Topological Lines in 1+1d Chiral Fermions

Physics and Mathematics of Generalized Symmetry
in QFTPhysics and Mathematics of Generalized Symmetry in QFT @ KIPMU

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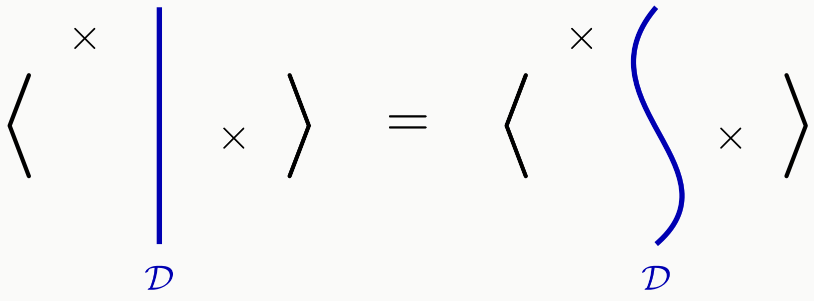
Motivation

Topological lines & the question

Topological line $\mathcal{D}$: $[\mathcal{D}, T] = [\mathcal{D}, \overline{T}] = 0$
$\Rightarrow$ **freely deformable**.



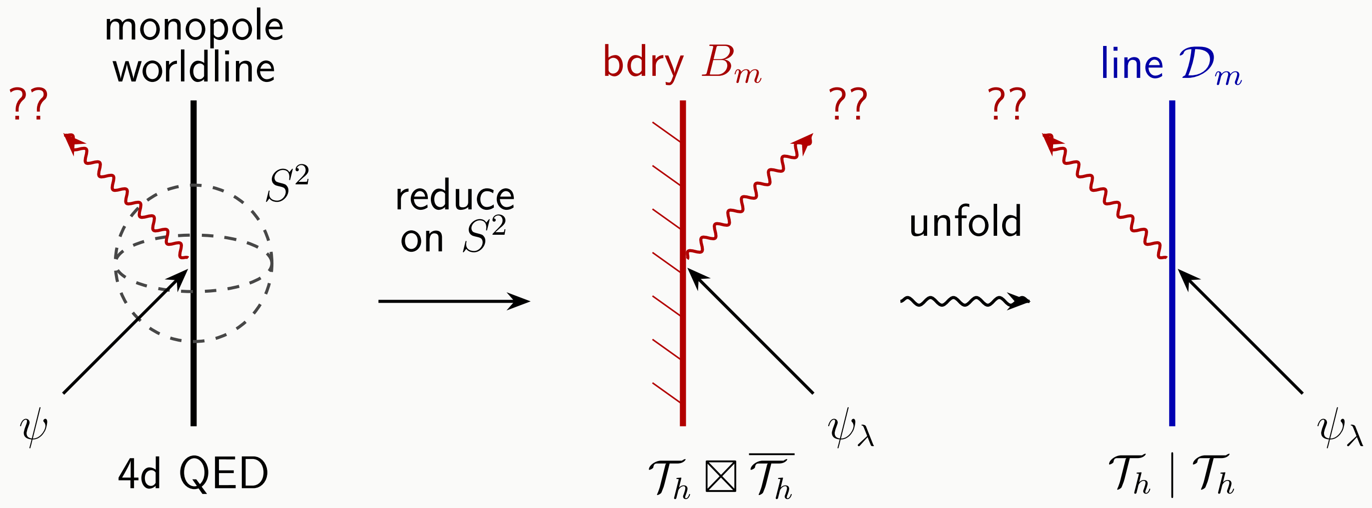
- $\mathcal{D}$ realizes **generalized (non-invertible) symmetry**.
- $\mathcal{S}$-rational CFT $\Rightarrow$ $\mathcal{S}$-preserving lines $=$ **Verlinde lines**: one L_λ per $\lambda \in \mathrm{Rep}(\mathcal{S})$.

- **This talk:** $\mathcal{D}$ intertwining two **different realizations** of a chiral algebra $\mathcal{A}$?

The monopole scattering problem (1)

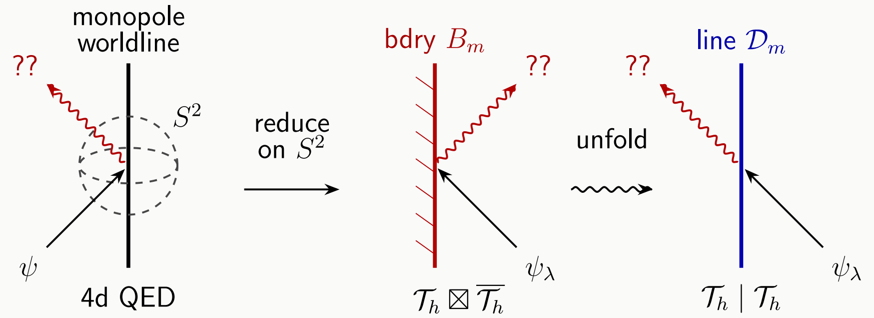
Monopole scattering $=$ a topological-line problem in disguise.

- **Callan–Rubakov** [Rubakov '82; Callan '83]: 4d fermion $+$ monopole
  $\xrightarrow{S^2}$ 1+1d non-chiral fermion $+$ boundary at $r = 0$.
- **unfold** $\Rightarrow$ (conf.) boundary $\mapsto$ **topological line** $\mathcal{D}$ in $\mathcal{T}_h$ = (chiral fermions).
  - conformal defect in chiral theory ($(T - \bar{T})_L = (T - \bar{T})_R$) is topological.



The monopole scattering problem (2)

- 4d N-flavor QED + monopole $\xrightarrow{S^2}$ 1+1d N Dirac fermions + boundary at $r = 0$.
  $\psi_L : (1, \Box) | \, \psi_R : (-1, \Box)$ of $G = U(1)^{\text{EM}} \times SU(N)^{\text{flavor}}$.
- "**Missing final state problem**" for $N \geq 3$
  - No outgoing state with the same G-charges as the incoming state.
    - $O(1)$ **violation of unitarity or charge conservation!**
    - $\psi_R = \psi_L$ nor $\psi_R = \psi_L^\dagger$ does not work.

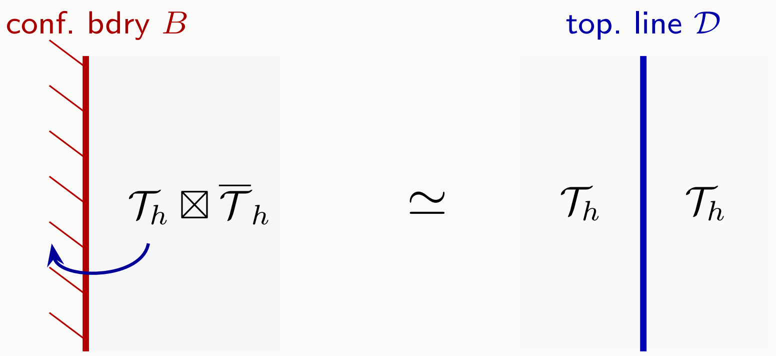


- symmetry G realized **differently across** $\mathcal{D}$ (in vs out)
  - How to construct $\mathcal{D}$ and B? [Maldacena and Ludwig '97; Boyle Smith and Tong '20; van Beest, Boyle Smith, Delmastro, Mouland, and Tong '24; van Beest, Boyle Smith, Delmastro, Komargodski, and Tong '25].

Symmetric boundary for anomaly-free G?

- **General question.** G with 't Hooft anomaly ω: $\omega \neq 0 \;\Rightarrow\;$ **no G-symmetric boundary** [Thorngren and Wang '21; Hellerman, Orlando, and Watanabe '21].
- **Converse?** $\omega = 0 \overset{?}{\Rightarrow}$ symmetric boundary — especially **chiral** G.
  - E.g. Two Dirac fermions $(\psi_L^{1,2}, \psi_R^{1,2})$ with $U(1)_L \times U(1)_R$, charges $Q_L; Q_R = (3,4;0,5)$
    - Anomaly $k_L = 3^2 + 4^2 = k_R = 0^2 + 5^2$ (Pythagorean) [Boyle Smith and Tong '20; van Beest, Boyle Smith, Delmastro, Komargodski, and Tong '25]
- **Unfolding.** G-symmetric boundary of $\mathcal{T}_h \boxtimes \overline{\mathcal{T}_h}$

  $\Longleftrightarrow$ topological line $\mathcal{D}$ of $\mathcal{T}_h$ intertwining **two realizations** f_1, f_2.

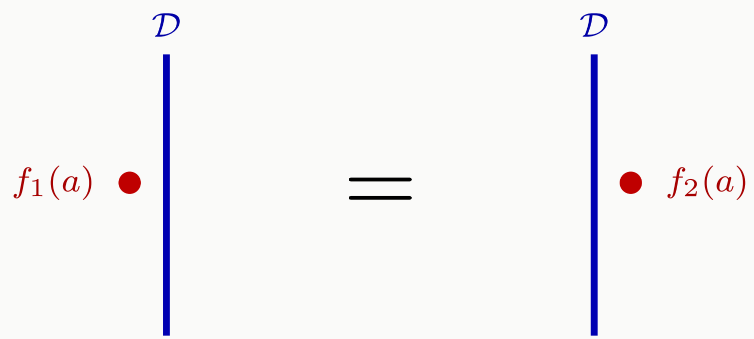


The Main Question

- preserved symmetry = rational **current algebra** $\mathcal{A} = \widehat{\mathfrak{g}}_k$
- two realizations $f_1, f_2 : \mathcal{A} \hookrightarrow \mathcal{T}_h$ (chiral fermions)
- Ex. (Pythagorean)
  - $\mathcal{T}_h$: 2 Weyl (chiral) fermions $\psi^{1,2}$
  - $\mathcal{A} = \widehat{\mathfrak{u}}(1)_{25}$, $f_1(j) = 3\psi^1\psi^{1\dagger} + 4\psi^2\psi^{2\dagger}$, $f_2(j) = 5\psi^2\psi^{2\dagger}$.

Main Question. Is there a **topological** line $\mathcal{D}$ with

$$f_1(a)\,\mathcal{D} = \mathcal{D}\,f_2(a) \qquad \forall\, a \in \mathcal{A}\,?$$

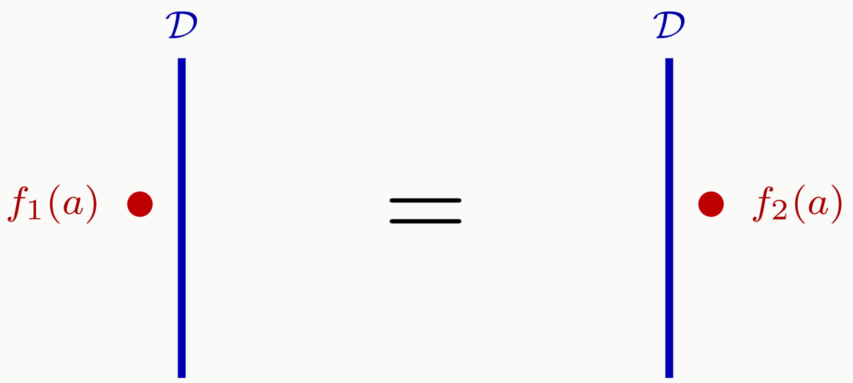


Construction

Main lemma [Etingof, Nikshych, Ostrik, and Meir '09; Davydov, Mueger, Nikshych, and Ostrik '10; Möller and Rayhaun '24]

- $\mathcal{T}_h$ rational holomorphic CFT;
- $\mathcal{S}$: a rational chiral algebra (VOA) with $c_{\mathcal{S}} = c_{\mathcal{T}_h}$;
- $f_1, f_2 : \mathcal{S} \hookrightarrow \mathcal{T}_h$ two **conformal embeddings**.
  $\Rightarrow$ there is a topological line $\mathcal{D}$ with

$$f_1(s)\,\mathcal{D} = \mathcal{D}\,f_2(s) \qquad \forall s \in \mathcal{S}.$$



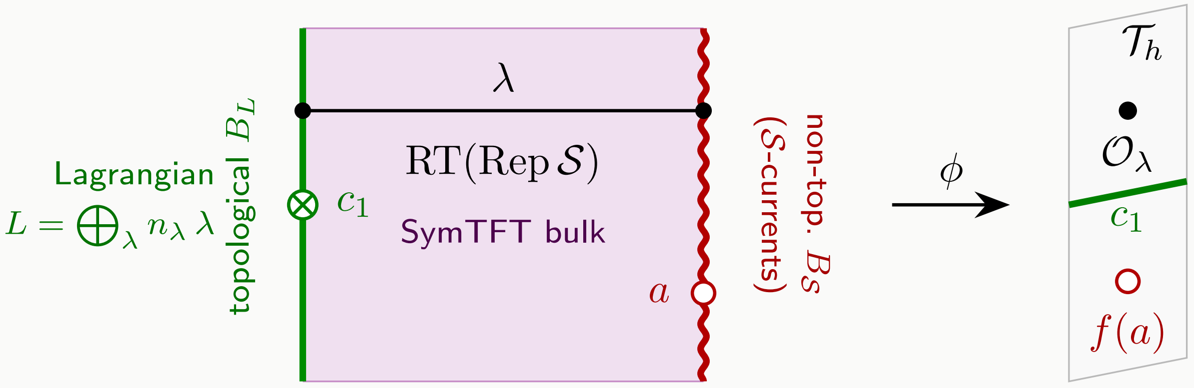
- **Main Question**: we want $\mathcal{A}$ (usually $c_{\mathcal{A}} < c_{\mathcal{T}_h}$);
  - **find** a rational **conformal extension** $\mathcal{A} \subseteq \mathcal{S} \hookrightarrow \mathcal{T}_h$ with $c_{\mathcal{S}} = c_{\mathcal{T}_h}$
    - Lack a universal way to find extension. gap: $c_{\mathcal{T}_h} - c_{\mathcal{A}}$ (c.f. [van Beest, Boyle Smith, Delmastro, Komargodski, and Tong '25]).
- built via **3d-TFT / 2d-RCFT** [Witten '89; Fuchs, Runkel, and Schweigert '02] as **SymTFT** [Gaiotto and Kulp '21; Freed, Moore, and Teleman '22; Apruzzi, Bonetti, García Etxebarria, Hosseini, and Schafer-Nameki '23].
- **Corollary.** $\mathcal{T}_h \boxtimes \overline{\mathcal{T}_h}$ has a conf. boundary B with $B\,f_1(s) = B\,\overline{f_2(s)}$.

Background: rational chiral algebra

- $\mathcal{S}$: **rational** $\iff$ finitely many "good" modules $\iff$ $\mathrm{Rep}(\mathcal{S})$ a **(super)MTC**.
- $\mathfrak{g}$-currents with **anomaly** k: $\widehat{\mathfrak{g}}_k \ni j^a, T_{\text{Sugawara}} = \sum_a j^a j^a$. $c_{\text{Sugawara}} = \dfrac{k \dim \mathfrak{g}}{k + h^\vee}$.
- $\widehat{\mathfrak{u}}(1)_k$ **simple-current extension**: generators $b^\pm$ of weight $k/2$, charge $\pm k$
  - (k odd $\Rightarrow$ super-VOA).
- **Example** (N Weyl, $(\psi_a, \psi_a^\dagger)$): $\widehat{\mathfrak{su}}(N)_1 \oplus \widehat{\mathfrak{u}}(1)_N$,
  - $j^a_{\mathfrak{su}(N)} \propto T^{a;ij}\, \psi_i \psi_j + \text{h.c.}$,
  - $j_{\mathfrak{u}(1)} \propto \sum_a \psi_a \psi_a^\dagger$,
  - $b^+ = \prod_a \psi^a$, $b^- = \prod_a \psi^{\dagger a}$.
- $\mathcal{S} \hookrightarrow \mathcal{T}_h$: **conformal free fermion realization** [Rayhaun '24]

Conformal embedding & the sandwich

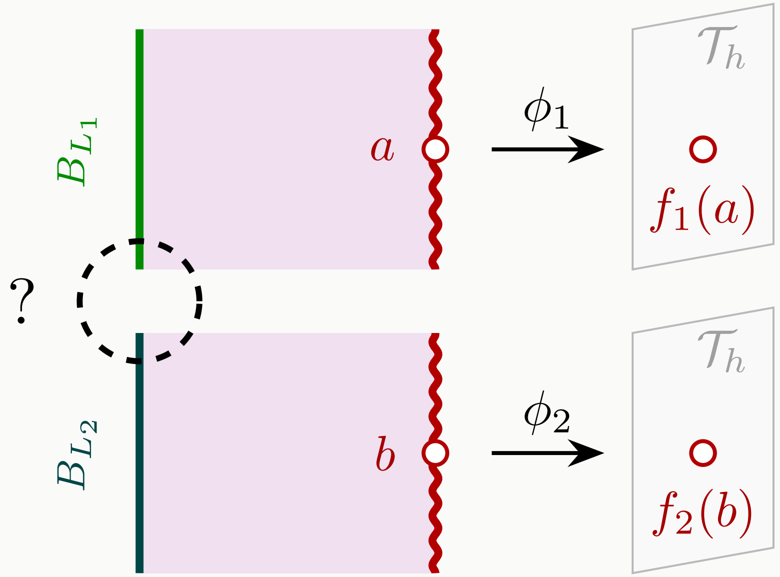
- $\mathcal{S} \hookrightarrow \mathcal{T}_h$ conformal $\iff c(\mathcal{S}) = c(\mathcal{T}_h)$ [Möller and Rayhaun '24];
  - $\phi : \bigoplus_\lambda n_\lambda V_\lambda \cong \mathcal{T}_h$. $\quad n_\lambda < \infty$
- **Sandwich:** bulk $\mathrm{RT}(\mathrm{Rep}\,\mathcal{S})$, G_k Chern-Simons theory for $\mathcal{S} = \widehat{\mathfrak{g}}_k$.
  - a **non-topological** boundary carrying the $\mathcal{S}$-currents,
  - and a **topological** boundary B_L; Lagrangian algebra $L = \bigoplus_\lambda n_\lambda \lambda$.
- $\mathcal{C}_L = \mathrm{Mod}_L(\mathrm{Rep}\,\mathcal{S})$ = **topological lines** on B_L [Cordova, García-Sepúlveda, and Harvey '25]
  - = the **$\mathcal{S}$-preserving topological lines** of $\mathcal{T}_h$.



$\otimes$ = a topological line on a boundary, seen end-on (into the page)

Two conformal embeddings

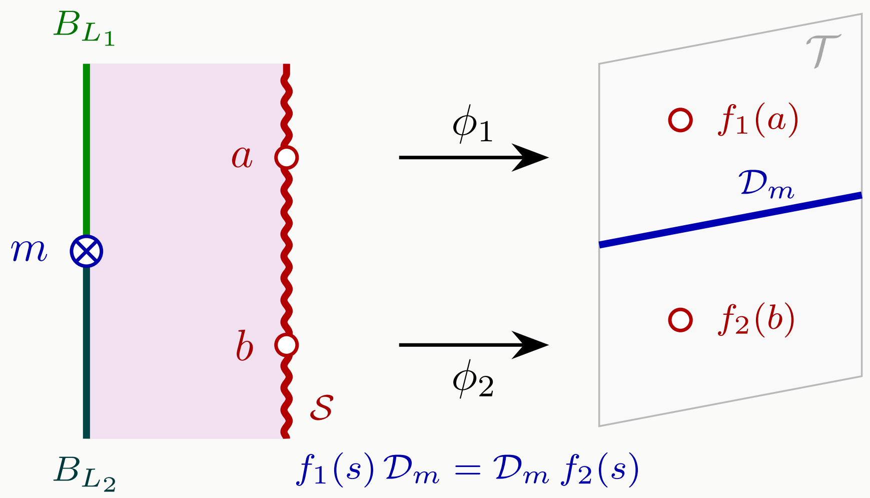
- **Two** embeddings $f_{1,2} : \mathcal{S} \hookrightarrow \mathcal{T}_h$
  - $\oplus_\lambda n_\lambda^{(1)} V_\lambda \overset{\phi_1}{\cong} \mathcal{T}_h, \quad \oplus_\lambda n_\lambda^{(2)} V_\lambda \overset{\phi_2}{\cong} \mathcal{T}_h, \quad \phi_i|_{\lambda=\mathrm{vac}} = f_i$
  - Two Lagrangian algebras $L_i = \oplus_\lambda n_\lambda^{(i)} \lambda \in \mathrm{Rep}(\mathcal{S})$.
  - $\mathcal{C}_i = \mathrm{Mod}_{L_i}(\mathrm{Rep}\,\mathcal{S})$, with $Z(\mathcal{C}_1) \cong \mathrm{Rep}\,\mathcal{S} \cong Z(\mathcal{C}_2)$ [Davydov, Mueger, Nikshych, and Ostrik '10].
    - preserving $f_i(\mathcal{S})$ for $i = 1, 2$



Connecting Junction

- **Topological junction** connecting B_{L_1} and B_{L_2} **always exists!** [Etingof, Nikshych, Ostrik, and Meir '09; Davydov, Mueger, Nikshych, and Ostrik '10]
- The junctions form an **invertible** $(\mathcal{C}_1, \mathcal{C}_2)$-bimodule $\mathcal{M}_{12} = \mathrm{Mod}_A(\mathrm{Rep}\,\mathcal{S})(\neq 0)$
  - $A \subset L_1 \otimes L_2$: an irreducible algebra in $\mathrm{Rep}(\mathcal{S})$.

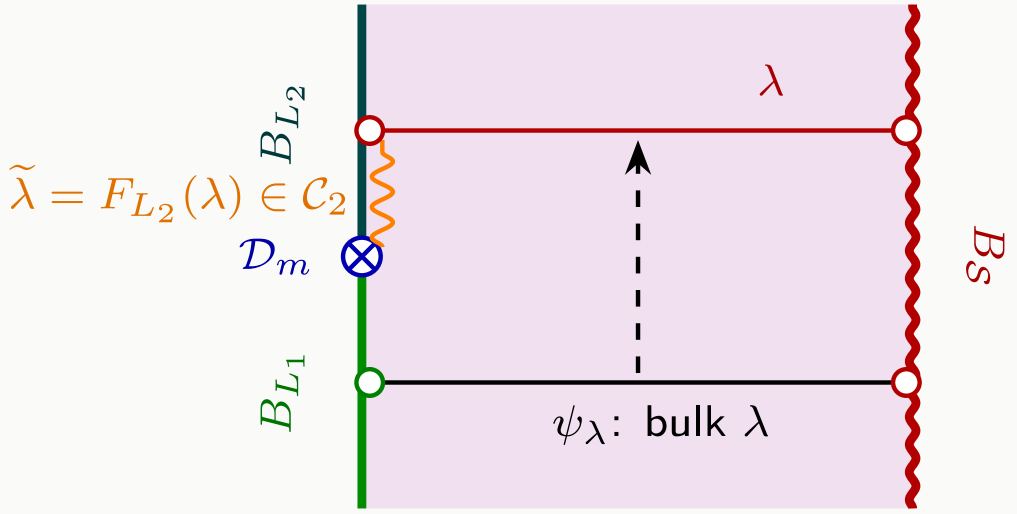
For every object $m \in \mathcal{M}_{12}$ there is a topological line $\mathcal{D}_m$ of $\mathcal{T}_h$ with $f_1(s)\,\mathcal{D}_m = \mathcal{D}_m\, f_2(s)$

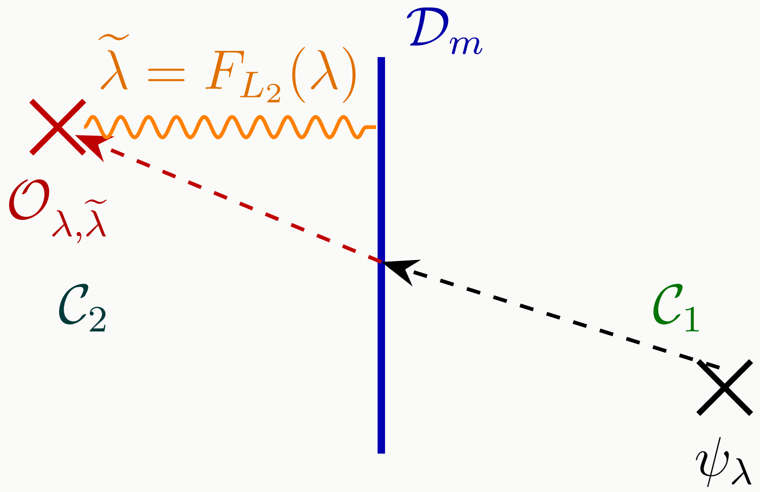


The missing final state in 2d

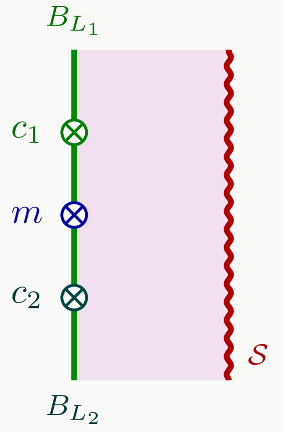
- **Back to the monopole puzzle.** Incoming radiation = a bulk module $\lambda \in \mathrm{Rep}\,\mathcal{S}$; the outgoing **"??"** is fixed by $\mathcal{D}_m$:

$$\widetilde{\lambda} = F_{L_2}(\lambda) = L_2 \otimes \lambda \;\in\; \mathcal{C}_2 = \mathrm{Mod}_{L_2}(\mathrm{Rep}\,\mathcal{S}).$$





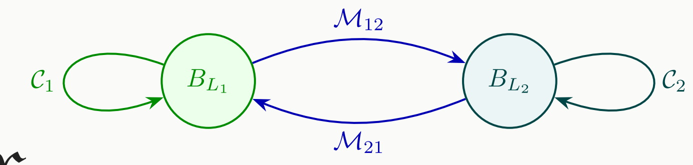
The symmetry category $\mathfrak{X}$



- Lines on/between B_{L_1} and B_{L_2} form a **universal** category $\mathfrak{X}$ [Gannon and Rayhaun '26]:

$$\mathfrak{X} = \langle\, \mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{M}_{12},\ \mathcal{M}_{21}\,\rangle$$

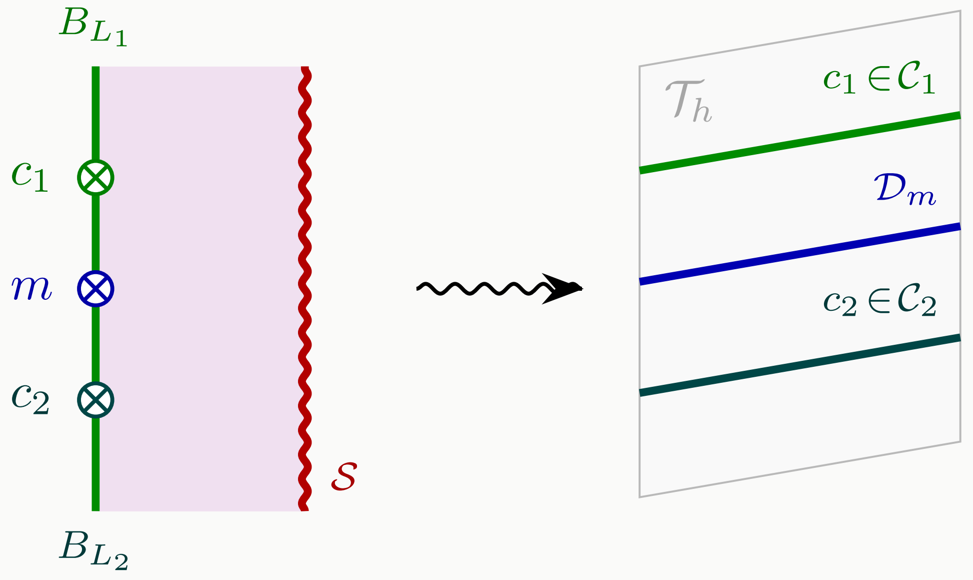
  - $\mathcal{C}_i = \mathrm{Mod}_{L_i}(\mathrm{Rep}\,\mathcal{S}),\ \mathcal{M}_{12} = \mathrm{Mod}_A(\mathrm{Rep}\,\mathcal{S}),\ \mathcal{M}_{21} = \mathrm{Mod}_{A^{\mathrm{op}}}(\mathrm{Rep}\,\mathcal{S})$
- **Fusion**: balanced tensor product over L_1 or L_2 when makes sense:
  - $c_1 \otimes_{L_1} m_{12} \in \mathcal{M}_{12}, m_{12} \otimes_{L_2} m_{21} \in \mathcal{C}_1, m_{12} \otimes_{L_2} c_2 \in \mathcal{M}_{12}$, etc.



- **Freely generated** otherwise.
  - $c_1 \otimes c_2 \otimes c_1,\ m_{12} \otimes m_{12},\ c_2 \otimes m_{12}\ \ldots \Rightarrow$ **infinite** category $\mathfrak{X}$
- $m \in \mathcal{M}_{12} \Rightarrow m^* \in \mathcal{M}_{21}$,
  - $m \otimes_{L_2} m^* = A \in \mathcal{C}_1$: **non-invertible** A-gauging duality defect [Choi, Lu, and Sun '24]
  - $m^* \neq m$ not Tambara-Yamagami. $m \otimes m \otimes m \cdots \neq 1$.

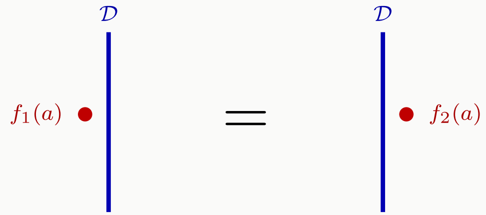
$\mathfrak{X}$ acting on $\mathcal{T}_h$: when is it finite?

- A realization $\phi_{1,2} : \mathcal{S} \hookrightarrow \mathcal{T}_h$ makes $\mathfrak{X}$ **act on $\mathcal{T}_h$** (as the lines $\{\mathcal{D}_m\}$), generally **non-faithfully**.
- The physical symmetry is the **faithful quotient** $\mathfrak{X} \twoheadrightarrow \mathfrak{X}_{\mathcal{T}_h}$ (depends on $\mathcal{T}_h, \phi_{1,2}$).
- $\mathfrak{X}_{\mathcal{T}_h}$ preserves $\mathcal{S}^\sigma = \{s : f_1(s) = f_2(s)\}$
  - $\mathfrak{X}_{\mathcal{T}_h}$ is **finite** $\iff$ **$\mathcal{S}^\sigma$ is rational and conformal**.
  - Otherwise infinite symmetry.

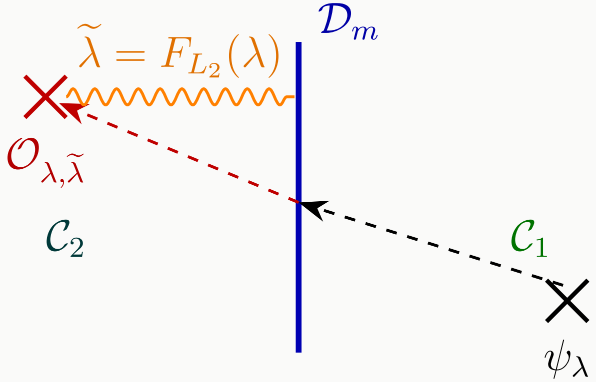


Examples

Ex 1 — Abelian / Pythagorean [Boyle Smith and Tong '20; Nagoya and Shimamori '23; van Beest, Boyle Smith, Delmastro, Komargodski, and Tong '25; Arias-Tamargo, Boyle Smith, Mouland, and Velásquez Cotini Hutt '26; Wei and Zheng '26]



- $\mathcal{T}_h$: 2 complex fermions ψ_1, ψ_2 $(c = 2)$.
  - $\mathcal{A} = \widehat{\mathfrak{u}}(1)_{25}, Q_1; Q_2 = (3,4); (0,5),\ c_{\mathcal{A}} = 1 < c_{\mathcal{T}_h}$
  - $\mathcal{S} = \widehat{\mathfrak{u}}(1)^2_{25}, Q_1 = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}, K = Q_1^T Q_1 = Q_2^T Q_2 = \mathrm{diag}(25, 25).$
- $\mathcal{C}_1 = \mathbb{Z}_{25}, \mathcal{C}_2 = (\mathbb{Z}_5 \times \mathbb{Z}_5)^\omega$: subgroups of $U(1)^2$ **fixing simple currents**.
  - $\mathcal{D}$: **gauging** $\mathbb{Z}_5 \subset \mathbb{Z}_{25}$.
  - $\mathcal{D}^*\mathcal{D} = \sum_{g \in \mathbb{Z}_5 \subset \mathbb{Z}_{25}} g \Rightarrow \dim \mathcal{D} = \sqrt{5}$.
  - $\mathcal{D}(j^1, j^2) = R(\arctan(3/4))(j^1, j^2) \Rightarrow$ **infinite order**.
- **shooting ψ into the wall**
  - $\Rightarrow$ a **defect operator** at the end of the $\mathbb{Z}_5$ line $\tilde{\lambda}$



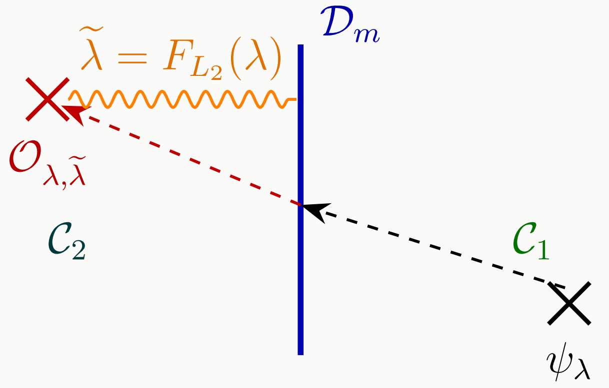
Ex 2 — Monopole

[Maldacena and Ludwig '97; van Beest, Boyle Smith, Delmastro, Komargodski, and Tong '25; Wei and Zheng '26]

- 4d QED, N flavors $+$ monopole
  - $\to$ 2d N Dirac; $\psi^L : (1, \square)$, $\psi^R : (1, \overline{\square})$ of $U(1)_{EM} \times SU(N)$.
  - Unfold $\to$ 2d N Weyl with a line **intertwining** $U(1) \times SU(N)$.
- $\mathcal{A} = \mathcal{S} = \widehat{\mathfrak{u}}(1)_N \oplus \widehat{\mathfrak{su}}(N)_1$:

$$f_1(j_{\mathfrak{u}(1)}) = f_2(j_{\mathfrak{u}(1)}), \qquad f_1(b^+) = f_2(b^+) = \textstyle\prod_a \psi^a,$$

$$f_1(j^a_{\mathfrak{su}(N)}) = -f_2(j^a_{\mathfrak{su}(N)}) \propto \sum_{i,j} T^a_{ij}\psi^i\psi^j + \text{h.c.}$$



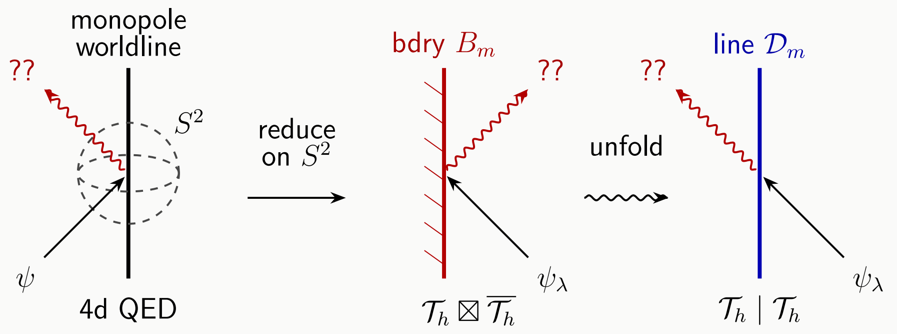
- $\mathcal{S}^\sigma = \widehat{\mathfrak{u}}(1)_N \oplus \widehat{\mathfrak{so}}(N)_2$ (**rational, conformal**) $\Rightarrow \mathfrak{X} = \mathrm{TY}(\mathbb{Z}_n)$ **finite**.
  - $n = N$ (N odd) or $N/2$ (N even).
- **Final state** $=$ **defect op** on the $\mathbb{Z}_n$ lines.

Ex 3 — chiral QED $+$ monopole **(new)** · setup

- 4d chiral QED, 10 Weyl of $U(1)$ charge $(1,1,1,1,1,\ 3,\ -2,-2,-2,-2)$
  - $\sum q = \sum q^3 = 0$ (anomaly-free, chiral).
- Monopole reduction $\Rightarrow$ 2d 8 Weyl; beyond $SU(2)^{\text{ang}} \times U(1)_{\text{EM}}$ there is an **anomaly-free global $\mathrm{Spin}(5)$** flavor symmetry:

$$\underbrace{(\mathbf{1},1,\ \mathbf{5})\oplus(\mathbf{3},3,\ \mathbf{1})}_{\psi^L}\ \Big|\ \underbrace{(\mathbf{2},-2,\ \mathbf{4})}_{\psi^R},\qquad \big(\mathbf{n}_{SU(2)^{\text{ang}}},\ q_{\text{EM}},\ \mathbf{r}_{\mathrm{Spin}(5)}\big).$$

- Preserve it all: $\mathcal{A} = \widehat{\mathfrak{su}}(2)^{\text{ang}}_4 \oplus \widehat{\mathfrak{u}}(1)_{32} \oplus \widehat{\mathfrak{so}}(5)_2,\ c = 2+1+4 = 7$
  - **just $c=1$ short** of a conformal embedding ($c_{\mathcal{T}} = 8$).



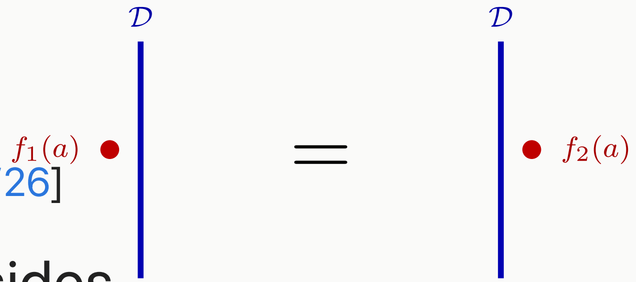
Ex 3 · enlargement & construction

- Fill the $c=1$ **gap** with an **orbifold chiral algebra** — $\widehat{\mathfrak{u}}(1)^{\mathbb{Z}_2}_{480}$:

$$\mathcal{S} = \underbrace{\widehat{\mathfrak{su}}(2)^{\text{ang}}_4 \oplus \widehat{\mathfrak{u}}(1)^{\text{EM}}_{32} \oplus \widehat{\mathfrak{so}}(5)_2}_{\mathcal{A}} \oplus \boxed{\widehat{\mathfrak{u}}(1)^{\mathbb{Z}_2}_{480}}\,, \qquad c = 7 + \underset{\text{orb}}{1} = 8 = c_{\mathcal{T}}$$

$$\underbrace{(\mathbf{1}, 1,\ \mathbf{5};\ \boxed{-9}\,) \oplus (\mathbf{3}, 3,\ \mathbf{1};\ \boxed{5}\,)}_{\psi^L} \;\Big|\; \underbrace{(\mathbf{2}, -2,\ \mathbf{4};\ \boxed{\tau_{\frac{1}{16}}}\,)}_{\psi^R}.$$

  - $\tau_{\frac{1}{16}}$: **twist field** of $\widehat{\mathfrak{u}}(1)^{\mathbb{Z}_2}_{480}$, $h = \frac{1}{16}$.
- $\Rightarrow \mathcal{D}$ intertwining $\mathcal{S}$ exists!
  - $\mathcal{D}^*\mathcal{D}$: non-invertible gauging duality [Choi, Lu, and Sun '24; Wei and Zheng '26]



- **non-trivial part**: $\widehat{\mathfrak{u}}(1)^{\mathbb{Z}_2}_{480}$ — a particular radius that works for both sides.
- Final state in scattering is not yet analyzed.

Summary

Main Question. Is there a **topological** line $\mathcal{D}$ with

$$f_1(a)\,\mathcal{D} = \mathcal{D}\,f_2(a) \qquad \forall\, a \in \mathcal{A}\,?$$

Main Lemma.
Yes, if you can find **conformal** extension $\mathcal{A} \hookrightarrow \mathcal{S}$ with $(c_{\mathcal{S}} = c_{\mathcal{T}_h})$, compatible with f_1, f_2.

- Nonabelian $\mathcal{A}$ ([Boyle Smith and Tong '20; van Beest, Boyle Smith, Delmastro, Komargodski, and Tong '25; Wei and Zheng '26])
- $\mathrm{Rep}\,\mathcal{S}$: finite MTC, B_{L_1}, B_{L_2}: topological boundaries
  - General nonsense tells (c.f. [Etingof, Nikshych, Ostrik, and Meir '09; Davydov, Mueger, Nikshych, and Ostrik '10]) $\mathcal{D}$ exists.
- $\mathcal{D}$: **non-invertible duality defect** in general (c.f. [Wei and Zheng '26])
- whole symmetry $\mathfrak{X}_{\mathcal{T}_h}$ is **infinite** in general.
- Chiral 4d QED $+$ monopole example

Open questions

Main Lemma.
Yes, if you can find **conformal** extension $\mathcal{A} \hookrightarrow \mathcal{S}$ with ($c_{\mathcal{S}} = c_{\mathcal{T}_h}$), compatible with f_1, f_2.

- Such extension always exists? Counterexample?
  - Chiral QED with $U(1)$ charges $(1, 5, 9, -7, -8)$ (the **1-5-7-8-9** model) [Boyle Smith and Tong '20; van Beest, Boyle Smith, Delmastro, Mouland, and Tong '24]
- Do we need such extension?
- **dynamics:** lattice/MPO transmission [Ueda, Linden, Lootens, Haegeman, Fendley, et al. '25], defect-anomaly scattering [Antinucci, Copetti, Galati, and Rizi '26], wavepackets [Tachikawa, Tsuji, and Watanabe '26].
- How do we organize **all** topological lines in free fermion theories?
- **4d uplift??** in monopole situation?
  - 2d end state is a defect operator on a **non-invertible line** in general

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Backup

Ex 3′: the transpose $(5, 1, 1, 1, -4, -4)$

- 4d QED $(5, 1, 1, 1, -4, -4)$ — **transpose** of Ex 3′s $(3, 1^5, -2^4)$: same $\sum |q| = 16$, same $\mathcal{T}_h$ ($c_{\mathcal{T}} = 8$).
- Manifest $\mathcal{A} = \widehat{\mathfrak{su}}(2)^{\text{ang}}_{20} \oplus \widehat{\mathfrak{u}}(1)^{\text{EM}}_{128} \oplus \widehat{\mathfrak{su}}(2)^{\text{fl}}_{4} \quad c \approx 5.73, \delta \approx 2.27$:

$$\underbrace{(\mathbf{5}, 5;\ \mathbf{1}) \oplus (\mathbf{1}, 1;\ \mathbf{3})}_{\psi^L} \Big| \underbrace{(\mathbf{4}, -4;\ \mathbf{2})}_{\psi^R}, \qquad (\mathbf{n}_{SU(2)^{\text{ang}}},\ q_{\text{EM}};\ \mathbf{r}_{SU(2)^{\text{fl}}}).$$

- Conformal $\mathcal{S} = \widehat{\mathfrak{so}}(5)_2 \oplus \widehat{\mathfrak{u}}(1)^{\text{EM}}_{128} \oplus \widehat{\mathfrak{su}}(2)^{\text{fl}}_{4} \oplus \widehat{\mathfrak{u}}(1)^{\mathbb{Z}_2}_{1920}$.

$$\underbrace{(\mathbf{5}, 5;\ \mathbf{1}; -3) \oplus (\mathbf{1}, 1;\ \mathbf{3}; 25)}_{\psi^L} \Big| \underbrace{(\mathbf{4}, -4;\ \mathbf{2}; \tau_{\frac{1}{16}})}_{\psi^R}.$$

  - $1920 = 30 * 8^2$.