Generalized Hecke Relations

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August 28, 2026

Modular forms and Hecke operators

- **Consider a weight-k (weakly) holomorphic modular form $f(\tau) \in \mathcal{M}_k(\Gamma)$,**

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \ , \qquad \forall \gamma \in \Gamma \ .$$

- **A natural set of mutually commuting linear operators acting on $\mathcal{M}_k(\Gamma)$ are the Hecke operators T_n,**

$$(\mathsf{T}_n f)(\tau) := n^{k-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{a\tau + b}{d}\right) \ .$$

- **For $n = p$ prime, we have**

$$(\mathsf{T}_p f)(\tau) = p^{k-1}\left[f(p\tau) + p^{-k}\sum_{b=0}^{p-1} f\left(\frac{\tau + b}{p}\right)\right] \ .$$

Physics of Hecke Operators

- Say that $f(\tau) = Z_A(\tau)$ is the partition function/elliptic genus for a theory A. One way to understand $(\mathsf{T}_p f)(\tau)$ is to consider the product $A^{\otimes p}$ and define the cyclic permutation $s : (1\,2\,\ldots\,p) \longmapsto (p\,1\,\ldots\,p-1)$.

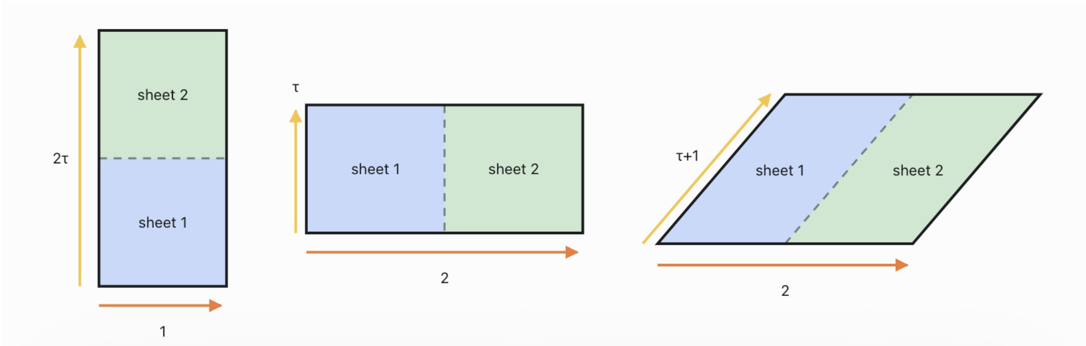
- Note that

$$(Z_{A^{\otimes p}})_0^s = \mathrm{Tr}_{\mathcal{H}^{\otimes p}}\left[\hat{s}\, q^{\sum_a L_0^{(a)} - \frac{pc}{24}}\right] = \sum_n q^{p\left(h_n - \frac{c}{24}\right)} = Z_A(p\tau)\ .$$

Modular transformations then give

$$(Z_{A^{\otimes p}})_s^0 = Z_A(\tau/p)\ , \qquad (Z_{A^{\otimes p}})_s^s = Z_A((\tau+1)/p)\ , \qquad \ldots$$

- Each of these describes the original theory on a degree-p covering torus,



Physics of Hecke Operators

- **For $p = 2$, we have**

$$(\mathsf{T}_2 Z_A)(\tau) = \frac{1}{2}\Big[(Z_{A^{\otimes 2}})_0^s(\tau) + (Z_{A^{\otimes 2}})_s^0(\tau) + (Z_{A^{\otimes 2}})_s^s(\tau)\Big]$$

and hence

$$Z_{A^{\otimes 2}/S_2}(\tau) = \frac{1}{2}(Z_A(\tau))^2 + (\mathsf{T}_2 Z_A)(\tau)\ .$$

- **Similarly, for $p = 3$ we have**

$$Z_{A^{\otimes 3}/S_3}(\tau) = \frac{1}{6}(Z_A(\tau))^3 + Z_A(\tau)(\mathsf{T}_2 Z_A)(\tau) + (\mathsf{T}_3 Z_A)(\tau)\ .$$

- **In general, symmetric product orbifolds are given by the DMVV formula** [Dijkgraaf-Moore-Verlinde-Verlinde '97]

$$\sum_{n\geq 0} p^n\, Z_{A^{\otimes n}/S_n} = \exp\left(\sum_{n\geq 1} p^n\, \mathsf{T}_n Z_A\right)\ .$$

Mathematics of Hecke Operators

- **Let us note two useful properties of Hecke operators:**

i) They are particular examples of double coset operators. Indeed, define $g_p := \mathrm{diag}(p,1)$. Note that

$$\Gamma g_p \Gamma = \bigsqcup_i \Gamma g_p r_i = \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \sqcup \bigsqcup_{b=0}^{p-1} \Gamma \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$$

Then we have

$$(\mathsf{T}_p f)(\tau) = p^{k-1} \sum_i f(\tau|_{g_p r_i})$$

ii) Hecke operators form an algebra,

$$\mathsf{T}_n \mathsf{T}_m = \sum_{d\,|\,\gcd(n,m)} d^{k-1} \mathsf{T}_{mn/d^2}$$

Harvey-Wu operators

- Next, consider a vector-valued modular function (vvmf), such as a character vector,

$$\boldsymbol{\chi}(\gamma\tau) = \rho(\gamma)\boldsymbol{\chi}(\tau)\ , \qquad \gamma \in SL(2,\mathbb{Z})\ , \qquad \rho : SL(2,\mathbb{Z}) \longrightarrow GL(d)\ .$$

- Harvey-Wu introduced a useful extension of the Hecke operators to operators acting on vvmfs [Harvey-Wu '18] . Writing N for the conductor (order of T-matrix), for any p coprime to N we have,

$$(\mathsf{T}_p^{\mathrm{HW}}\boldsymbol{\chi})(\tau) := G_p\boldsymbol{\chi}(p\tau) + \sum_{b=0}^{p-1}\boldsymbol{\chi}\left(\frac{\tau + bN}{p}\right)\ .$$

where we have defined

$$G_p := \rho(\sigma_p)\ , \qquad \sigma_p \equiv \begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix} \pmod{N}\ , \qquad p\bar{p} \equiv 1 \pmod{N}\ .$$

- The output is again a d-dimensional vector-valued modular function,

$$(\mathsf{T}_p^{\mathrm{HW}}\boldsymbol{\chi})(\gamma\tau) = \rho^{(p)}(\gamma)(\mathsf{T}_p^{\mathrm{HW}}\boldsymbol{\chi})(\tau)\ ,$$

where $\rho^{(p)}$ is a Galois conjugate of ρ.

Harvey-Wu operators

- **As a simple example, consider the $SU(2)_1$ characters**

$$\begin{aligned}\chi_{\mathbf{1}}^{A_1}(\tau) &= q^{-1/24}\left(1+3q+4q^2+\cdots\right)\ ,\\ \chi_{\mathbf{2}}^{A_1}(\tau) &= q^{5/24}\left(2+2q+6q^2+\cdots\right)\ .\end{aligned}$$

- **The modular representation is**

$$\rho_{A_1}(S)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\ ,\qquad \rho_{A_1}(T)=\begin{pmatrix}e^{-\pi i/12} & 0\\ 0 & e^{5\pi i/12}\end{pmatrix}\ ,\qquad N_{A_1}=24\ .$$

- **Consider the action of $\mathsf{T}_7^{\mathrm{HW}}$,**

$$\begin{aligned}(\mathsf{T}_7^{\mathrm{HW}}\boldsymbol{\chi}^{A_1})(\tau) &= \boldsymbol{\chi}^{A_1}(7\tau)+\sum_{b=0}^{6}\boldsymbol{\chi}^{A_1}\left(\frac{\tau+24b}{7}\right)\\ &= \begin{pmatrix}q^{-7/24}\left(1+133q+1673q^2+\cdots\right)\\ q^{11/24}\left(56+968q+7504q^2+\cdots\right)\end{pmatrix}\ .\end{aligned}$$

- **These are precisely the characters of $(E_7)_1$. The modular data is obtained by applying the Galois automorphism $\zeta_{24}\longmapsto\zeta_{24}^7$.**

Harvey-Wu operators

- **Other examples include**

$$\mathsf{T}_7^{\mathrm{HW}}\chi^{\mathrm{LY}} = \chi^{G_{2,1}}\ , \qquad \mathsf{T}_{13}^{\mathrm{HW}}\chi^{\mathrm{LY}} = \chi^{F_{4,1}}\ ,$$

$$\mathsf{T}_{31}^{\mathrm{HW}}\chi^{\mathrm{Ising}} = \chi^{(\mathrm{E}_8)_2}\ ,$$

$$\mathsf{T}_{47}^{\mathrm{HW}}\chi^{\mathrm{Ising}} = \chi^{\mathrm{Baby\ \ Monster}}\ ,$$

$$\mathsf{T}_p^{\mathrm{HW}}\chi^{\mathrm{Ising}} = \chi^{\mathrm{Spin}(p)_1}\ , \qquad p = 5, 7, 11, 13, 17, 19, 23\ .$$

Rephrasing Harvey-Wu

- **For fixed p, first form the extended character vector**

$$\widehat{\boldsymbol{\chi}}_p(\tau) := \begin{pmatrix} \boldsymbol{\chi}(p\tau) \\ \boldsymbol{\chi}(\tau/p) \\ \boldsymbol{\chi}((\tau+N)/p) \\ \vdots \\ \boldsymbol{\chi}((\tau+(p-1)N)/p) \end{pmatrix} .$$

- **The Harvey-Wu operator is the rectangular projection**

$$\mathsf{T}_p^{\rm HW}\boldsymbol{\chi} = P_{\rm HW}\,\widehat{\boldsymbol{\chi}}_p\ , \qquad P_{\rm HW} = \left(G_p \,|\, 1 \,|\, \cdots \,|\, 1\right)\ .$$

- **The extended character vector has its own $(p+1)d$-dimensional modular action,**

$$\widehat{\boldsymbol{\chi}}_p(\gamma\tau) = \widehat{\rho}_p(\gamma)\widehat{\boldsymbol{\chi}}_p(\tau)\ .$$

and the projection intertwines this representation with the Galois-conjugate representation,

$$P_{\rm HW}\,\widehat{\rho}_p(\gamma) = \rho^{(p)}(\gamma)\,P_{\rm HW}\ .$$

Extending Harvey-Wu

- **This suggests a straightforward extension of the Harvey-Wu construction:**
  - Construct the extended character vector $\widehat{\boldsymbol{\chi}}_p$ and determine its modular representation $\widehat{\rho}_p$.
  - Decompose $\widehat{\rho}_p$ into irreducible representations, and check whether any of them gives a legitimate RCFT character.
- **This gives a large class of generalized Hecke relations (see also** [Duan-Lee-Sun '22; Bouchard-Creutzig-Joshi '18]**).**

Extending Harvey-Wu: Example 1

- Consider the characters of $SU(3)_1$,

$$\begin{aligned} \chi^{A_2}_{\mathbf{1}}(\tau) &= q^{-1/12}\left(1+8q+17q^2+\cdots\right) \,, \\ \chi^{A_2}_{\mathbf{3}}(\tau) = \chi^{A_2}_{\bar{\mathbf{3}}}(\tau) &= q^{1/4}\left(3+9q+27q^2+\cdots\right) \,. \end{aligned}$$

- After identifying the two equal characters, the modular representation is

$$\rho_{A_2}(S) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \,, \qquad \rho_{A_2}(T) = \begin{pmatrix} e^{-\pi i/6} & 0 \\ 0 & i \end{pmatrix} \,, \qquad N_{A_2} = 12 \,.$$

- For $p=3$, construct

$$\widehat{\boldsymbol{\chi}}_{A_2,3}(\tau) = \begin{pmatrix} \boldsymbol{\chi}_{A_2}(3\tau) \\ \boldsymbol{\chi}_{A_2}(\tau/3) \\ \boldsymbol{\chi}_{A_2}((\tau+1)/3) \\ \boldsymbol{\chi}_{A_2}((\tau+2)/3) \end{pmatrix}$$

and determine $\widehat{\rho}_{A_2,3}$.

Extending Harvey-Wu: Example 1

- **Under $T : \tau \longmapsto \tau + 1$, the four blocks transform as**

$$\begin{aligned} \chi_{A_2}(3\tau) &\longmapsto \rho_{A_2}(T)^3 \chi_{A_2}(3\tau) \ , \\ \chi_{A_2}(\tau/3) &\longmapsto \chi_{A_2}((\tau+1)/3) \ , \\ \chi_{A_2}((\tau+1)/3) &\longmapsto \chi_{A_2}((\tau+2)/3) \ , \\ \chi_{A_2}((\tau+2)/3) &\longmapsto \rho_{A_2}(T) \chi_{A_2}(\tau/3) \ . \end{aligned}$$

- **Hence**

$$\widehat{\rho}_{A_2,3}(T) = \begin{pmatrix} \rho_{A_2}(T)^3 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & \rho_{A_2}(T) & 0 & 0 \end{pmatrix} ,$$

and similarly for $\widehat{\rho}_{A_2,3}(S)$.

- **This is an eight-dimensional representation, which decomposes as**

$$8 = 2 + 2 + 4 \ .$$

Extending Harvey-Wu: Example 1

- Choose one of the two-dimensional pieces via

$$P = \left(\begin{array}{cc|cc|cc|cc} 1 & -1 & 0 & 1 & 0 & i & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & e^{-5\pi i/6} & 0 & e^{\pi i/3} \end{array}\right) .$$

- Acting with this gives

$$\begin{aligned} P\widehat{\boldsymbol{\chi}}_{A_2,3} &= \begin{pmatrix} \chi_{\mathbf{1}}^{A_2}(3\tau) - \chi_{\mathbf{3}}^{A_2}(3\tau) + \chi_{\mathbf{3}}^{A_2}(\tau/3) + i\chi_{\mathbf{3}}^{A_2}((\tau+1)/3) - \chi_{\mathbf{3}}^{A_2}((\tau+2)/3) \\ \chi_{\mathbf{3}}^{A_2}(\tau/3) + e^{-5\pi i/6}\chi_{\mathbf{3}}^{A_2}((\tau+1)/3) + e^{\pi i/3}\chi_{\mathbf{3}}^{A_2}((\tau+2)/3) \end{pmatrix} \\ &= \begin{pmatrix} q^{-1/4}\left(1 + 78q + 729q^2 + 4382q^3 + \cdots\right) \\ q^{5/12}\left(27 + 378q + 2484q^2 + 12312q^3 + \cdots\right) \end{pmatrix} . \end{aligned}$$

This is the character vector of $(E_6)_1$. So we get a Hecke-like relationship, even when we cannot use the results of Harvey-Wu.

Extending Harvey-Wu: Example 2

- **As another example, consider $\mathrm{Spin}(8)_1$, with $N_{D_4} = 6$. For $p = 2$, form**

$$\widehat{\boldsymbol{\chi}}_{D_4,2}(\tau) = \begin{pmatrix} \boldsymbol{\chi}_{D_4}(2\tau) \\ \boldsymbol{\chi}_{D_4}(\tau/2) \\ \boldsymbol{\chi}_{D_4}((\tau+1)/2) \end{pmatrix} .$$

- **The extended modular representation is given by**

$$\widehat{\rho}_{D_4,2}(T) = \begin{pmatrix} \rho_{D_4}(T)^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \rho_{D_4}(T) & 0 \end{pmatrix} , \qquad \widehat{\rho}_{D_4,2}(S) = \begin{pmatrix} 0 & \rho_{D_4}(S) & 0 \\ \rho_{D_4}(S) & 0 & 0 \\ 0 & 0 & \rho_{D_4}\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \end{pmatrix} .$$

- **The six-dimensional representation decomposes as $6 = 1 + 2 + 3$. Projecting onto the one-dimensional piece gives**

$$P\widehat{\boldsymbol{\chi}}_{D_4,2} = \chi_0^{D_4}(2\tau) - \chi_v^{D_4}(2\tau) + 2\chi_v^{D_4}(\tau/2) + 2\omega\chi_v^{D_4}((\tau+1)/2) = \chi_0^{E_8}(\tau) .$$

- **Many other examples, e.g.**

$$\mathcal{M}(5,6) \xrightarrow{p=2} \mathcal{M}(2,5): \qquad \begin{array}{l} \chi_{1,1}^{(2,5)}(\tau) = \chi_{1,2}^{(5,6)}(2\tau) + \chi_{1,4}^{(5,6)}(2\tau) \\ \chi_{1,2}^{(2,5)}(\tau) = \chi_{2,2}^{(5,6)}(2\tau) + \chi_{2,4}^{(5,6)}(2\tau) \end{array}$$

Zhou-Zhu operators

- **Consider a scalar modular function with multiplier ε_A,**

$$f(\gamma\tau) = (c\tau + d)^k \varepsilon_A(\gamma) f(\tau) \ , \qquad \gamma \in \Gamma \ .$$

- **Fix some g. Suppose there is another multiplier ε_B satisfying**

$$\varepsilon_A(g\lambda g^{-1}) = \varepsilon_B(\lambda) \ , \qquad \lambda \in \Gamma \cap g^{-1}\Gamma g \ ,$$

and decompose the double coset as

$$\Gamma g \Gamma = \bigsqcup_i \Gamma g r_i \ .$$

- **Zhou-Zhu define** [Zhou-Zhu '23]

$$(\mathsf{T}_{[g]}^{\varepsilon_A \to \varepsilon_B} f)(\tau) = \sum_i \varepsilon_B(r_i)^{-1} f(g r_i \tau) \ .$$

It maps elements of $\mathcal{M}_k(\Gamma, \varepsilon_A)$ to elements of $\mathcal{M}_k(\Gamma, \varepsilon_B)$.

Zhou-Zhu operators for vvmfs

- **We can easily generalize this to vvmfs. Let**

$$\rho_A : \Gamma \longrightarrow GL(V_A) \ , \qquad \rho_B : \Gamma \longrightarrow GL(V_B) \ ,$$

and introduce a map $Q : V_A \longrightarrow V_B$ **such that**

$$Q\rho_A(g\lambda g^{-1}) = \rho_B(\lambda)Q \ , \qquad \lambda \in \Gamma \cap g^{-1}\Gamma g \ .$$

- **The vector-valued Zhou-Zhu operator is**

$$(\mathsf{T}^{\rho_A \to \rho_B}_{[g],Q}\boldsymbol{f})(\tau) = \sum_i \rho_B(r_i)^{-1} Q\, \boldsymbol{f}(g r_i \tau) \ .$$

It maps a vvmf with representation ρ_A **to one with representation** ρ_B**.**

Zhou-Zhu operators for vvmfs

- This can be written in a more streamlined way. Define

$$\widehat{\boldsymbol{f}}_{[g]}(\tau) := \begin{pmatrix} \boldsymbol{f}(gr_1\tau) \\ \vdots \\ \boldsymbol{f}(gr_m\tau) \end{pmatrix}~, \qquad P_{[g],Q} := \big(\rho_B(r_1)^{-1}Q \,|\, \cdots \,|\, \rho_B(r_m)^{-1}Q\big)~.$$

  Writing $\widehat{\rho}_{A,g}$ for the modular action on $\widehat{\boldsymbol{f}}_g$,

$$\mathsf{T}^{\rho_A\to\rho_B}_{[g],Q}\boldsymbol{f} = P_{[g],Q}\widehat{\boldsymbol{f}}_{[g]}~, \qquad P_{g,Q}\,\widehat{\rho}_{A,g}(\gamma) = \rho_B(\gamma)\,P_{g,Q}~.$$

- For $\Gamma = SL(2,\mathbb{Z})$, one can reduce (up to scalar and double-coset equivalence) to $g = \mathrm{diag}(n,1)$. We can then check that this captures all of the relations introduced above.

Defect-decorated Zhou-Zhu operators

- **We can generalize further. Introduce a second vvmf**

$$\mathbb{F}(\gamma\tau) = \sigma(\gamma)\mathbb{F}(\tau)\ , \qquad \sigma : \Gamma \longrightarrow GL(W)\ .$$

- **Take a map $Q : W \otimes V_A \longrightarrow V_B$ satisfying**

$$Q\left(\sigma(\lambda)\otimes\rho_A(g\lambda g^{-1})\right) = \rho_B(\lambda)Q\ , \qquad \lambda \in \Gamma \cap g^{-1}\Gamma g\ .$$

- **Define**

$$\begin{aligned} \widehat{\boldsymbol{f}}_{[g],\mathbb{F}}(\tau) &:= \begin{pmatrix} \mathbb{F}(r_1\tau)\otimes \boldsymbol{f}(gr_1\tau) \\ \vdots \\ \mathbb{F}(r_m\tau)\otimes \boldsymbol{f}(gr_m\tau) \end{pmatrix}, \\ P_{[g],Q} &:= \left(\rho_B(r_1)^{-1}Q \,|\, \cdots \,|\, \rho_B(r_m)^{-1}Q\right)\ . \end{aligned}$$

- **Then we have the more general operator**

$$\mathsf{T}_{[g],\mathbb{F},Q}(\boldsymbol{f}) = P_{[g],Q}\widehat{\boldsymbol{f}}_{[g],\mathbb{F}}\ .$$

Defect-decorated Zhou-Zhu operators: Example

- **As an example, one obtains the $\mathrm{Spin}(18)_1$ characters from the characters of $(A_7)_1$ and the three $p=2$ images of $(A_1)_1$:**

$$
\begin{aligned}
\chi^{D_9}_{\mathbf{1}}(\tau) &= \left(\chi^{A_7}_{\mathbf{1}}(\tau) - \chi^{A_7}_{\mathbf{70}}(\tau)\right)\chi^{A_1}_{\mathbf{1}}(2\tau) \\
&\quad + \left(\chi^{A_7}_{\mathbf{8}}(\tau) + \chi^{A_7}_{\mathbf{56}}(\tau)\right)\chi^{A_1}_{\mathbf{1}}(\tau/2) \\
&\quad - e^{\pi i/24}\left(\chi^{A_7}_{\mathbf{8}}(\tau) - \chi^{A_7}_{\mathbf{56}}(\tau)\right)\chi^{A_1}_{\mathbf{1}}((\tau+1)/2)\ , \\
\chi^{D_9}_{\mathbf{18}}(\tau) &= \left(\chi^{A_7}_{\mathbf{1}}(\tau) - \chi^{A_7}_{\mathbf{70}}(\tau)\right)\chi^{A_1}_{\mathbf{2}}(2\tau) \\
&\quad + \left(\chi^{A_7}_{\mathbf{8}}(\tau) + \chi^{A_7}_{\mathbf{56}}(\tau)\right)\chi^{A_1}_{\mathbf{1}}(\tau/2) \\
&\quad + e^{\pi i/24}\left(\chi^{A_7}_{\mathbf{8}}(\tau) - \chi^{A_7}_{\mathbf{56}}(\tau)\right)\chi^{A_1}_{\mathbf{1}}((\tau+1)/2)\ , \\
\chi^{D_9}_{\mathbf{256}}(\tau) &= \left(\chi^{A_7}_{\mathbf{8}}(\tau) + \chi^{A_7}_{\mathbf{56}}(\tau)\right)\chi^{A_1}_{\mathbf{2}}(\tau/2) \\
&\quad + e^{19\pi i/24}\left(\chi^{A_7}_{\mathbf{8}}(\tau) - \chi^{A_7}_{\mathbf{56}}(\tau)\right)\chi^{A_1}_{\mathbf{2}}((\tau+1)/2)\ .
\end{aligned}
$$

Algebra of Zhou-Zhu operators

- **Thus far we took $\Gamma = SL(2,\mathbb{Z})$. Instead, consider $\Gamma = \Gamma(2)$ and introduce**

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ \tau \longmapsto \tau + 1 \ , \qquad W = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} : \ \tau \longmapsto -\frac{1}{2\tau} \ .$$

- **The double coset $\Gamma T \Gamma = \Gamma T$ defines a Zhou-Zhu operator,**

$$\mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[T]} f = f|_T \ .$$

- **The other double coset decomposes as**

$$\Gamma W \Gamma = \Gamma W \sqcup \Gamma T^{-1} W \ ,$$

and defines

$$\mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[W]} f = \frac{1}{\sqrt{2}} \left(f|_W + f|_{T^{-1}W} \right) \ .$$

Algebra of Zhou-Zhu operators

- **Although $T \notin \Gamma$, we have $T^2 \in \Gamma$. Hence the double coset $\Gamma T \Gamma$ defines an order-two Zhou-Zhu operator,**

$$\left(\mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[T]}\right)^2 = 1 \ .$$

- **Similarly, we find the following,**

$$\begin{aligned} \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[T]} \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[W]} &= \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[W]} \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[T]} = \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[W]} \ , \\ \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[W]} \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[W]} &= 1 + \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[T]} \ . \end{aligned}$$

Thus $\eta := \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[T]}$ and $\mathcal{N} := \mathsf{T}^{\varepsilon_1 \to \varepsilon_1}_{[W]}$ obey the Ising fusion algebra.

Algebra of Zhou-Zhu operators

- **Now consider the non-trivial multiplier**

$$\varepsilon_\eta(\gamma) = (-1)^{c/2} \ , \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2) \ .$$

- **We may define $\mathsf{T}_{[g]}^{\varepsilon_A \to \varepsilon_B}$ whenever**

$$\varepsilon_A(g\lambda g^{-1}) = \varepsilon_B(\lambda) \ , \qquad \lambda \in \Gamma \cap g^{-1}\Gamma g \ .$$

- **A simple computation shows that:**
  - For $g = T$, the source and target multipliers must agree.
  - For $g = W$, all source-target combinations are allowed.
- **Thus we have six operators:**

$$\mathsf{T}_{[T]}^{\varepsilon_1 \to \varepsilon_1} \ , \ \mathsf{T}_{[T]}^{\varepsilon_\eta \to \varepsilon_\eta} \ , \ \mathsf{T}_{[W]}^{\varepsilon_1 \to \varepsilon_1} \ , \ \mathsf{T}_{[W]}^{\varepsilon_1 \to \varepsilon_\eta} \ , \ \mathsf{T}_{[W]}^{\varepsilon_\eta \to \varepsilon_1} \ , \ \mathsf{T}_{[W]}^{\varepsilon_\eta \to \varepsilon_\eta} \ .$$

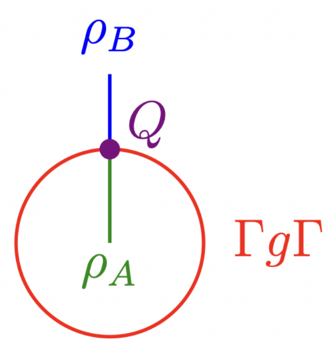
Algebra of Zhou-Zhu operators

- With products read from right to left, one finds e.g.

$$\begin{aligned} \mathsf{T}^{\varepsilon\eta\to\varepsilon\eta}_{[T]}\mathsf{T}^{\varepsilon 1\to\varepsilon\eta}_{[W]} &= \mathsf{T}^{\varepsilon 1\to\varepsilon\eta}_{[W]} \,, \\ \mathsf{T}^{\varepsilon 1\to\varepsilon\eta}_{[W]}\mathsf{T}^{\varepsilon\eta\to\varepsilon 1}_{[W]} &= 1 + \mathsf{T}^{\varepsilon\eta\to\varepsilon\eta}_{[T]} \,. \end{aligned}$$

- These are the 1- and η-twisted portions of the Ising tube algebra. The $\mathcal{N}$-twisted portions are similarly obtained by using two-dimensional vvmfs. The choice of Q corresponds to the choice of junction.

$$\mathsf{T}^{\rho_A\to\rho_B}_{g,Q} \;:$$



- This motivates generalization to $\mathsf{T}^{([g_1],\rho_1)\to([g_2],\rho_2)}_{([g_3],\rho_3);Q}$ where each ρ_i is a representation of $\Gamma\cap g_i^{-1}\Gamma g_i$. It can easily be worked out.

Algebra of Zhou-Zhu operators

- The origin of this structure is roughly as follows. Consider $\Gamma(2) \subset \Gamma_1^*(2)$ and try to "gauge" it. To do so, note that the following group

$$K = \bigcap_{x \in \Gamma_1^*(2)} x\Gamma(2)x^{-1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2) \mid c \equiv 0 \mod 4 \right\}$$

leaves each gr_i invariant, and we have the following

$$\Gamma(2)/K \cong \mathbb{Z}_2 \ , \qquad \Gamma_1^*(2)/K \cong D_8 \ .$$

- Thus the double cosets are given by

$$\Gamma(2) \backslash \Gamma_1^*(2) / \Gamma(2) \ \cong \ \mathbb{Z}_2 \backslash D_8 / \mathbb{Z}_2 \ .$$

As complex algebras, we have $\mathbb{Z}_2 \backslash D_8 / \mathbb{Z}_2 = \text{Ising fusion ring}$.

- As for little groups, both 1 and η cosets have $\mathbb{Z}_2$ little group, so they correspond to two elements (trivial vs. sign rep.) in the $\mathbb{Z}_2$-gauged D_8 theory. The sign rep. pulls back to ε_η.

Eigenforms of the Zhou-Zhu algebra

- **Explicit irreps can be written in terms of the Eisenstein series E_4,**

$$\begin{aligned} f_1(\tau) &= E_4(\tau) + 4E_4(2\tau)\ , \\ f_\epsilon(\tau) &= E_4(\tau) - 4E_4(2\tau)\ , \\ f_\sigma(\tau) &= E_4(\tau/2) - E_4((\tau+1)/2)\ , \\ f_\mu(\tau) &= 64E_4(4\tau) - 4E_4(\tau+1/2)\ . \end{aligned}$$

- **Then we find that**

$$\begin{array}{lll} \mathsf{T}^{\varepsilon 1\to\varepsilon 1}_{[T]} f_1 = f_1\ , & \mathsf{T}^{\varepsilon 1\to\varepsilon 1}_{[W]} f_1 = \sqrt{2} f_1\ , & \mathsf{T}^{\varepsilon 1\to\varepsilon \eta}_{[W]} f_1 = 0\ , \\ \mathsf{T}^{\varepsilon 1\to\varepsilon 1}_{[T]} f_\epsilon = f_\epsilon\ , & \mathsf{T}^{\varepsilon 1\to\varepsilon 1}_{[W]} f_\epsilon = -\sqrt{2} f_\epsilon\ , & \mathsf{T}^{\varepsilon 1\to\varepsilon \eta}_{[W]} f_\epsilon = 0\ . \\ \mathsf{T}^{\varepsilon 1\to\varepsilon 1}_{[T]} f_\sigma = -f_\sigma\ , & \mathsf{T}^{\varepsilon 1\to\varepsilon 1}_{[W]} f_\sigma = 0\ , & \mathsf{T}^{\varepsilon 1\to\varepsilon \eta}_{[W]} f_\sigma = \sqrt{2} f_\mu\ , \\ \mathsf{T}^{\varepsilon \eta\to\varepsilon \eta}_{[T]} f_\mu = f_\mu\ , & \mathsf{T}^{\varepsilon \eta\to\varepsilon 1}_{[W]} f_\mu = \sqrt{2} f_\sigma\ , & \mathsf{T}^{\varepsilon \eta\to\varepsilon \eta}_{[W]} f_\mu = 0\ . \end{array}$$

Non-invertible vvmfs

- This is a non-invertible analog of a well-known construction. For example, we have

$$\mathrm{SL}(2,\mathbb{Z})/\Gamma(2) \cong S_3$$

so we can easily construct a vector of modular forms transforming in a representation of S_3, e.g.

$$(\theta_2^8(\tau), \theta_3^8(\tau), \theta_4^8(\tau)) \ .$$

- Similar things can be done using

$$\mathrm{SL}(2,\mathbb{Z})/\pm\Gamma(3) \cong A_4 \ , \quad \mathrm{SL}(2,\mathbb{Z})/\pm\Gamma(4) \cong S_4 \ , \quad \mathrm{SL}(2,\mathbb{Z})/\pm\Gamma(5) \cong A_5 \ .$$

- In the current case, we begin with the Fricke group $\Gamma_1^*(N)$ and quotient by $\pm\Gamma(N)$. However, $\pm\Gamma(N)$ is not normal in $\Gamma_1^*(N)$, so we must consider the double-coset algebra

$$\pm\Gamma(N)\backslash\Gamma_1^*(N)/\pm\Gamma(N) \ \cong \ \mathrm{TY}(\mathbb{Z}_N)\text{ fusion ring} \ .$$

Conclusions

- **Zhou-Zhu operators and their generalizations give rise to a large number of relations between RCFT characters.**

- **For $\Gamma = \pm\Gamma(N)$, the algebra of a certain subsector of Zhou-Zhu operators reproduces the tube algebra of $\mathrm{TY}(\mathbb{Z}_N)$.**

- **What can we use this for?**
  - Applications to modular flavor symmetry? (discussions with T. Kai, K. Kina, H. Otsuka)
  - Useful for spurion analysis?
  - Extension to MTCs? (see also [Harvey-Hu-Wu '19])

The End (for now)

Thank you!