Dimensional reduction of topological operators

Based on arXiv:2603.03438 with Pavel Putrov

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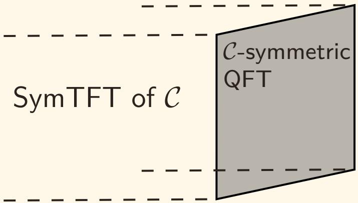
Motivation

- (Finite) non-invertible symmetries in 1+1d QFTs implemented by topological line operators have a rich description in terms of fusion categories. (See talks by Yu Nakayama, Kantaro Ohmori, Yichul Choi.)
- In higher dimensions, a QFT contains topological operators of different codimensions, forming a higher-group in the invertible case, and a higher-fusion category in the non-invertible case.
- Despite this increasing complexity, non-invertible symmetries are highly constrained in higher dimensions.
- I will discuss one perspective on these constraints coming from dimensional reduction of topological operators.

Non-invertible symmetries in $d > 2$

- Expectation: Unlike in $d = 2$, the structure of non-invertible symmetries in $d > 2$ is simpler because SymTFTs (see talks by Pavel, Iñaki and Mirjam) simplify in $d > 3$.

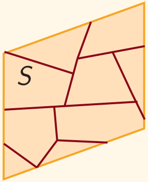


An intuition emphasized by several people. For example, Kantaro's 2021 GGI talk.

- But topological operators of QFTs are not as constrained as topological operators of TQFTs. Symmetric QFTs live at the boundary of SymTFTs. In what sense do topological operators of the boundary QFT "simplify" in higher dimensions?

Some classes of non-invertible symmetries in $d > 2$

- Some examples of non-invertible symmetries in higher-dimensional QFTs are
  1. Condensation operators: Obtained from gauging a higher-form symmetry on a sub-manifold. [Kapustin, Saulina (2010)][Fuchs, Schweigert, Valentino (2013)] [Roumpedakis, Seifnashri, Shao (2022)]



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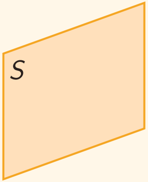
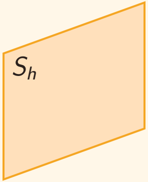
  2. Group-theoretical non-invertible symmetries: Obtained from gauging a subsymmetry of $H \leq G$ 0-form symmetry. [Bhardwaj, Schafer-Nameki, Wu (2022)] [Hsin, Kobayashi, Zhang (2024,2025)]



$$\sim \sum_{h\in[h],[h]\in H\backslash G/H}$$



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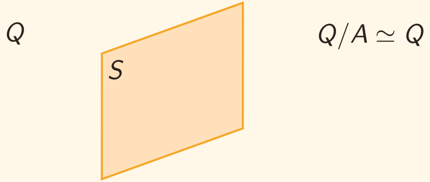

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- Among these, both condensation operators and duality operators act invertibly on local operators. Group-theoretical non-invertible symmetries act non-invertibly, but their action can be rendered invertible by gauging a symmetry.
- **Main question:** What is the general story?

Dimensional reduction of topological operators

- Consider a p-dimensional topological operator supported on $\mathbb{R}^q \times \Sigma^{p-q}$ where Σ^{p-q} is a $(p-q)$-dimensional compact manifold.

- Shrinking Σ^{p-q} to a point produces a topological operator supported on $\mathbb{R}^q$.

High energy physics: Similar to dimensional reduction of QFTs, but now for operators within a QFT.

Condensed matter physics: Appeared in the study of 3d topological orders where closed string-like excitations can be shrunk to point-like excitations.

[Lan, Kong, Wen (2017)] [Lan, Wen (2018)]

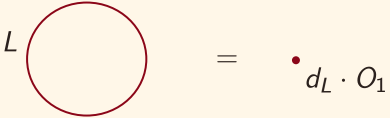
Mathematics: Dimensional reduction of X is $ev_X \circ coev_X$ where ev_X and $coev_X$ are the evaluation and coevaluation maps in a spherical higher-fusion category.

[N. Reshetikhin, V. Turaev (1990, 1991)][Johnson-Freyd, Yu (2020)]

Dimensional reduction of topological line operators

- Consider a QFT with the only topological point operator being the identity operator, say O_1.

- Consider a topological line operator L on a circle S^1. Shrinking the circle gives $d_L \cdot O_1$ where d_L is some number called the **quantum dimension** of L.



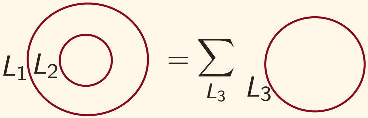
- This can be interpreted as the action of L on O_1.

Constraints from fusion rules of line operators

- Consider the fusion rule

$$L_1 \times L_2 = \sum_{L_3} N_{L_1 L_2}^{L_3} \; L_3$$

- Dimensional reduction must be compatible with fusion:



$$d_{L_1} \times d_{L_2} = d_{L_1 \times L_2} = \sum_{L_3} N_{L_1 L_2}^{L_3} \; d_{L_3}$$

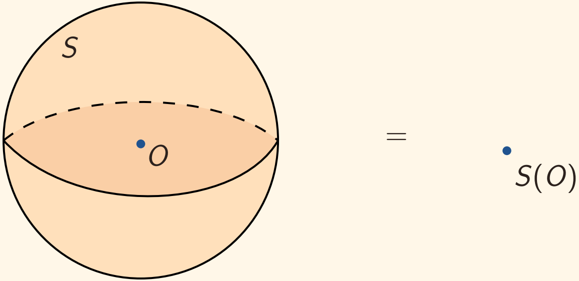
- For example, in the Ising CFT with $\psi^2 = \mathbf{1}$ and $\sigma \times \sigma = \mathbf{1} + \psi$ we have $d_{\mathbf{1}} = d_\psi = 1$ and $d_\sigma = \sqrt{2}$.

- If $d_L = 1$ and $d_{\overline{L}} = 1$, then $d_{L \times \overline{L}} = 1 \implies L \times \overline{L} = \mathbf{1} \implies L$ is invertible. (Only true for unitary theories.)

Topological Operators in $2+1$d

0-form Symmetry

0-form symmetries are implemented by topological surface operators.



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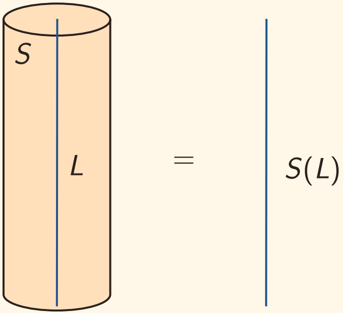
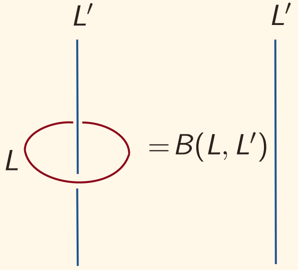


Topological Operators in $2+1$d

1-form Symmetry

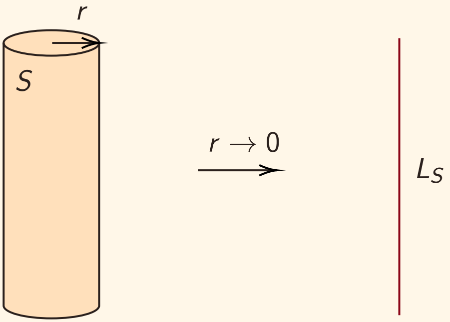
- 1-form symmetries are implemented by topological line operators.



- While topological line operators cannot act on local operators, they constrain the action of topological surface operators on local operators.

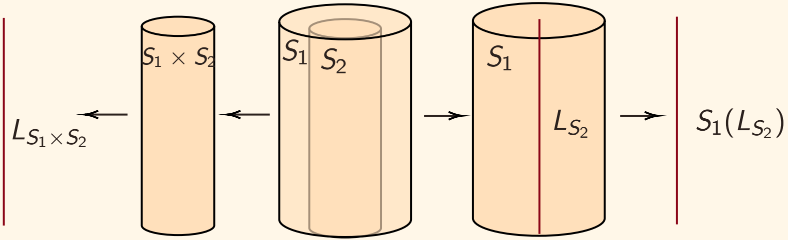
Dimensional reduction of topological surfaces

Consider a topological operator S on the 2-manifold $S^1 \times \mathbb{R}$. Shrinking S^1 to a point, we get a line operator L_S.



Constraints from fusion rules of surface operators

Dimensional reduction of topological operators must be compatible with their fusion rules.



$$\boxed{L_{S_1 \times S_2} = S_1(L_{S_2})}$$

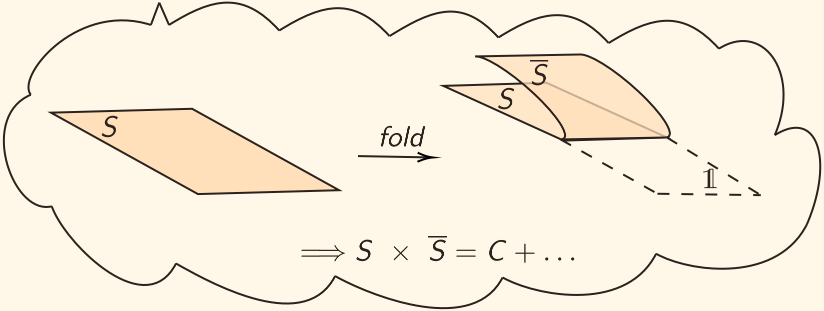
(Note: $L_{S_1 \times S_2} \neq L_{S_1} \times L_{S_2}$)

A diagnostic for invertible surface operators

- Assume $L_S = L_{\overline{S}} = \mathbf{1}$ (analog of assuming $d_L = 1$ for line operators). We will show that S is an invertible operator.

  Using $L_{S_1 \times S_2} = S_1(L_{S_2})$, we find $L_{S \times \overline{S}} = \mathbf{1}$.

  Consider the fusion rule $S \times \overline{S} = C + \ldots$



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  Using $L_S = L_{\overline{S}} = \mathbf{1}$, we get $S \times \overline{S} = C$. S and $\overline{S}$ acts trivially on $\mathbf{1}$, and so does C. This shows that C is invertible $\implies$ S is an invertible operator.

- The assumption above is satisfied when the QFT does not have any non-trivial topological line operators.
  In the absence of any 1-form symmetry, 0-form symmetry must be invertible!

Action on Local Operators in $2+1$d QFTs: General Case

- Consider a 2+1d QFT $\mathcal{Q}$ containing a collection of topological line operators $\mathcal{B}$.
- The action of the topological surface operators in $\mathcal{Q}$ on local operators can be rendered invertible by gauging a sub-symmetry of $\mathcal{B}$.
- A sketch of the argument:

  Suppose $\mathcal{B}$ is non-anomalous, then gauging it gives a new QFT $\mathcal{Q}/\mathcal{B}$ without topological line operators. Therefore, surface operators in $\mathcal{Q}/\mathcal{B}$ are invertible.

  If $\mathcal{B}$ is anomalous, we can choose a maximal sub-symmetry $\mathcal{D}$ of $\mathcal{B}$ that is non-anomalous. It can be argued that surface operators in $\mathcal{Q}/\mathcal{D}$ are either invertible or condensation operators, implying that the action on local operators in $\mathcal{Q}/\mathcal{D}$ is invertible.

Invertible Action on Local Operators in $3+1$d

- Consider a 3d topological membrane operator M on $S^2 \times \mathbb{R}$. Shrinking S^2 to a point, we get a line operator L_M.
- **Assumption:** No non-trivial topological line operators $\implies L_M = \mathbf{1}$ for all M.
- Consistency with fusion rules imply $L_{M\times\overline{M}} = \mathbf{1} \implies M \times \overline{M} = C$ for some condensation operator C.
- C acts trivially on local operators $\implies M$ acts invertibly on local operators.

Note: Unlike in $2+1$d, absence of topological line operators does not imply that M is an invertible operator! M can still be an intrinsically non-invertible symmetry as defined in [Kaidi, Zafrir, Zheng (2022)].

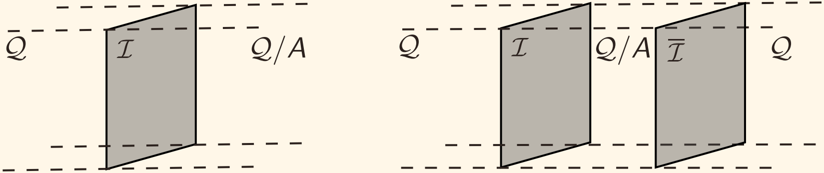
Conclusion

- The structure of non-invertible symmetries in higher-dimensional QFTs, especially their action on local operators, is highly constrained.
- We showed that in higher-dimensional QFTs without topological line operators, the codimension-1 operators must act invertibly on local operators.
- This result can be generalized to the case of QFTs with topological lines. In this case, the action on local operators can be non-invertible, but it can be rendered invertible by gauging an appropriate higher-form symmetry.
- The anomaly of the effective invertible symmetry group acting on local operators is a diagnostic for the anomaly of the full non-invertible symmetry. (More in our paper!)

Non-invertible Symmetries from Gauging Interface

- Consider a $2+1$d QFT $\mathcal{Q}$ with non-anomalous 1-form symmetry A. Gauging A produces a new QFT $\mathcal{Q}/A$.

- Gauging A on half of the spacetime defines a topological interface $\mathcal{I}$. Gauging A on a strip defines a **condensation operator** $C_A := \overline{\mathcal{I}} \circ \mathcal{I}$.



- Condensation operators $\Longleftrightarrow$ Topological operators that admit topological boundary conditions. They act trivially on local operators.

[A. Kapustin, N. Saulina (2010)][J. Fuchs, C. Schweigert, A. Valentino (2013)]
[K. Roumpedakis, S. Seifnashri, S.-H. Shao (2022)]

Example: $2+1$d $\mathbb{Z}_2$ Gauge Theory

Invertible Symmetries

- The 2+1d $\mathbb{Z}_2$ gauge theory has the action

$$\frac{i2}{2\pi}\int BdA$$

where A, B are $U(1)$ gauge fields.

- This theory has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry implemented by the topological line operators

$$L_1 = e^{i\int_{\Sigma^{(1)}} A}\,,\ L_2 = e^{i\int_{\Sigma'^{(1)}} B}$$

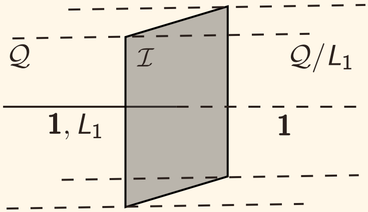
satisfying

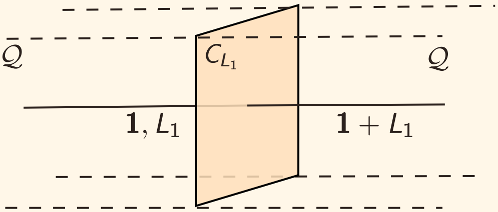
$$\boxed{L_1 L_2 = \mathbf{exp}\left(\frac{2\pi i I(\Sigma^{(1)},\Sigma'^{(1)})}{2}\right) L_2 L_1.}$$

Example: $2+1$d $\mathbb{Z}_2$ Gauge Theory

A Non-invertible Symmetry

- Let $\mathcal{Q}$ be the 2+1d $\mathbb{Z}_2$ gauge theory. It has a non-anomalous $\mathbb{Z}_2$ 1-form symmetry generated by L_1.
- Gauging L_1 produces a gauging interface $\mathcal{I}$ between the $\mathbb{Z}_2$ gauge theory and the trivial theory from which we can construct the non-invertible symmetry $C_{L_1} := \overline{\mathcal{I}} \circ \mathcal{I}$.





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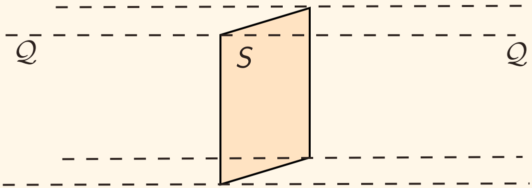
- Let $\mathcal{B}$ be the collection of topological line operators in a QFT $\mathcal{Q}$.
- Suppose $\mathcal{B}$ is non-anomalous. Gauging $\mathcal{B}$ produces a new QFT $\mathcal{Q}/\mathcal{B}$ which does not contain any topological line operators. In $\mathcal{Q}/\mathcal{B}$, all topological surface operators act invertibly on local operators.



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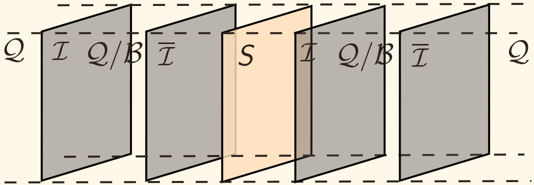

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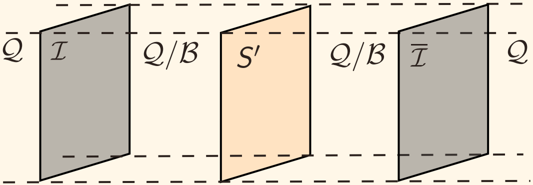


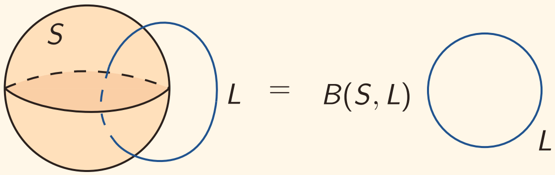
- The action of a topological surface operator S in $\mathcal{Q}$ on a local operator O is given by

$$\boxed{S(O) = \overline{\mathcal{I}} \circ S' \circ \mathcal{I}(O)}$$

where S' is a surface operator in $\mathcal{Q}/\mathcal{B}$. S' is a sum of invertible surface operators.

Topological Operators in $3 + 1$d

- **3d membrane operators:** Implements 0-form symmetry. Acts on all operators.
- **2d surface operators:** Implements 1-form symmetry. Cannot act on local operators. Acts on line operators by braiding.



- **1d line operators:** Implements 2-form symmetry. Cannot act on local operators or line operators. Acts on surfaces by braiding.