Exact Large *N* Expansion of Partition function of 3d Superconformal Chern-Simons Theories

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Introduction and Motivation

- $U(N)_k$ superconformal quiver Chern-Simons theory

  = worldvolume theory of *N* M2-branes on $Y_7 \times \mathbb{R}_+$

- example: ABJM theory $\longrightarrow$ $Y_7 = S^7/\mathbb{Z}_k$

- By localization technique,

$$Z(N) \approx \exp\left[-N^{3/2}\sqrt{\frac{2\pi^6}{27vol(Y_7)}}\right] \quad (N \to \infty)$$

[Herzog-Klebanov-Pufu-Tesileanu]

: Consistent with AdS/CFT (duality to $\mathrm{AdS}_4 \times Y_7$)

Beyond Large *N* limit ?

= Beyond classical SUGRA, **M-theory** (*k*: fixed)

- perturbative corrections in $1/N$
- non-perturbative effects: $e^{-\sqrt{N/k}}, e^{-\sqrt{kN}}$

fundamental ***Membrane*** winding Y_7

(exponent∼vol(Membrane))

[Becker-Becker-Strominger]

- 't Hooft expansion ($\lambda = N/k$) is not enough.

Fermi Gas formalism

$Z(N)$ = "partition function of 1d quantum stat. system"

[Marino-Putrov]

- In r.h.s, $\hbar \sim k$

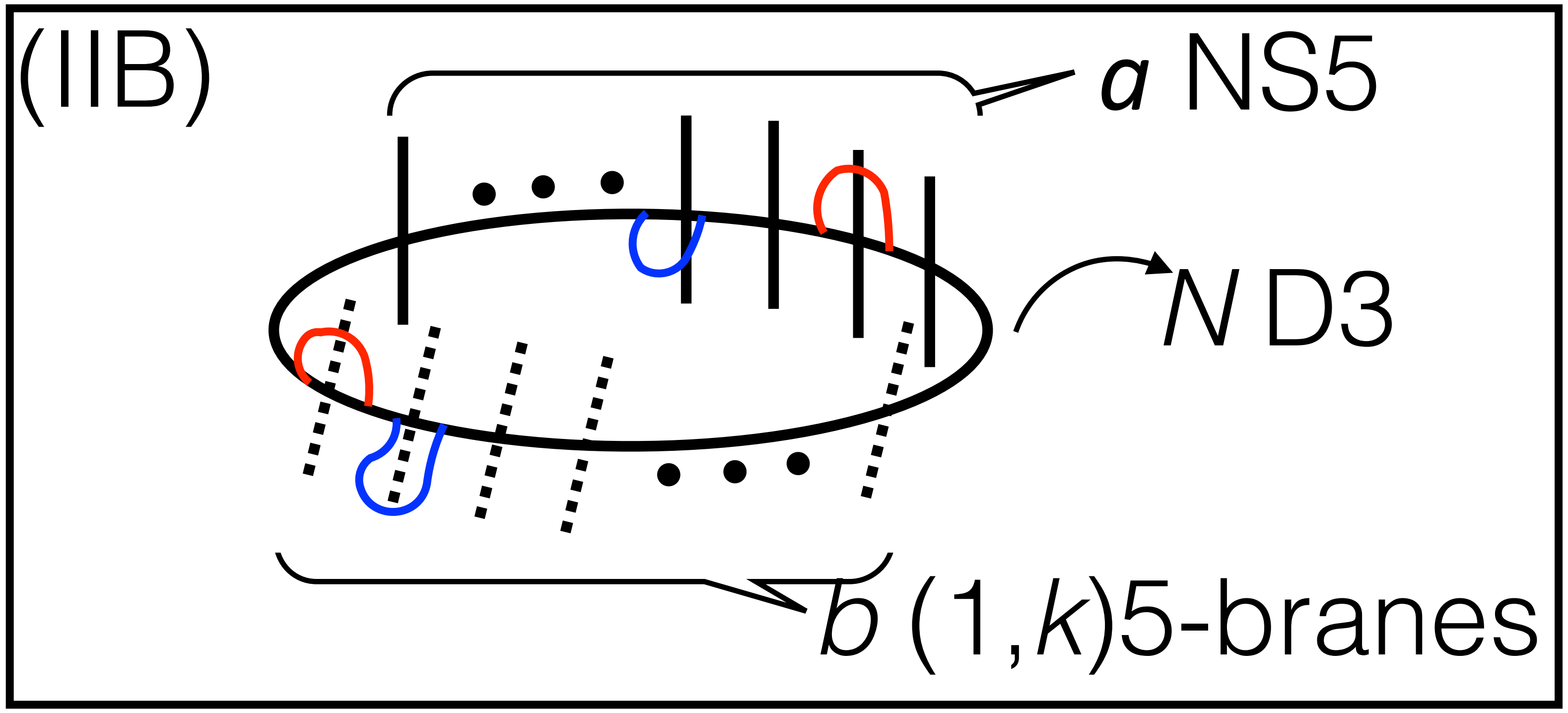
$k \to 0$ is ***easy*** !

: complement to 't Hooft exp.

Results :

- **All order** perturbative corrections in $1/N$
- "membrane Instantons": $e^{-\sqrt{kN}}$
  - reflecting finer structure of Y_7

M2-brane setup



→ CS gauge fields
+
bifund. matter fields

T-dual + M-lift

(D3→M2
5-branes→KK monopole)

N M2-branes on $(\mathbb{C}^2/\mathbb{Z}_a \times \mathbb{C}^2/\mathbb{Z}_b)/\mathbb{Z}_k$ [Imamura-Kimura]

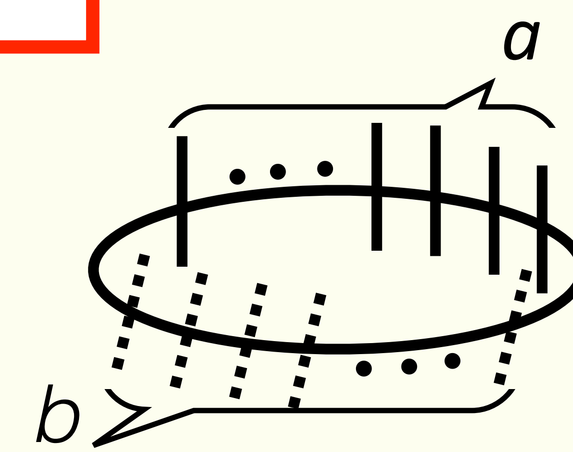
➡ membrane instantons: $e^{-\bigcirc\sqrt{kN}/a}, e^{-\bigcirc\sqrt{kN}/b}$

Fermi Gas formalism for Partition Function

$$J(\mu) \equiv \sum_{N\geq 0} e^{\mu N} Z(N) = \mathrm{Tr}\log(1+e^{\mu}\widehat{\rho})$$

with one-particle density matrix

$$\widehat{\rho} = \left(2\cosh\frac{\widehat{Q}}{2}\right)^{-a}\left(2\cosh\frac{\widehat{P}}{2}\right)^{-b}$$



- Inverse trsf.: $Z(N) = \int \frac{d\mu}{2\pi i} e^{J-\mu N}$

  To get $Z(N) \approx e^{-\bigcirc\sqrt{k}N^{3/2}}$ by saddle approx., $\mu \sim \sqrt{kN}$. ➡ $N \gg 1 \leftrightarrow \mu \gg 1$

- $[\widehat{Q},\widehat{P}] = i\hbar \quad (\hbar = 2\pi k)$

  ➡ $J(\mu)$ can be computed by ***small k-expansion***

$k \to 0$: "classical limit"

$$J(\mu) = \mathrm{Tr}\log\left[1 + e^{\mu}\frac{1}{\left(2\cosh\frac{\widehat{Q}}{2}\right)^{a}\left(\cosh\frac{\widehat{P}}{2}\right)^{b}}\right]$$

$$J(\mu) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{n\mu}\int\frac{dQdP}{2\pi\hbar}\frac{1}{\left(2\cosh\frac{Q}{2}\right)^{na}\left(2\cosh\frac{P}{2}\right)^{nb}}$$

$$= \frac{\Gamma\left(\frac{na}{2}\right)^2\Gamma\left(\frac{nb}{2}\right)^2}{\Gamma(na)\Gamma(nb)}$$

large μ expansion

$$J(\mu) = \frac{C}{3}\mu^3 + B\mu + A + \mathcal{O}(e^{-\mu})$$

$$Z^{\mathrm{perturb}}(N)$$

$$= e^{A}C^{-\frac{1}{3}}\mathrm{Ai}[C^{-\frac{1}{3}}(N-B)]$$

: **All order** perturbative

$$e^{-\sqrt{kN}}$$

: membrane instantons

Explicit result

$$
\begin{aligned}
J(\mu) = J^{\mathrm{pert}} + \frac{1}{\hbar}\Bigg[&\sum_{m_1\geq 1} \binom{2m_1}{m_1} \frac{1}{m_1 \sin\frac{2\pi m_1}{a}} \frac{\Gamma\left(-\frac{bm_1}{a}\right)^2}{\Gamma\left(-\frac{2bm_1}{a}\right)} e^{-\frac{2m_1\mu}{a}} \\
&+ \sum_{m_2\geq 1} \binom{2m_2}{m_2} \frac{1}{m_2 \sin\frac{2\pi m_2}{b}} \frac{\Gamma\left(-\frac{am_2}{b}\right)^2}{\Gamma\left(-\frac{2am_2}{b}\right)} e^{-\frac{2m_2\mu}{b}} \\
&+ \sum_{m_3\geq 1} \frac{(-1)^{m_3-1}}{2\pi m_3} \frac{\Gamma\left(-\frac{am_3}{2}\right)^2 \Gamma\left(-\frac{bm_3}{2}\right)^2}{\Gamma(-am_3)\Gamma(-bm_3)} e^{-m_3\mu} \Bigg]
\end{aligned}
$$

[Moriyama-TN]

- coefficients have ***poles*** in $a, b \in \mathbb{N}$
- However, a divergence is always canceled by another instanton (consider $\lim_{\epsilon\to 0} J(a+\epsilon, b+\epsilon)$)

Mysterious instanton $e^{-\mu}$

instanton = Membrane winding on $S^7/\Gamma_{a,b}$

$(\mathbb{C}^4/\Gamma_{a,b} = (\mathbb{C}^2/\mathbb{Z}_a \times \mathbb{C}^2/\mathbb{Z}_b)/\mathbb{Z}_k)$

winding in :

$$\sum_{m_1 \geq 1} \binom{2m_1}{m_1} \frac{1}{m_1 \sin \frac{2\pi m_1}{a}} \frac{\Gamma\left(-\frac{bm_1}{a}\right)^2}{\Gamma\left(-\frac{2bm_1}{a}\right)} e^{-\frac{2m_1\mu}{a}} \quad \Rightarrow \quad S^1/\mathbb{Z}_a$$

$$+ \sum_{m_2 \geq 1} \binom{2m_2}{m_2} \frac{1}{m_2 \sin \frac{2\pi m_2}{b}} \frac{\Gamma\left(-\frac{am_2}{b}\right)^2}{\Gamma\left(-\frac{2am_2}{b}\right)} e^{-\frac{2m_2\mu}{b}} \quad \Rightarrow \quad S^1/\mathbb{Z}_b$$

$$+ \sum_{m_3 \geq 1} \frac{(-1)^{m_3-1}}{2\pi m_3} \frac{\Gamma\left(-\frac{am_3}{2}\right)^2 \Gamma\left(-\frac{bm_3}{2}\right)^2}{\Gamma(-am_3)\Gamma(-bm_3)} e^{-m_3\mu} \quad \Rightarrow \quad ???$$

- ***Never appears*** with distinct exponent if $(a, b) \in \mathbb{N}^2$
- ***Required*** for pole cancellations

Summary

- M2 partition function has rich structure in M-theoretical region $(k < \infty)$

- $Z^{\rm pert}(N) = e^A C^{-\frac{1}{3}} {\rm Ai}[C^{-\frac{1}{3}}(N-B)]$

  → All order in 1/*N*. coefficients explicitly determined

- three membrane instantons: $e^{-2\mu/a}, e^{-2\mu/b}, e^{-\mu}$

  → divergent, pole cancellation

But less understood in gravity side:

- $N-B$ → shift in charge quantization ?
- interpretation of inst. coefs ; why diverges ?
- "*ghost-like*" $e^{-\mu}$?

Further directions

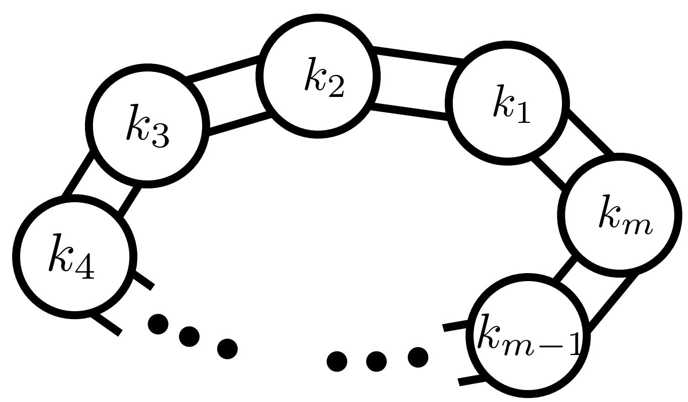
① Exact partition function for ***finite k*** ?

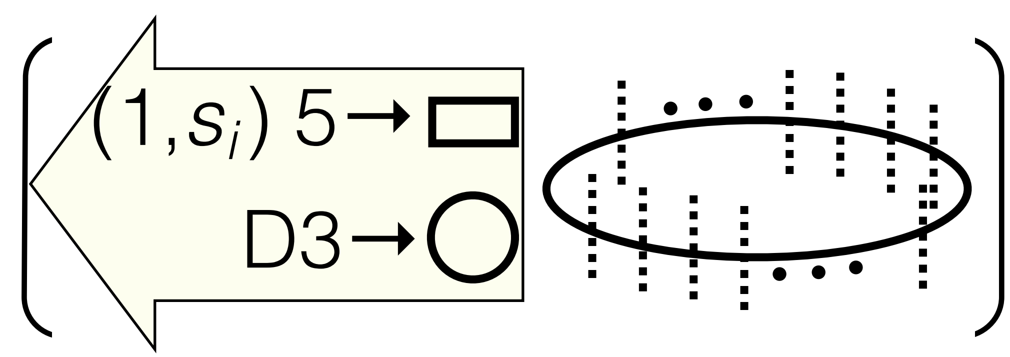
- Direct tools was developed [Moriyama-TN, 1412]

② Prop: M2 worldvolume theory $\Longrightarrow$ Fermi Gas ?

Checked:

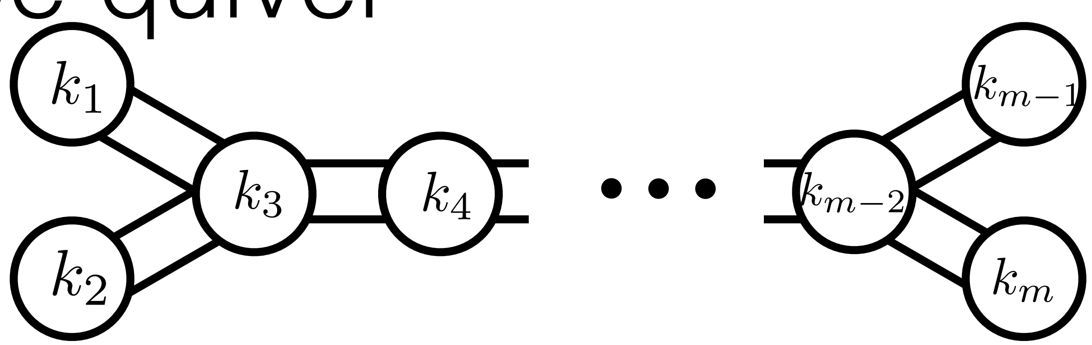
- $\widehat{A}$ -type quiver





[Marino-Putrov]

- $\widehat{D}$ -type quiver



[Assel-Drukker-Felix]
[Moriyama-TN, 1504]