DERIVATIVE-*DEPENDENT* METRIC TRANSFORMATIONS AND SCALAR-TENSOR THEORIES

新学術領域「なぜ宇宙は加速するのか？
-徹底的究明と将来への挑戦-」発足シンポジウム
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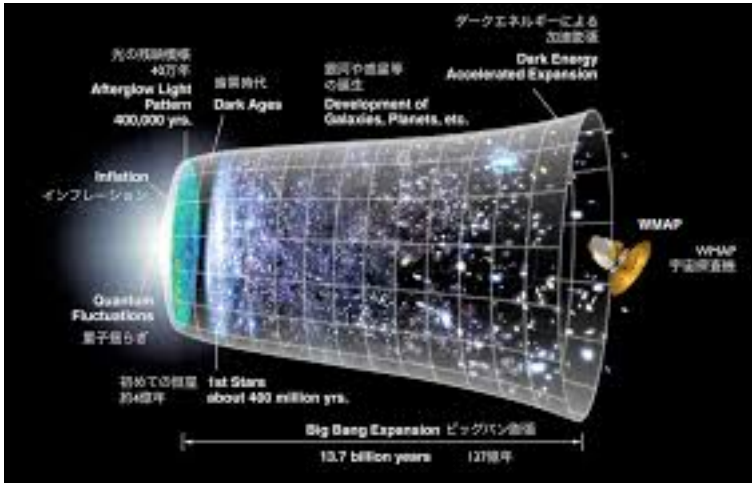
Talk plan

1. Introduction and motivation
2. Derivative-dependent transformation: Simple example
3. General analysis
4. Summary

What is the universe accelerated by?

- Cosmological constant
- Graviton mass
- Matter condensation
- Unknown scalar field(s)
- …



Horndeski theory Horndeski '74, Nicolis et al & Deffayet et al '09, Kobayashi et al '11

- Most-general single scalar field and gravity theory, which field equations contain derivatives only up to the second order

$$S=\sum_{i=2}^{5}\int d^4x\sqrt{-g}\mathcal{L}_i \;0 \quad \longrightarrow \quad f(\ddot{g}_{\mu\nu},\dot{g}_{\mu\nu},g_{\mu\nu},\ddot{\phi},\dot{\phi},\phi)=0$$

$$\mathcal{L}_2=K(\phi,X)$$

$$\mathcal{L}_3=-G_3(\phi,X)\Box\phi$$

$$\mathcal{L}_4=G_4(\phi,X)R+G_{4X}\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right],$$

$$\mathcal{L}_5=G_5(\phi,X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi-\frac{G_{5X}}{6}\left[(\Box\phi)^3-3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2+2(\nabla_\mu\nabla_\nu\phi)^3\right]$$

$$X=-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

Horndeski theory doesn't suffer from the Ostrogradski's theorem

Ostrogradski's theorem Ostrogradski (1850), Woodard '07

- "If the higher order time derivative Lagrangian is non-degenerate, there is at least one linear instability in the Hamiltonian of this system"

- example

$$L = \frac{1}{2}R^2 + S(R - \ddot{Q}) \longrightarrow \frac{d^4Q}{dt^4} = 0$$

$$H_{phys} = \boxed{P_Q(P_S - P_R)} + S^2 + SR + \frac{1}{2}R^2$$

$$P_R \approx 0, \ \ R + S \approx 0$$

An extra d.o.f appears!

# of degree of freedom = (6-2)/2 = 2 = 1 **+ 1**

Horndeski theory

- Most-general single scalar field and gravity theory, which field equations contain derivatives only up to the second order

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0, \ \frac{\delta S}{\delta \phi} = 0 \quad \Longrightarrow \quad f(\ddot{g}_{\mu\nu}, \dot{g}_{\mu\nu}, g_{\mu\nu}, \ddot{\phi}, \dot{\phi}, \phi) = 0$$

No extra d.o.f appear.

Horndeski theory doesn't suffer from the *Ostrogradski's theorem*

Counter-example(?) of the theorem

- Derivative−***dependent*** metric transformation

$$\bar{g}_{\mu\nu} = \Omega^2(X,\phi) g_{\mu\nu}$$

Bekenstein '93,
Zumalacarregui et al '14

$$\mathcal{L}_E = \frac{\sqrt{-\bar{g}}}{16\pi G}\bar{R}[\bar{g}] + \sqrt{-g}(\mathcal{L}_\phi(g_{\mu\nu},\phi) + \mathcal{L}_m(g_{\mu\nu},\phi))$$

$$\Downarrow$$

$$\mathcal{L}_C = \frac{\sqrt{-g}}{16\pi G}(\Omega^2 R + \underline{6\Omega_{,\alpha}\Omega^{,\alpha}}) + \sqrt{-g}(\mathcal{L}_\phi + \mathcal{L}_m)$$

$$\boxed{\text{Explicitly beyond Horndeski term appears!}}$$

Field equations in Jordan frame

- 4th order term by beyond Horndeski

$$\Omega^2 G_{\mu\nu} + 2\Omega(g_{\mu\nu}\Box\Omega - \Omega_{;\mu\nu}) + (6\Box\Omega - \Omega R)\Omega_{,X}\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\Omega_{,\alpha}\Omega^{,\alpha} + 4\Omega_{,\mu}\Omega_{,\nu} = 8\pi G(T^{\phi}_{\mu\nu} + T^{m}_{\mu\nu}), \quad (41)$$

$$\nabla_\mu(\Omega_{,X}\phi^{,\mu}(\Omega R - 6\Box\Omega)) + \Omega_{,\phi}(\Omega R - 6\Box\Omega) + \frac{1}{2}\frac{\delta\mathcal{L}_\phi}{\delta\phi} = 0,$$

Substitute it back into EoMs

- Trace of (41)

$$(6\Box\Omega - \Omega R)(\Omega - 2\Omega_{,X}X) = 8\pi GT$$

Field equations in Jordan frame

- 4th order term by beyond Horndeski

$$\Omega^2 G_{\mu\nu} + 2\Omega(g_{\mu\nu}\Box\Omega - \Omega_{;\mu\nu}) + T_{\mathrm{K}}\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\Omega_{,\alpha}\Omega^{,\alpha}$$
$$+ 4\Omega_{,\mu}\Omega_{,\nu} = 8\pi G T^{\mathrm{tot}}_{\mu\nu}, \qquad \text{l)}$$

$$\nabla_\mu(\phi^{,\mu} T_{\mathrm{K}}) + \frac{\Omega_{,\phi}}{\Omega_{,X}} T_{\mathrm{K}} - \frac{1}{2}\frac{\delta\mathcal{L}_\phi}{\delta\phi} = 0. \qquad 0,$$

- Trace o

Reduced to the 2nd order EoMs

Short summary of transformed theory

$$\mathcal{L}_C = \frac{\sqrt{-g}}{16\pi G}(\Omega^2 R + 6\Omega_{,\alpha}\Omega^{,\alpha}) + \sqrt{-g}(\mathcal{L}_\phi + \mathcal{L}_m)$$

Fact	Expectation
EoMs are higher order than the 2nd order, and the theory will suffer from the Ostrogradski's theorem	Linear instability of the Hamiltonian Extra degrees of freedom
EoMs could be reduced to the 2nd order by the trace of field equation	Counter-example of the theorem and stable New scalar-tensor theory??

2. Derivative-dependent transformation

- Conformal + Disformal transformation   Bekenstein '93

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \mathcal{A}(\phi, X) g_{\mu\nu} + \mathcal{B}(\phi, X) \partial_\mu \phi \partial_\nu \phi$$

$$\det g \neq 0 \ : \quad \mathcal{A}(\mathcal{A} - \mathcal{B}X) \neq 0$$

$$\det \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} \neq 0 \ : \ \mathcal{A}(\mathcal{A} - \mathcal{A}_X X + \mathcal{B}_X X^2) \neq 0$$

- Transforming E-H + K-essence in tilde system

$$\tilde{I}_{\rm total} = \tilde{I}_{\rm EH} + \int d^{d+1}x \sqrt{-\tilde{g}}\, \tilde{P}(\phi, \tilde{X})$$

$$\tilde{I}_{total}[\tilde{g}_{\mu\nu}, \phi] = I_{total}[g_{\mu\nu}, \phi]$$

- Transformed action in general gauge

$$\tilde{I}_{\rm EH} = \frac{M_p^2}{2} \int d^{d+1}x \sqrt{-g}\, \mathcal{A}^{(d-1)/2} \left(1 - \frac{\mathcal{B}}{\mathcal{A}} X\right)^{1/2}$$

$$\times \left\{ R + \frac{\mathcal{B}}{\mathcal{A} - \mathcal{B}X} \left[\nabla_\mu \nabla_\nu \phi\, \nabla^\mu \nabla^\nu \phi - \left(\nabla^2 \phi\right)^2\right] + \frac{d(d-1)}{4} \left[\nabla_\mu \ln \mathcal{A}\, \nabla^\mu \ln \mathcal{A} - \frac{\mathcal{B}}{\mathcal{A} - \mathcal{B}X} \left(\nabla^\mu \phi\, \nabla_\mu \ln \mathcal{A}\right)^2\right] \right.$$

$$- \frac{d-1}{2} \frac{\mathcal{B}}{\mathcal{A} - \mathcal{B}X} \nabla_\mu \ln \mathcal{A} \left[X \nabla^\mu \ln\left(\frac{\mathcal{B}}{\mathcal{A}}\right) + \nabla^\mu \phi\, \nabla^\nu \phi\, \nabla_\nu \ln\left(\frac{\mathcal{B}}{\mathcal{A}}\right)\right]$$

$$\left. - \frac{\mathcal{B}}{\mathcal{A} - \mathcal{B}X} \left(\frac{1}{2} \nabla^\mu X + \nabla^\mu \phi\, \nabla^2 \phi\right) \left[(d-1)\, \nabla_\mu \ln \mathcal{A} + \nabla_\mu \ln\left(\frac{\mathcal{B}}{\mathcal{A}}\right)\right] \right\}, \tag{11}$$

Many beyond Horndeski terms!

Hamiltonian analysis in Unitary gauge

- Unitary gauge

$$\phi = t$$

- ADM variables

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

- Transformation

$$\tilde{N}^2 = \mathcal{A}N^2 - \mathcal{B}, \quad \tilde{N}^i = N^i, \quad \tilde{\gamma}_{ij} = \mathcal{A}\gamma_{ij}$$

$$\mathcal{A} = \mathcal{A}(t, N),\ \mathcal{B} = \mathcal{B}(t, N)$$

- Action in unitary gauge

$$\tilde{I}^{\text{unitary}}_{\text{total}} = \tilde{I}^{\text{unitary}}_{\text{EH}} + \int dt d^d x \, N\sqrt{\gamma}\, A_2(t,N)$$

$$\tilde{I}^{\text{unitary}}_{\text{EH}} = \int dt\, d^d x N\sqrt{\gamma}\left[A_4(t,N)\left(\underline{K^2 - K^i{}_j K^j{}_i + (d-1)KL + \frac{d(d-1)}{4}L^2}\right) - U(t,N,\gamma)\right]$$

$$K_{ij} \equiv \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right) \ , \quad K \equiv \gamma^{ij} K_{ij}$$

$$\boxed{L \equiv \frac{\mathcal{A}_N}{\mathcal{A}}\left(\frac{\dot{N}}{N} - \frac{N^i}{N} D_i N\right) + \frac{\mathcal{A}_t}{\mathcal{A}N}} \, ,$$

$$U(t,N,\gamma) \equiv -B_4(t,N)\left[R^{(d)} - (d-1)D^2 \ln\mathcal{A} - \frac{(d-1)(d-2)}{4} D_i \ln\mathcal{A}\, D^i \ln\mathcal{A}\right]$$

$$A_4(t,N) \equiv -\frac{M_p^2}{2}\frac{N\mathcal{A}^{d/2}}{\sqrt{\mathcal{A}N^2 - \mathcal{B}}} \, ,$$

$$B_4(t,N) \equiv \frac{M_p^2}{2N}\mathcal{A}^{(d-2)/2}\sqrt{\mathcal{A}N^2 - \mathcal{B}} \, ,$$

The Hamiltonian & The Constraints

- Hamiltonian

$$\pi_\Phi = \frac{\delta I_{total}}{\delta \dot{\Phi}}$$

$$H = \int d^d x \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_N \dot{N} - \mathcal{L}\right) = \int d^d x \left(\mathcal{H}_\perp + N^i \mathcal{H}_i^N\right)$$

$$\mathcal{H}_\perp = -N\sqrt{\gamma}\left[\frac{1}{A_4}\left(\frac{\pi^{ij}\pi_{ij}}{\gamma} - \frac{1}{d-1}\frac{\pi^2}{\gamma}\right) + \frac{\mathcal{A}_t}{N\mathcal{A}}\frac{\pi}{\sqrt{\gamma}} + A_2 - U(t, N, \gamma)\right]$$

$$\mathcal{H}_i^N = \mathcal{H}_i + \pi_N D_i N \ ,$$

$$\mathcal{H}_i \equiv -2\sqrt{\gamma} D_j \left(\frac{\pi^j{}_i}{\sqrt{\gamma}}\right) \ ,$$

- Constraints

$$\pi_i \approx 0\ , \quad \pi_N - \frac{\mathcal{A}_N}{\mathcal{A}}\pi \equiv \tilde{\pi}_N \approx 0$$

$$\dot{\pi}_i(x) \approx \{\pi_i(x), H'\}_P = -\mathcal{H}_i^N(x) \approx 0$$

$$\dot{\tilde{\pi}}_N(x) \approx \frac{\partial}{\partial t}\tilde{\pi}_N(x) + \{\tilde{\pi}_N(x), H'\}_P \equiv \mathcal{C} \approx 0$$

$$\mathcal{C} = \sqrt{\gamma}D_i\left(N^i\frac{\tilde{\pi}_N}{\sqrt{\gamma}}\right) + \frac{1}{\sqrt{\gamma}}\left[\left(\frac{N}{A_4}\right)_N + \frac{d\mathcal{A}_N N}{2\mathcal{A}A_4}\right]\left(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2\right) + \mathcal{C}_U[t, N, \gamma, A_{2N}, B_{4N}]$$

$$\mathcal{C}_U = \left(\frac{\delta}{\delta N(x)} - \frac{\mathcal{A}_N}{\mathcal{A}}\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}(x)}\right)\int d^d y N\sqrt{\gamma}\,\left(A_2 - U(t, N, \gamma)\right)\ .$$

$$\text{1st class}: \mathcal{H}_i^N,\ \pi_i \qquad \text{2nd class}: \tilde{\pi}_N,\ \mathcal{C}$$

- # of degrees of freedom

No extra d.o.f. appear!

$$\# = \frac{1}{2}(20 - 6 \times 2 - 2) = 3 = 2 + 1$$

- Constraints

$$\pi_i \approx 0\,, \quad \pi_N - \frac{\mathcal{A}_N}{\mathcal{A}}\pi \equiv \tilde{\pi}_N \approx 0$$

$$\dot{\pi}_i(x) \approx \{\pi_i(x), H'\}_P = -\mathcal{H}_i^N(x) \approx 0$$

$$\dot{\tilde{\pi}}_N(x) \approx \frac{\partial}{\partial t}\tilde{\pi}_N(x) + \{\tilde{\pi}_N(x), H'\}_P \equiv \mathcal{C} \approx 0$$

~~New scalar-tensor theory?~~

$$\text{1st class} : \mathcal{H}_i^N,\ \pi_i \qquad \text{2nd class} : \tilde{\pi}_N,\ \mathcal{C}$$

- # of degrees of freedom

No extra d.o.f. appear!

$$\# = \frac{1}{2}(20 - 6\times 2 - 2) = 3 = 2 + 1$$

3. General analysis

- Consider a scalar-tensor theory which contains up to m-th order g's derivatives and up to n-th order φ's derivatives:

$$
\begin{aligned}
I' =& \int d^{d+1}x \, \Big[L(g_{\mu\nu}, \mathcal{R}_{\alpha\beta\gamma\delta}, \mathcal{R}_{\mu\alpha\beta\gamma\delta}, \cdots, \mathcal{R}_{\mu_1\cdots\mu_m\alpha\beta\gamma\delta}, \phi, \phi_\mu, \cdots, \phi_{\mu_1\cdots\mu_n}) + \Lambda^{\alpha\beta\gamma\delta}(\mathcal{R}_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta}) \\
&+ \Lambda^{\mu\alpha\beta\gamma\delta}(\mathcal{R}_{\mu\alpha\beta\gamma\delta} - \nabla_\mu \mathcal{R}_{\alpha\beta\gamma\delta}) + \cdots + \Lambda^{\mu_1\cdots\mu_m\alpha\beta\gamma\delta}(\mathcal{R}_{\mu_1\cdots\mu_m\alpha\beta\gamma\delta} - \nabla_{(\mu_m}\mathcal{R}_{\mu_1\cdots\mu_{m-1})\alpha\beta\gamma\delta}) \\
&+ \lambda^\mu(\phi_\mu - \nabla_\mu\phi) + \cdots + \lambda^{\mu_1\cdots\mu_n}(\phi_{\mu_1\cdots\mu_n} - \nabla_{(\mu_m}\phi_{\mu_1\cdots\mu_{m-1})})\Big] \, ,
\end{aligned}
$$

- The action can be cast into the form

$$
I = \int d^{d+1}x \left[\frac{1}{2}\mathcal{K}_{AB}\dot{\Phi}^A\dot{\Phi}^B + M_A\dot{\Phi}^A - V\right]
$$

$$
\tilde{I}_{\rm EH}^{\rm unitary} = \int dt\, d^dx N\sqrt{\gamma}\left[A_4(t,N)\left(K^2 - K^i{}_j K^j{}_i + (d-1)KL + \frac{d(d-1)}{4}L^2\right) - U(t,N,\gamma)\right]
$$

- Transformation would be also cast into the derivative-*independent* form if it is regular:

$$\tilde{\Phi}^A = F^A(\Phi, t), \quad (A = 1, 2, \cdots, \mathcal{N})$$

$$\det F^A_B \neq 0, \ \infty \qquad F^A_B = \frac{\partial F^A}{\partial \Phi^B}$$

$$\tilde{N}^2 = \mathcal{A} N^2 - \mathcal{B}, \quad \tilde{N}^i = N^i, \quad \tilde{\gamma}_{ij} = \mathcal{A}\gamma_{ij}$$

This transformation is a point transformation included in canonical transformation as long as F^A is regular. So, the physics in the two different frame should be the same.

- We can easily find the generator by comparing the Hamiltonians in the different frames

$$\tilde{H} = H + \frac{\partial \mathcal{G}}{\partial t} \qquad \mathcal{G}[\Pi, \tilde{\Phi}; t] = -\int d^d x \, \Pi_A \, G^A(\tilde{\Phi}, t),$$

$$\Phi^A = G^A(\tilde{\Phi}, t), \quad (A = 1, 2, \cdots, \mathcal{N})$$

$$\{\Phi^A(\vec{x}), \Phi^B(\vec{y})\}_P = 0, \quad \{\Phi^A(\vec{x}), \Pi_B(\vec{y})\}_P = \delta^A_B \delta^3(\vec{x} - \vec{y}), \quad \{\Pi_A(\vec{x}), \Pi_B(\vec{y})\}_P = 0.$$

$$\left\{\tilde{\Phi}^A(\vec{x}), \tilde{\Phi}^B(\vec{y})\right\}_P = 0, \quad \left\{\tilde{\Phi}^A(\vec{x}), \tilde{\Pi}_B(\vec{y})\right\}_P = \delta^A_B \delta^3(\vec{x} - \vec{y}), \quad \left\{\tilde{\Pi}_A(\vec{x}), \tilde{\Pi}_B(\vec{y})\right\}_P = 0,$$

We would never find new scalar-tensor theories as long as we consider the regular transformation, although the transformed theory may have quite non-trivial beyond Horndeski terms.

4. Summary

- Derivative-dependent transformation, though it can create beyond Horndeski term, does not make any new scalar-tensor theory from known theories as long as the transformation is invertible and regular.

$$\mathcal{A}(\mathcal{A}-\mathcal{B}X)(\mathcal{A}-\mathcal{A}_X X+\mathcal{B}_X X^2)\neq 0$$

- In unitary gauge, regular derivative-dependent trf. reduces to point trf. included in the canonical trf.

- The result looks very non-trivial in general gauge, but quite natural in unitary gauge.

- *Singular* transformation can create some new scalar-tensor theories (ex. mimetic DM Chamseddine et al'13)

$$\tilde{g}_{\mu\nu}=Xg_{\mu\nu}$$