# Unoriented

PART I: HOLOGRAPHY - **WIP with J. Liu & I. Melnikov**

PART II: COMPACTIFICATIONS - **WIP with I. Melnikov**

## M-theory on non-orientable spaces and $m_c$ structures

$\mathrm{Pin}^+$ structure $\quad\Rightarrow\quad$ suitable tangential structure needed for gravitino

$m_\mathrm{c}$ structure on a $\mathrm{Pin}^+$ manifold - choice of *twisted* integer lift of $w_4(TM) \in H^4(M;\mathbb{Z}_2)$

- twisted integer cohomology class - element of $H^\bullet(M;\widetilde{\mathbb{Z}})$ (or $H^\bullet(M;\widetilde{\mathbb{R}})$)
- $\widetilde{\mathbb{Z}}$ - constant sheaf of integers on $M$ twisted by the orientation bundle of $M$
- $\mathrm{Pin}^+$ manifold $M$ admits $m_\mathrm{c}$ structure iff $\tilde{\beta} w_4(TM) = 0$,
  - ⋆ $\tilde{\beta}$ - Bockstein homomorphism $H^4(M;\mathbb{Z}_2) \to H^5(M;\tilde{\mathbb{Z}})$ associated to short exact sequence $\quad 0 \to \widetilde{\mathbb{Z}} \xrightarrow{2} \widetilde{\mathbb{Z}} \to \mathbb{Z}_2 \to 0$
- $G_4$ - closed twisted 4-form on $M_{11}$ $\qquad$ ($G_4 \in \widetilde{\Omega}^4_d(M_{11})$ and $[G_4]_{\mathrm{dR}} \in H^4(M_{11};\widetilde{\mathbb{R}})$)
  - ◇ $\exists\, a_4$ s.t. $\quad 2[G_4]_{\mathrm{dR}} = \widetilde{\varrho}(a_4)$
  - ◇ $\widetilde{\varrho} : H^\bullet(M_{11};\widetilde{\mathbb{Z}}) \to H^\bullet(M_{11};\widetilde{\mathbb{R}})$ - natural map induced by $\widetilde{\mathbb{Z}} \to \widetilde{\mathbb{R}}$
  - ◇ $a_4 \in H^4(M_{11};\widetilde{\mathbb{Z}})$ - twisted integral class - twisted lift of $w_4(TM_{11})$
- $$\int_{\mathcal{C}_4} G_4 = \int_{\mathcal{C}_4} \tfrac{1}{2} a_4 \overset{\text{mod } \mathbb{Z}}{=} \tfrac{1}{2}\int_{\mathcal{C}_4} w_4(TM_{11})$$
  - ◇ Spin $M_{11}$: $\quad$ familiar $\int_{\mathcal{C}_4} G_4 \overset{\text{mod } \mathbb{Z}}{=} \frac{1}{2}\int_{\mathcal{C}_4} \lambda(TM_{11})$ (for $2\lambda(TM_{11}) = p_1(TM_{11})$)

## M-theory (limited view)

- Top. couplings:

$$\mathcal{I}_{12} = -\frac{1}{6} G_4 \wedge G_4 \wedge G_4 - G_4 \wedge X_8 \ ,$$

## Consistency of M-theory and action and cubic form for $\mathrm{Pin}^+$ spaces

Half-integer flux quantisation

- $G_4 = a_4 - \frac{1}{2} v_4$
  - ◇ $a_4$ - closed 4-form with integral periods
- Consistent action for M-theory:

$$\int_{M_{12}} \left\{ \left[ -\frac{1}{6} \left( a_4 - \frac{1}{2} v_4 \right)^3 - \left( a_4 - \frac{1}{2} v_4 \right) X_8 \right] - (a_4 \to 0) \right\}$$

  - ◇ $v_4$ - twisted integral lift of $w_4$ that specifies the $m_{\rm c}$ structure
  - ◇ In orientable setting $v_4 \longrightarrow \lambda$ (provides canonical integral lift of $w_4$)
  - ◇ In $\mathrm{Pin}^+$setting - integer-valued cubic form: $k_M(C) = \frac{1}{6} C^3 + C\, X_8 + \frac{1}{2} RS'(M)$
- Phase factor $\Leftrightarrow$ ambiguity in the Pfaffian of Rarita-Schwinger operator

## PART I: HOLOGRAPHY

*AdS/CFT:* $AdS_{d+1} \times X^{D-d-1} \Leftrightarrow CFT_d$ (for $D = 10, 11$ (M-theory/IIB))

### Max susy:

- Near horizon of supersymmetric branes $\Leftrightarrow$ type A
  - ◇ M5 in 11D: $AdS_7 \times S^4 \Leftrightarrow$ d=6 (2,0) theory
  - ◇ D3 in IIB: $AdS_5 \times S^5 \Leftrightarrow$ d=4 max. super-YM
  - ◇ SUSY: Killing spinors on spheres
- D-type theories
  - ◇ ADE classification d=6 (2,0) theories - purely geometric holographic realization for type D expected
  - ◇ allowed by $AdS/CFT$ dictionary
  - ◇ $\mathcal{N} = 4$ SYM $\Leftrightarrow$ $AdS_5 \times \mathbb{RP}^5$ Less supersymmetry - ?
- Question:
  - ◇ D-type (B / C) theories $\Leftrightarrow$ **new/what** geometry/topology (*beyond holography*)?

## $AdS_7 \times \mathbb{RP}^4$ is maximally supersymmetriic

- $S^n$: $\delta_{AB} y^A y^B = 1 \qquad (A, B = 1, ..., n+1)$
- Stereographic coordinates $x^i$ $(i = 1, \ldots, n)$: $y^i = \frac{2x^i}{1+x\cdot x}$ and $y^{n+1} = \frac{x\cdot x - 1}{1+x\cdot x}$
- Antipodal involution $\sigma$: $\sigma : y^A \mapsto -y^A$ and $\sigma : x^i \mapsto -\frac{x^i}{x\cdot x}$
- Action of $\sigma$ on sections of the spinor bundle on $S^n$ $\psi(x) \quad \Rightarrow$

$$
\begin{aligned}
n &= 0 \mod 4 : & w_1(T\mathbb{RP}^n) &= \alpha\,, & w_2(T\mathbb{RP}^n) &= 0\,, & &(\text{Pin}^+)\\
n &= 1 \mod 4 : & w_1(T\mathbb{RP}^n) &= 0\,, & w_2(T\mathbb{RP}^n) &= \alpha^2\,, & &(\text{Spin}^c)\\
n &= 2 \mod 4 : & w_1(T\mathbb{RP}^n) &= \alpha\,, & w_2(T\mathbb{RP}^n) &= \alpha^2\,, & &(\text{Pin}^-)\\
n &= 3 \mod 4 : & w_1(T\mathbb{RP}^n) &= 0\,, & w_2(T\mathbb{RP}^n) &= 0\,. & &(\text{Spin})
\end{aligned}
$$

$\alpha$ - generator of $H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (Spin, Pin$^\pm$) $\quad (H^2(\mathbb{RP}^n; \mathbb{Z}) \cong \mathbb{Z}_2$ for $n \geq 2$ (Spin$^c$))

- $\psi(x)$ descends to a section of Pin$^+$ bundle on $\mathbb{RP}^4$ if $(\sigma\psi)(x) = \pm\psi(x)$
- Killing spinors on $S^n$ $\quad (D_i\epsilon = \frac{i}{2}\gamma_i\epsilon)$ : $\epsilon(x) = \frac{1}{\sqrt{1+x\cdot x}}(\mathbb{I} + ix\cdot\gamma)\epsilon_0$

$(\sigma\epsilon)(x) = \epsilon(x) \qquad \Rightarrow \qquad$ Killing $(\mathrm{Pin}^+)$-pinors on $\mathbb{RP}^4$

$M_{11} = M_7 \times \mathbb{RP}^4$ ($M_7$ - orientable and Spin)

$$H^\bullet(\mathbb{RP}^4;\mathbb{Z}) = \{\mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2\}$$

$$H^\bullet(\mathbb{RP}^4;\widetilde{\mathbb{Z}}) = \{0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \mathbb{Z}\}$$

$$H^\bullet(\mathbb{RP}^4;\mathbb{Z}_2) = \{\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2\}$$

- $t_1$ - generator of $H^1(\mathbb{RP}^4;\widetilde{\mathbb{Z}})$
- $t_2 := t_1^2,\ t_3 := t_1^3\ , t_4 := t_1^4$ - generators of $H^2(\mathbb{RP}^4;\mathbb{Z})$, $H^3(\mathbb{RP}^4;\widetilde{\mathbb{Z}})$, $H^4(\mathbb{RP}^4;\mathbb{Z})$
- Poincaré duality exchanges twisted and untwisted coefficients):

$$H_p(\mathbb{RP}^4;\mathbb{Z}) \cong H^{4-p}(\mathbb{RP}^4;\widetilde{\mathbb{Z}})\ , \quad H_p(\mathbb{RP}^4;\widetilde{\mathbb{Z}}) \cong H^{4-p}(\mathbb{RP}^4;\mathbb{Z})\ , \quad p = 0, \ldots, 4$$

- $v_4$ - generator of $H^4(\mathbb{RP}^4;\widetilde{\mathbb{Z}})$ ($\int_{\mathbb{RP}^4} v_4 = 1$)
- $v_4$ - twisted integral lift of $w_4(T\mathbb{RP}^4)$ (fixing $m_c$ structure) : $v_4 = w_4(T\mathbb{RP}^4) \mod 2$
- total SW class: $w(T\mathbb{RP}^4) = (1+\alpha)^5 = 1 + \alpha + \alpha^4$ ($\alpha \in H^1(\mathbb{RP}^4;\mathbb{Z}_2) \cong \mathbb{Z}_2$)
- background flux: $G_4 = \left(Q - \frac{1}{2}\right) v_4$

$\Rightarrow$ 6D $(2,0)$ SymTFTs:

- For $A_{Q-1}$ type: $S_{\text{7d TQFT}} = -2\pi \frac{N}{2} \int_{M_7} c_3 \wedge dc_3$
- For $D_N$ type $(c_3 \mapsto \mathsf{c}_3/2...)$:

$$S_{\text{7d TQFT}} = 2\pi \int_{M_7} \left[ -\frac{N}{8} \mathsf{c}_3 \cup \delta \mathsf{c}_3 + \frac{1}{2} \mathsf{c}_3 \cup \delta \mathsf{b}_3 + \frac{1}{2} \mathsf{a}_1 \cup \delta \mathsf{a}_5 + \frac{1}{4} \mathsf{a}_1 \cup \mathsf{c}_3 \cup \mathsf{c}_3 \right] + (?)$$

- The SymTFT fields are not in the $AdS$ spectrum

$\Rightarrow$ 6D $(2,0)$ Anomalies: ( $I_6^{(1)} = d^{-1} \delta d^{-1} I_8$ )

- $A_{Q-1}$ : $I^{(2,0)}_{A_{Q-1}} = (Q-1) I_{\text{M5}}(Q=1) + \frac{Q^3-Q}{24} p_2(\mathcal{N})$ ($\mathcal{N}$ - SO(5) bundle)
  - ◇ removing one free (2,0) (COM multiplet) anomaly

$$I_{\text{M5}}(Q=1) = \frac{1}{48}\left(\frac{1}{4}(p_1(TW)^2 + p_1(\mathcal{N})^2 - 2p_1(TW)p_1(\mathcal{N})) - p_2(TW) + p_2(\mathcal{N})\right)$$

- $D_N$ : $I^{(2,0)}_{D_N} = N I_{\text{M5}}(Q=1) + \frac{N(2N-1)(2N-2)}{24}$
- $I^{(2,0)}_G = r_G I_{\text{M5}}(Q=1) + \frac{d_G \cdot h^\vee_G}{24} p_2(\mathcal{N})$
  - ◇ $r_G, d_G, h^\vee_G$ - rank, dimension, dual Coxeter

## Smooth holographic duals to type D $\mathcal{N}=1$ SYM

Candidate (quotient) geometries

- IIB: Spin$^c$ 5-manifolds $M_5$ : $\mathcal{C}(M_5)$ - Kähler
- M-theory: Pin$^+$ 6-manifolds
  - ◇ GMSW geometries (for type A)

$$\begin{cases} \text{Mink}_4 \times_w \mathcal{C}(M_6) \quad \Rightarrow \quad AdS_5 \times M_6 \\ M_6 - \text{compact complex} - S^2 \text{ bundle over Kähler } K_4 \end{cases}$$

◇ Two cases for $K_4$:

- ○ $K_4$ - Einstein-Kähler (full classification)
- ○ $K_4 \sim S^2 \times \mathbb{T}^2$
  - ◎ can be reduced to IIA and T-dualised
  - ◎ infinite family of SE 5-dim geometries: $Y^{p,q}$
- ○ Killing spinor: $\xi = e^{i\psi/2} e^{-\frac{i}{2}\zeta\gamma^6}(1+\gamma^5)\hat{\xi}(K_4)$

⊕ $\xi$ can be made invariant under orientation reversing free involution !

- ◎ All case (i) solutions, restrictions for case (ii)

## M-theory with an involution (case(i))

$$(\mathrm{d}s^2)_{11} = e^{2\lambda}\left(\mathrm{d}s^2(\mathsf{AdS}_5) + \mathrm{d}s^2(M_6)\right)$$

$$\mathrm{d}s^2(M_6) = \frac{1}{2}e^{-6\lambda}(1-y^2)\mathrm{d}s^2(K_4) + e^{-6\lambda}\sec^2\zeta\mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta(\mathrm{d}\psi + P)^2$$

$$G_{(4)} = \partial_y(e^{6\lambda})E^1\wedge E^2\wedge E^3\wedge E^4 + \frac{3m}{2}e^{3\lambda}(2y\partial_y\lambda - 1)J_{ab}E^a\wedge E^b\wedge E^5\wedge E^6$$

- $E^a = \frac{1}{\sqrt{3}}e^{-3\lambda}\sqrt{1-y^2}e^a, \qquad E^5 = e^{-3\lambda}\sec\zeta\mathrm{d}y, \qquad E^6 = \frac{1}{3m}\cos\zeta(\mathrm{d}\psi + P)$
- $S^2$ $(y,\psi)$: $\psi\sim\psi + 2\pi$ and $y_1\le y\le y_2$ - real roots of a quartic
- $\lambda$ and $\cos^2\zeta$ can be chosen to be invariant under $y\to -y$
- Killing spinor: $\xi = e^{i\psi/2}e^{-\frac{i}{2}\zeta\gamma^6}(1+\gamma^5)\hat{\xi}(K_4)$
- free involution $f$: $f(y,\psi,p) = (-y, \psi\mp\pi, F(p))$ $\rightarrow$ $\begin{cases} f^*G_{(4)} = -G_{(4)} \\ f^*\xi(M_6) = \xi(M_6)\end{cases}$
- Kähler infolution $F$: $F^*\begin{pmatrix}\boldsymbol{e}^1\\ \boldsymbol{e}^2\end{pmatrix} = U_F\begin{pmatrix}\boldsymbol{e}^1\\ \boldsymbol{e}^2\end{pmatrix}$ s.t. $\det U_F = -1$
- Such $F$ can be shown to exist for all KE 4-manifolds
- Minimal susy $AdS_5\times M_6/\mathbb{Z}_2 \leftarrow$ unorienable

case (ii): $M_6$ has a $\mathbb{T}^2$ factor and can be reduced to IIA and T-dualised to IIB:

- $AdS_5 \times Y^{p,q}$, rational parameter in M-theory solution $c = p/q$
- $p$ and $q$ - coprime - infinite family of SE metrics on $S^2 \times S^3$
- $c = 0 \quad \Rightarrow \quad T^{1,1}$ space $\sim S^2 \times S^3$

$$\mathrm{d}s^2(T^{1,1}) = \frac{\ell^2}{6}\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\varphi_1^2\right) + \frac{\ell^2}{6}\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\varphi_2^2\right) + \frac{\ell^2}{9}\left(\mathrm{d}\psi - \cos\theta_1 \mathrm{d}\varphi_1 - \cos\theta_2 \mathrm{d}\varphi_2\right)^2$$

▷ Conifold $\mathcal{C}$ - a non-compact Calabi-Yau variety with a Ricci–flat conical Kähler metric and a non-degenerate holomorphic $3$–form defined on $\mathcal{C} \setminus \{0\}$

▷ homogeneous space $T^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\,\mathrm{U}(1)$ - base for the conical metric of $\mathcal{C}$

- $\mathcal{C} = \mathbb{C}^4//\,\mathrm{U}(1)$ - toric variety with coordinates $(z_1, z_2, z_3, z_4)$
- equiv. relation: $(z_1, z_2, z_3, z_4) \sim (e^{\mathrm{i}\lambda} z_1, e^{\mathrm{i}\lambda} z_2, e^{-\mathrm{i}\lambda} z_3, e^{-\mathrm{i}\lambda} z_4)$
- moment map: $\mu : \mathbb{C}^4 \to \mathbb{R}\,, \qquad \mu = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2$

case (ii): M-theory solution with $c = 0$

- orientation and $G_4$ reversing “universal" invlolution $y \to -y$ (+ more)
- in IIA: $T^{1,1}/\langle \theta_2 \to \pi - \theta_2, \quad \psi \to \psi + \pi, \quad \varphi_1 \to \varphi_1 + \pi, \ \rangle$
  - $y = \cos\theta_2$
  - NOT action by isometry singular space
  - appearance of worldsheet parity $\Pi$

1. Worldsheet parity $\Pi$ (sign of D-series)
   - Reflection symmetry of M-theory circle $\Leftrightarrow$ $(-1)^{F_L}$
   - GSO $\Rightarrow$ reflection in 10D accompanied by $\Pi$
   - $\Pi$ stays after T-diality
2. Singularity without planes/branes
   - subtleties of T-duality on quotients

## **T-duality** digression

Coord-independent $O(n,n)$ transformation (perturbative symmetry) in a background with $n$ isometries $v^i$ leading to **topology change**:

$$\begin{tikzcd}
 & X \times_M \tilde{X} \arrow[dl, "\tilde{p}"] \arrow[dr, "p"] & \\
\tilde{X} \arrow[dr, "\tilde{\pi}"'] & & X \arrow[dl, "\pi"] \\
 & M &
\end{tikzcd}$$

**Correspondence space** $Y = X \times_M \tilde{X}$:

- $\diamond$ a circle bundle over $X$ with first Chern class $\pi^*(c_1(\tilde{X}))$
- $\diamond$ a circle bundle over $\tilde{X}$ with first Chern class $\tilde{\pi}^*(c_1(\tilde{X}))(\mathcal{L}_{\mathsf{v}} H = 0 \;\Rightarrow\; \mathrm{d}(\imath_{\mathsf{v}} H) = 0)$

T-duality:

$$\pi_* H = c_1(\tilde{X}) \quad \tilde{\pi}_* \tilde{H} = c_1(X) \qquad \in H^2(M, \mathbb{Z})$$

## Correspondence Space and singular spaces:

$\oplus$ Flat smooth $\mathbb{T}^5/\mathbb{Z}_2 : \langle x^a \to -x^a, \quad x^5 \to x^5 + 1/2 \rangle \quad \Leftrightarrow \quad$ Singular $\mathbb{T}^4/\mathbb{Z}_2 \times S^1$ with flat equivariant vertical $B$-field

- $\ominus$ original $c_1$ is pure torsion
- $\ominus$ gerbe holonomy freezes the same moduli as the shift along $x^5$

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T-duality on quotients:

- $\circ$ $\Pi^2 =$ id and $\mathcal{T}_i^2 = (-1)^{\boldsymbol{F_R}}$ (for T-duality in the $i$-th direction on the torus $\mathcal{T}_i$)
- $\circ$ $\mathcal{T}_i \Pi \mathcal{T}_i^{-1} = \mathcal{R}_i \Pi \quad \& \quad \mathcal{R}_i \Pi = (-1)^{\boldsymbol{F}} \Pi \mathcal{R}_i$ ($\mathcal{R}_i$ - reflection in $i$-th direction)
- $\circ$ $\mathcal{R}_i^2 = (-1)^{\boldsymbol{F}}$ , $\mathcal{T}_i \mathcal{R}_i \mathcal{T}_i^{-1} = \mathcal{R}_i$ , $\mathcal{T}_i \mathcal{R}_j \mathcal{T}_i^{-1} = (-1)^{\boldsymbol{F_R}} \mathcal{R}_j$ , $\mathcal{R}_i \mathcal{R}_j = (-1)^{\boldsymbol{F}} \mathcal{R}_j \mathcal{R}_i$
- $\circ$ $\mathcal{T}_i \mathcal{S}_j^{(n)} \mathcal{T}_i^{-1} = \mathcal{S}_j^{(n)}$ for $i \neq j$ or $\mathcal{G}_j^{(n)}$ for $i = j$ ($\mathcal{S}_i/\mathcal{G}_i$ - shift/gerbe action)

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**Good** singular backgrounds in sugra:

- $\circ$ **smooth** correspondence space
- $\circ$ Enlarging the space of supersymmetric backgrounds (beyond (smooth) spin geometries) $\Rightarrow$ ...???

Hyperconifolds: $M_\Gamma = T^{1,1}/\Gamma \quad (\Gamma \simeq \mathbb{Z}_n)$

▷ Kähler isometry group Isom$(\mathcal{C}) = \{(\mathrm{U}(2) \times \mathrm{U}(2)) \rtimes \mathbb{Z}_2\} \,/\, \mathrm{U}(1)$

- $\mathbb{Z}_2$ - exchange two $\mathrm{U}(2)$ elements
- quotient is by the diagonal $\mathrm{U}(1)$ subgroup of $\mathrm{U}(2) \times \mathrm{U}(2)$
- Large class of quotient geometries - subgroup $\Gamma \subset$ Isom$(\mathcal{C})$

$\Gamma_{n,k} \simeq \mathbb{Z}_n$ $(n \neq 4)$ - $M_{n,k} = T^{1,1}/\Gamma_{n,k} \simeq (S^2 \times S^3)/\Gamma'_{n,k}$

- $n$ and $k$ - relatively prime: $\gamma_{n,k}(z_1, z_2, z_3, z_4) = (\zeta_{2n}^{k+1} z_1, \zeta_{2n}^{-k-1} z_2, \zeta_{2n}^{1-k} z_3, \zeta_{2n}^{k-1} z_4)$
  - ◇ $\zeta_m = e^{2\pi \mathrm{i}/m}$ a primitive $m$-th root of unity
  - ◇ $\mathcal{C}/\Gamma_{n,k} \simeq \mathcal{C}/\Gamma_{n',k'} \iff n' = n,$ and $k' = \pm k^{\pm 1} \mod n$
- $\Gamma'_{n,k}$: azimuthal rotation by $2\pi(k-1)/n$ on$S^2$ and lens action on $S^3$
- $\gamma_{n,k}(\theta_1, \varphi_1, \theta_2, \varphi_2, \psi) = \left(\theta_1, \varphi_1 - 2\pi\frac{k+1}{n}, \theta_2, \varphi_2 + 2\pi\frac{k-1}{n}, \psi\right)$

$\check{\Gamma} \simeq \mathbb{Z}_4$ - exceptional cyclic action - $\check{M} = T^{1,1}/\check{\Gamma}$

- $\check{\gamma}(z_1, z_2, z_3, z_4) = (z_3, \mathrm{i}z_4, \mathrm{i}z_1, z_2)$
- $\check{\gamma}(\theta_1, \varphi_1, \theta_2, \varphi_2, \psi) = \left(\theta_2, \varphi_2 + \frac{\pi}{2}, \theta_1, \varphi_1 - \frac{\pi}{2}, \psi + \pi\right)$

## Hyperconifolds - T-dualised and lifted to M-theory

- regular geometry (smooth quotients of case (ii)) for $n \neq 4, k = \pm 1$ after $\mathcal{T}_{\varphi_1/\varphi_2}$
- singular geometries + 3-form (M-theiry lift of a flat gerbe) for $n \neq 4, k \neq \pm 1$
- non-geometric for exceptional $n = 4$

In the basis of ground state momenta and windings (smooth $\theta_1, \theta_2 : \sim \mathbb{R}^2 \times \mathbb{T}^3_{\varphi_1,\varphi_2,\psi}$ ):

$$\mathcal{T}_{\varphi_2} \gamma_{n,k} \mathcal{T}_{\varphi_2}^{-1} |\theta_1, \theta_2; m_1, w_1, m_2, w_2, m_3, w_3\rangle = \exp\left[\tfrac{2\pi \mathrm{i}}{n}\left((k-1)w_2 - (k+1)m_1\right)\right] \times$$
$$\times\ |\theta_1, \theta_2; m_1, w_1, m_2, w_2, m_3, w_3\rangle$$

- multiplication by phases - shifts and gerbe actions
- no exchange of winding and momenta

$$\mathcal{T}_{\varphi_2} \check{\gamma} \mathcal{T}_{\varphi_2}^{-1} |\theta_1, \theta_2; m_1, w_1, m_2, w_2, m_3, w_3\rangle = \exp\left[\tfrac{\mathrm{i}\pi}{2}(m_1 - w_2 + m_3)\right] \times$$
$$\times\ |\theta_2, \theta_1; w_2, m_2, w_1, m_1, m_3, w_3\rangle$$

- exchange of base coordinates and circles
- exchange of momentum and winding quantum numbers (winding strings/M2s)

## $Spin^c$ involutions in IIB

⊕ Kähler $\mathbb{Z}_4$ isometry of $\mathcal{C}$: $\lambda(z_1, z_2, z_3, z_4) = (z_1, -z_2, \mathrm{i}z_4, \mathrm{i}z_3)$

- $\lambda(\theta_1, \theta_2, \varphi_1, \varphi_2, \psi) = (\theta_1, \pi - \theta_2, \varphi_1 + \pi, -\varphi_2, \psi + 2\pi)$
- $\lambda$ acts freely away from the tip of the cone
- $\lambda^* \Omega = -\Omega$

Supersymmetry preserved $\Rightarrow$ $\lambda$ combined with $\Pi$ and $(-1)^{\boldsymbol{F_L}}$

Worldsheet realization of the symmetry:

$$\lambda_{\text{ws}} = \lambda_{\text{base}} \mathcal{R}_{\varphi_2} \Pi (-1)^{\boldsymbol{F_L}}$$

- $\lambda_{\text{base}}$ includes $\theta_2$ reflection
- $\omega = \Pi(-1)^{\boldsymbol{F_L}}, \quad \omega^2 = (-1)^{\boldsymbol{F}}, \quad \omega^4 = 1$ ($\omega$ - element -1 of center of $SL(2,\mathbb{Z})$)

T-duality along $\varphi_1$ or $\varphi_2$ and lift to M-theory - unoriented

e.g. $\mathcal{T}_{\varphi_2} \lambda_{\text{ws}} \mathcal{T}_{\varphi_2}^{-1} = \lambda_{\text{base}} \Pi$

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Exists a “parallel" world of quotients (smooth / with flat vertical gerbes) leading holographically to theories with exiting symmetry structures

## PART II: COMPACTIFICATIONS

Bigger (higher-dimensional) Klein bottles

$$
\begin{tikzcd}
 & \text{K2: } S^1 \times S^1/\langle s, r\rangle \arrow[dl] \arrow[dr] & & \\
X_6 \times S^1/\langle g, r\rangle & & X_6 \times S^1/\langle \tau, s\rangle \arrow[r] & X_6 \times S^1/\langle \tau, r\rangle
\end{tikzcd}
$$

- $g$ - free action by a group $G$
- $\tau$ - (smooth) antiholomorhic involutions ( $\omega_2\, to\, -\omega_2,\ \Omega_3 \to \Omega_3$)
- $r$ and $s$ - reflection and shift

The result:

- $X_6 \times S^1/\langle g, r\rangle$ - Pin$^+$ $\checkmark$
- $X_6 \times S^1/\langle \tau, s\rangle$ - Pin$^-$ $\cancel{\checkmark}$
- $X_6 \times S^1/\langle \tau, r\rangle$ - barely $G_2$
  - non susy M/$X_6/\langle g\rangle$ with $(-1)^{\boldsymbol{F}}$ consistent (NOT w/o $/\langle g\rangle$!)

Toy examples: M-theory on flat smooth susy $\mathbb{T}^5/\mathbb{Z}_2 : \langle x^a \to -x^a, \quad x^5 \to x^5 + 1/2 \rangle$

- M-theory circle $x^5$
  - IIA on singular $\mathbb{T}^4/\mathbb{Z}_2 : \langle x^a \to -x^a \rangle$ + flat (nontrivial) RR $C_1$
- M-theory circle $x^4$
  - IIA $\mathbb{Z}_2$ : orbifold on $\mathbb{T}^4$ with generator: $\mathcal{R}_1 \Pi \mathcal{R}_2 \Pi \mathcal{R}_3 \Pi (-1)^{\boldsymbol{F_L}} \mathcal{S}_5$ (Pin$^-$)
  - IIB dual - $O_3^2 \times S^1$ with $\Pi(-1)^{\boldsymbol{F_L}}$
    - non-susy Bieberbach $O_3^2 = \mathbb{T}^3/\mathbb{Z}_2 : \langle x^{1,2} \to -x^{1,2},\ x^3 \to x^3 + 1/2 \rangle$
  - Similar for M-theory on FHSV CY
    - IIA on Enr$_4 \times S^1/\mathbb{Z}_2$ with $\Pi(-1)^{\boldsymbol{F_L}}$ Pin$^-$
    - IIB on Enr$_4 \times S^1/$ with $\Pi(-1)^{\boldsymbol{F_L}}$ $Spin^c$
- M-theory/ FHSV$\times S^1$ has the same massless spectrum as unorientable comp. on
  - $K3 \times \mathbb{T}^3/\mathbb{Z}_2 : \langle \kappa, x^a \to -x^a \rangle$
  - $\kappa$ - Enriques involution
- $\mathcal{N} = 1$ constructions? $X_6 \times S^1/\langle g, r \rangle$

X - CY threefold admits free Kähler isometry $G$ - worldsheet theory $\mathcal{C}(X)$

Y = X/G - smooth CY 3f-ld - worldsheet theory $\mathcal{C}(Y)$

$$H^{p,q}(X) = H_{\text{triv}}(X)^{p,q} \oplus \oplus_{\rho\neq\text{triv}} H^{p,q}_{\rho}(X) \qquad H^{p,q}(Y) \simeq H^{p,q}_{\text{triv}}(X)$$

Spacetime $\mathbb{Z}_2$ symmetry of type II: $(-1)^{\boldsymbol{F_L}}$ - assigns -1 to R-R and R-NS (NS-NS and NS-R - inv.)

$$(-1)^{\boldsymbol{F_L}} = e^{2\pi \mathrm{i} J_{\boldsymbol{L}0}}\,, \qquad (-1)^{\boldsymbol{F_R}} = e^{2\pi \mathrm{i} J_{\boldsymbol{R}0}} \tag{1}$$

- $J_{\boldsymbol{L},\boldsymbol{R}}$ - currents associated to the $\mathrm{U}(1)_{\boldsymbol{L}} \times \mathrm{U}(1)_{\boldsymbol{R}}$ symmetry of the (2,2) SCFT
- $J_{\boldsymbol{L}/\boldsymbol{R},0}$ - conserved charges ($e^{\mathrm{i}\pi J_{\boldsymbol{L}0}}$ and $e^{\mathrm{i}\pi J_{\boldsymbol{R}0}}$ - SCFT fermion numbers)

Quotient of the worldsheet theory by $(-1)^{\boldsymbol{F_L}}$ - flip of the GSO projection IIA $\leftrightarrow$ IIB

New action by $\hat{G} \simeq G$: $\hat{G} = \{g(-1)^{\boldsymbol{F_L}\phi(g)} \mid g \in G\}$ (parity $\phi(g) \in \{0,1\}\ \forall g \in G$)

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\phi} \mathbb{Z}_2 \longrightarrow 1$$

- $\mathbb{Z}_2$ - additive $\mathbb{Z}/2\mathbb{Z}$ group, the subgroup $H \subset G$ - all elements $g$ with $\phi(g) = 0$
- $H$ - normal and maximal in $G$, contains : $[G,G] \subseteq H \subset G$
- nontrivial homomorphism $\phi$ $\Rightarrow$ new susy vacua

View from M-theory : M/$Z$ = M/$(\overline{Z}/K)$

- $\overline{Z} = X \times S^1$
- $K$: $\kappa_g(p,\theta) = (g(p), (-1)^{\phi(g)}\theta)$
- $\kappa_g^*\left(\frac{\mathrm{i}}{8}\Omega \wedge \bar{\Omega} \wedge \mathrm{d}\theta\right) = -\frac{\mathrm{i}}{8}\Omega \wedge \bar{\Omega} \wedge \mathrm{d}\theta$

▷ $\overline{M}_{11} = \mathbb{R}^{1,3} \times X \times S^1$ - spin & $\overline{\mathcal{P}}_+ \to \overline{M}_{11}$ - fibers: $32$–component Dirac spinors

▷ $\mathcal{P}_+ = \overline{\mathcal{P}}_+/K$ is a pin$^+$ bundle over $M_{11} = \overline{M}_{11}/K$

▷ Can show for Majorana spinor $\Psi$ to preserve susy for all $g \in G$:

$$\Psi = \mathcal{C}_{11}\Gamma^0\Psi^* = .... = \hat{\kappa}_g\Psi = ... = (\gamma_5 \otimes \rho_7)^{\phi(g)}\Psi \;\Leftrightarrow\; \eta = \gamma_5\eta$$

▷ 1/2 susy preserved wrt susy preserved by $X$

Large new class of $\mathcal{N} = 1$ compactifications

- ⋆ Explicit examples of **smooth geometries** constructed
- ⋆ EA for massless fields known exactly at string tree-level - (2,2) worlsheet SCFT
- ⋆ non-free $G$ ? - analogous to barely $G_2$ vs full $G_2$