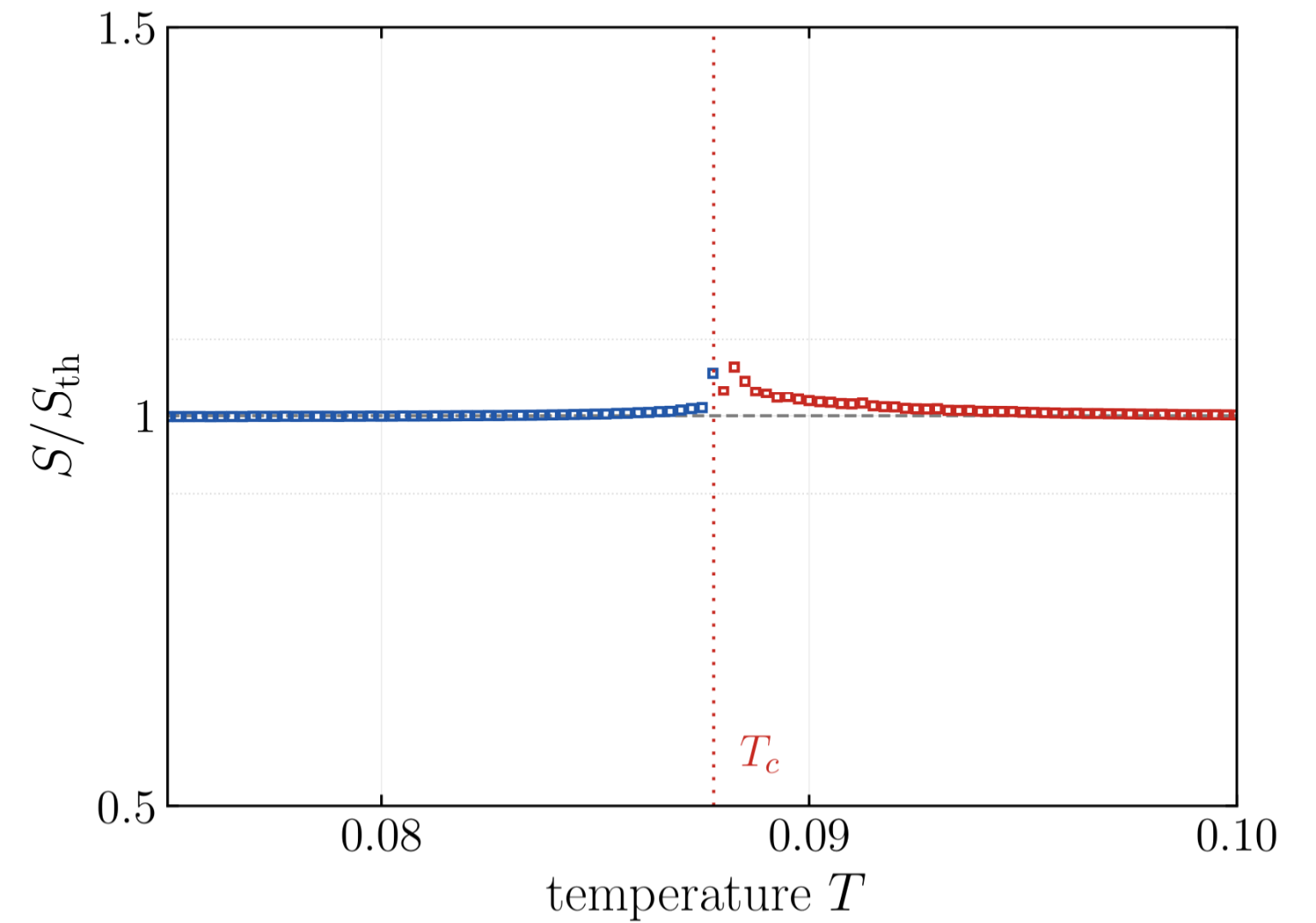
SPHALERON · HOT $T > T_c \simeq 0.0878$					
T	A_{th}	A	Δ %	b	S/S_{th}
0.08800	0.094286	0.094312	0.0280	-0.461	1.033
0.08900	0.093226	0.093252	0.0277	-0.056	1.030
0.09000	0.092190	0.092216	0.0275	-0.028	1.019
0.09100	0.091177	0.091203	0.0282	-0.022	1.015
0.09200	0.090186	0.090211	0.0278	-0.013	1.011
0.09300	0.089217	0.089240	0.0268	-0.011	1.009
0.09400	0.088267	0.088292	0.0280	-0.007	1.006
0.09500	0.087338	0.087361	0.0264	-0.005	1.005
0.09600	0.086429	0.086452	0.0267	-0.004	1.003
0.09700	0.085538	0.085560	0.0259	-0.003	1.003
0.09800	0.084665	0.084687	0.0259	-0.002	1.002
0.09900	0.083809	0.083830	0.0246	-0.002	1.001
0.10000	0.082971	0.082992	0.0252	-0.001	1.001

Same extraction as the cubic case; b changes sign across T_c . Deviations concentrate at T_c , where the two leading singularities are near-degenerate. Every fourth grid point shown.

Exponent and Stokes ratio



$b(T)$: again $\frac{1}{2} \leftrightarrow 0$, departing only across T_c .



$S/S_{th} \approx 1$; delicate where two singularities near-degenerate.

■ SUMMARY

What perturbation theory knows

BOREL DATA

SEMICLASSICAL MEANING

$A(T)$

dominant saddle action

$b(T)$

zero-mode structure ($\frac{1}{2} \leftrightarrow 0$)

$S(T)$

one-loop (Affleck) prefactor

Recovered from large orders around the false vacuum — **no semiclassical input:**

- the transition temperature T_c ;
- the order of the transition;
- the dominant saddle action;
- the zero-mode & one-loop data.

Caveat — extraction is delicate only in a narrow window around T_c , where two Borel singularities are near-degenerate.

■ OUTLOOK

Outlook

- 01 **Exact WKB** — identify $\mathcal{A}$ with periods on the spectral curve; what are the fold and the first-order transition there?
- 02 **More degrees of freedom** — coupled oscillators, matrix models, quantum field theory (regularized on fuzzy sphere?).
- 03 **Coleman–De Luccia / Hawking–Moss** — thermal and gravitational settings (mini-superspace?).
- 04 **The payoff** — recover hard one-loop determinants from high-order perturbation theory.

If perturbation theory can recover one-loop saddle data, it opens a new route to problems where determinants are hard.