Manifest Resurgence Structure in Sine-Gordon and CPN models

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Path integral & Saddle Points

$$Z = \int \mathcal{D} \exp(-S[\phi]) = \sum_{\sigma \in \{\text{saddle}\}} Z_\sigma$$

trivial saddle (pert. vacuum)

$$Z_0 = \sum_{q=0}^{\infty} a_q g^q$$

perturbative series

non-trivial saddle
(e.g. instanton) $\frac{\delta S}{\delta \phi} = 0$

$$e^{-S_{\rm sol}} \sim e^{-\frac{A}{g^2}}$$

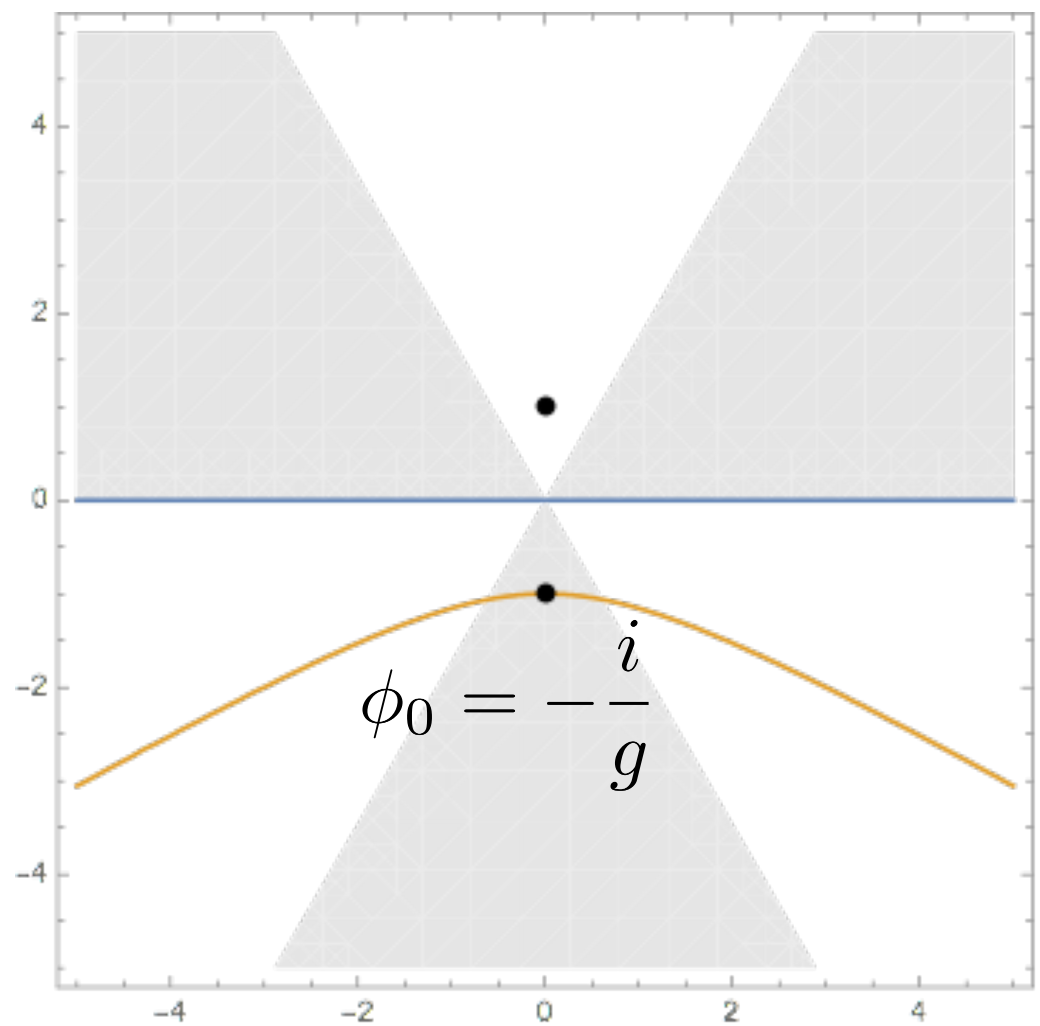
non-perturbative contributions

Complex Saddle Points needed

Saddle points out of original integration path can also contribute to the integral

- **Airy integral**

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$
$$= \sqrt{\frac{g}{4\pi}} \exp\left(-\frac{2}{3g^2}\right)$$



$\arg[g^2] = 0+$

Complex Saddle Points needed

complex saddle points in QM path integral

Behtash, Dunne, Schäfer, Sulejmanpasic, Ünsal (15)

⇨ complex bion

instanton-anti instanton pair with ``complex separation”

Extended resurgence (including complex saddles)

$$Z = \underbrace{Z_0}_{\text{pert.}} + \underbrace{Z_1}_{\text{Saddles}} + \cdots$$

cancelation of all the imaginary ambiguities

full partition function
: real and no ambiguity

Final goal is field theory (CPN models)
but as an exercise we begin with sine-Gordon QM

Sine-Gordon QM

- **SG Hamiltonian**

$$H_{SG} = -g^2\partial_\theta^2 + \frac{m^2}{4g^2}\sin^2\theta - \epsilon m\cos\theta$$

$$= H_{\mathbb{C}P^1}^{l=0} + \frac{g^2}{\tan\theta}\partial_\theta$$

ϵ : # of fermion d.o.f.

- $\epsilon = \frac{1}{2}$ **SUSY case**

$$\Psi = \exp\left(\frac{m}{2g^2}\cos\theta\right) \Rightarrow H\Psi = 0$$

Exact wave function

Zero Ground State Energy

Witten Index $\neq 0$

Sine-Gordon QM

- **SG Hamiltonian**

$$H_{SG} = -g^2 \partial_\theta^2 + \frac{m^2}{4g^2} \sin^2\theta - \epsilon m \cos\theta$$

$$= H_{\mathbb{C}P^1}^{l=0} + \frac{g^2}{\tan\theta} \partial_\theta$$

ϵ : # of fermion d.o.f.

- $\epsilon \approx \frac{1}{2}$ **near-SUSY case** [Fujimori, Kamata, TM, Nitta, Sakai(16)]

$$\delta H = H - H\big|_{\epsilon=\frac{1}{2}}$$

Hamiltonian $\delta\epsilon = \epsilon - \frac{1}{2}$

$$\Rightarrow \quad E = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle} + \frac{\langle \delta\psi|\delta H|\delta\psi\rangle}{\langle 0|0\rangle} + \mathcal{O}(\delta\epsilon^3)$$

Nonzero Ground State Energy

Sine-Gordon QM

- **Near-SUSY Energy**

[Fujimori, Kamata, TM, Nitta, Sakai(16)]

$$E = E^{(1)}\delta\epsilon + E^{(2)}\delta\epsilon^2 + \mathcal{O}(\delta\epsilon^3)$$

$$\Rightarrow \quad E^{(1)} = -m\frac{I_1(m/g^2)}{I_0(m/g^2)} = \boxed{-g^2 m\partial_m \log I_0(m/g^2)}$$

$$\Rightarrow \quad E^{(1)} = E^{(1)}_{\rm pert} + E^{(1)}_{\rm bion}$$

Perturbative part **Saddle-point part**

Sine-Gordon QM

- **Perturbative part as asympt. expansion**

$$E^{(1)}_{\mathrm{pert}} = -g^2 m \partial_m \log e^{\frac{m}{g^2}} \sqrt{\frac{g^2}{2\pi m}} \sum_{n=0}^{\infty} \frac{[(2n-1)!!]^2}{n!} \left(\frac{g^2}{8m}\right)^n$$

$$\Rightarrow \quad E^{(1)}_{\mathrm{pert}} = -g^2 m \frac{\partial}{\partial m} \log \left[I_0(m/g^2) \pm \frac{i}{\pi} K_0(m/g^2) \right]$$

Borel resum.

This is consistent with the known perturbative calculation!

Verbaarschot, West, Wu (90)

Behtash, Dunne, Schäfer, Sulejmanpasic, Ünsal (15)

Sine-Gordon QM

- **Saddle point parts**

$$E^{(1)}_{\rm bion} = E^{(1)} - E^{(1)}_{\rm pert} = g^2 m \frac{\partial}{\partial m} \log \left[1 \pm \frac{i}{\pi} \frac{K_0(m/g^2)}{I_0(m/g^2)} \right]$$

$$\Rightarrow \quad E^{(1)}_{\rm bion} = \underbrace{\boxed{\mp 2ime^{-\frac{2m}{g^2}}}}_{\text{single bions}} + \underbrace{\mathcal{O}\left(e^{-\frac{4m}{g^2}}\right)}_{\text{multi bions}}$$

This is exactly consistent with contributions from real and complex bions!

Sine-Gordon QM

- **Contributions from real and complex bions**

$$E_{\rm bion} = -\lim_{\beta\to\infty}\frac{1}{\beta}\frac{Z_1}{Z_0} \approx -\lim_{\beta\to\infty}\frac{1}{\beta}\int d\tau_0 d\tau_r \sqrt{\frac{\det\Delta_0}{\det''\Delta_{k\bar{k}}}}\exp(-V_{\rm SG})$$

Sine-Gordon QM

- **Contributions from real and complex bions**

$$[\mathcal{I}\bar{\mathcal{I}}] = \int_{\mathcal{C}_{\mathbb{R}}} d\tau \, e^{-V_{\rm SG}(\tau)} \qquad V_{\rm SG}(\tau) \equiv -\frac{4m}{g^2} e^{-m\tau} + 2\epsilon m\tau$$

Quasi-moduli integral

- **Relative distance between instantons is only nearly-massless mode**
- **The complex quasi-moduli integral corresponds to thimble integral**

Saddle points

$$\tau_\sigma \equiv \frac{1}{m}\left[\log\frac{2m}{\epsilon g^2} + (2\sigma - 1)\pi i - i\theta\right]$$

$$\theta = \arg[g^2]$$
$$\sigma = 0, 1$$

Thimbles

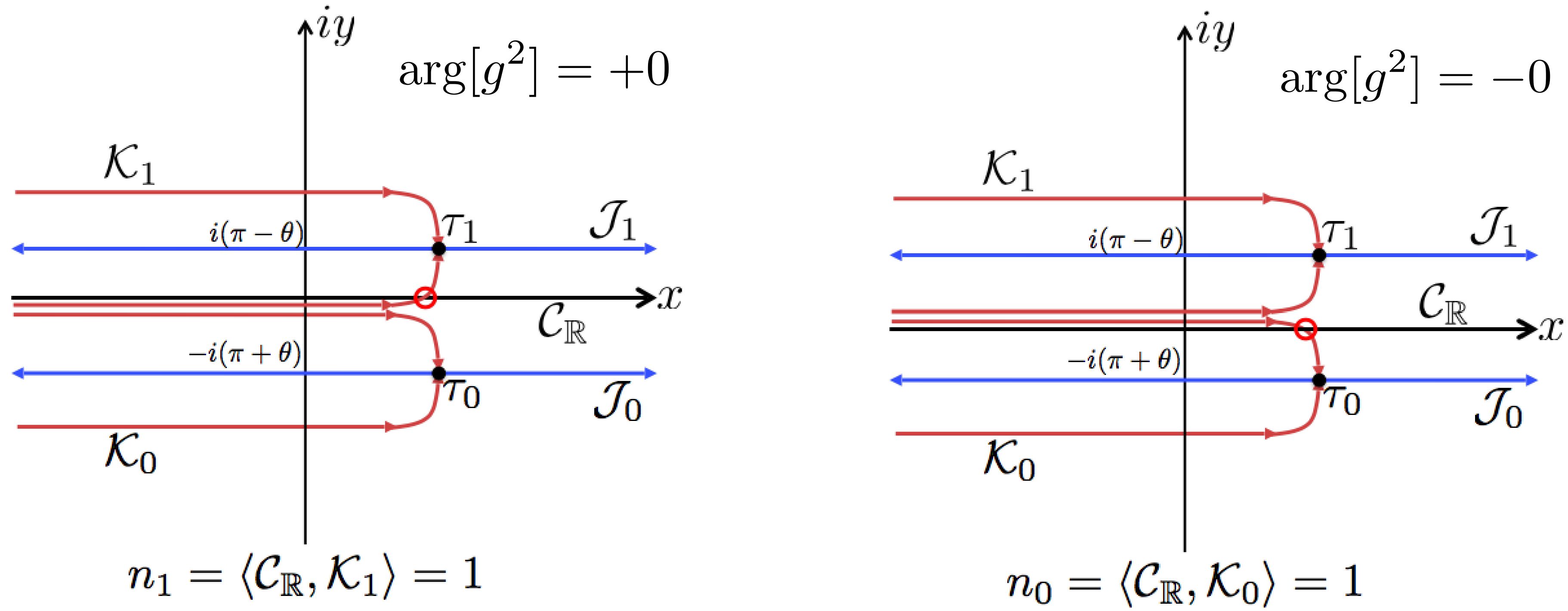
$$\tau(t) = \frac{1}{m}\log\left[\frac{2m}{\epsilon g^2}\frac{\sin(a - be^{-\epsilon m t} - \theta)}{be^{-\epsilon m t}}\right] - \frac{i}{m}(a - be^{-\epsilon m t})$$

Sine-Gordon QM

- **Contributions from real and complex bions**

$$[\mathcal{I}\bar{\mathcal{I}}] = \int_{\mathcal{C}_\mathbb{R}} d\tau\, e^{-V_{\mathrm{SG}}(\tau)} \qquad V_{\mathrm{SG}}(\tau) \equiv -\frac{4m}{g^2}e^{-m\tau} + 2\epsilon m\tau$$

Quasi-moduli integral



$$\Rightarrow \quad E_{1\mathrm{bions}} = (1+e^{\pm 2\pi i\epsilon})\frac{m\Gamma(2\epsilon)}{\pi}\left(\frac{g^2}{4m}\right)^{2\epsilon-1} e^{-\frac{2m}{g^2}}$$

$$= \boxed{\mp 2ime^{-\frac{2m}{g^2}}}\,\delta\epsilon + \mathcal{O}\left(\delta\epsilon^2\right)$$

Sine-Gordon QM

- **Explicit resurgence structure in SG QM**

$$E^{(1)} = E^{(1)}_{\rm pert} + E^{(1)}_{\rm bion}$$

Imaginary ambiguities cancel between pert and nonpert parts, and we end up with the exact result !

$$E^{(1)}_{\rm pert} = -g^2 m \frac{\partial}{\partial m} \log \left[I_0(m/g^2) \pm \frac{i}{\pi} K_0(m/g^2) \right]$$

$$E^{(1)}_{\rm bion} = \mp 2ime^{-\frac{2m}{g^2}} + \mathcal{O}\left(e^{-\frac{4m}{g^2}}\right)$$

single bions **multi bions**

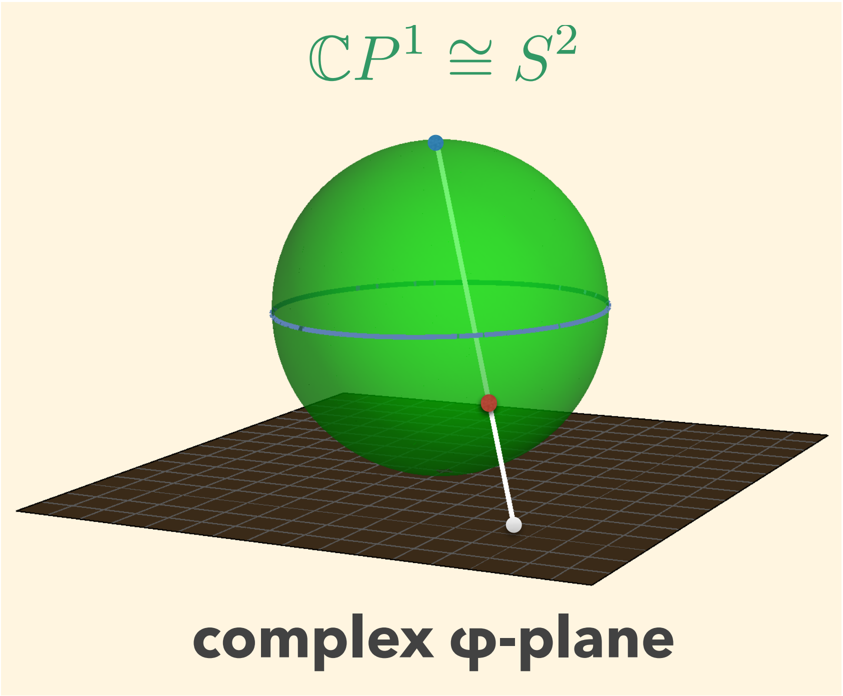
can be obtained from multi-bion solutions will be announced in a forthcoming paper

CP^N-1 models

CP1 Sigma model

- **CP1 sigma model on R1 x S1**

$$\mathcal{L} = \frac{1}{g^2}\frac{|\partial_\mu \varphi|^2}{(1+|\varphi|^2)^2}$$



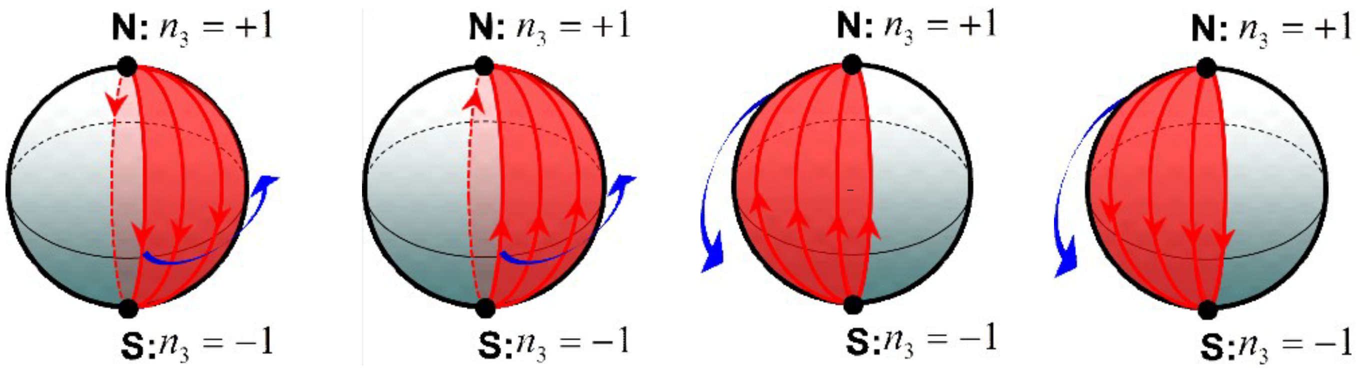
- **Twisted boundary conditions**

$$\varphi(y+L) = e^{imL}\varphi(y)$$

$m=\pi$: Z_2 twisted b.c.

→ BPS Fractional instantons

cf.) $m=\pi$



CP1 QM via dimensional reduction

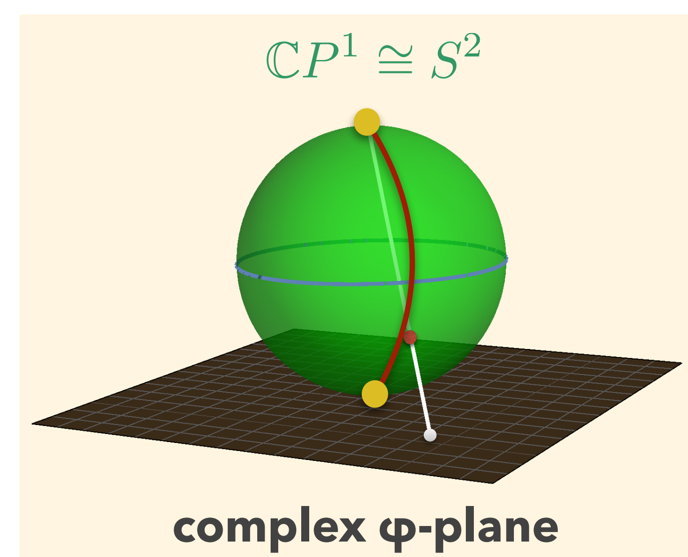
- CP1 QM Lagrangian

$$L = \frac{1}{g^2} G \Big[\partial_t \varphi \partial_t \bar{\varphi} - m^2 \varphi \bar{\varphi} + i \bar{\psi} \mathcal{D}_t \psi + \epsilon m (1 + \varphi \, \partial_\varphi \log G) \bar{\psi} \psi \Big]$$

$$G = \frac{1}{(1+\varphi\bar{\varphi})^2}, \qquad \mathcal{D}_t \psi = \Big[\partial_t + \partial_t \varphi \, \partial_\varphi \log G \Big] \psi$$

- Potential with two minima due to t.b.c.

North and South poles



- Kink solutions

Tunneling between two minima

Eliminating fermion

· **Fermionic part of Lagrangian**

$$L = \cdots + i\psi\partial_t\psi + F(|\varphi|)\bar{\psi}\psi$$

· **Fermion number** $f = \bar{\psi}\psi$ **: conserved charge**

$$Z = Z_{f=0} + Z_{f=1}$$

· **Partition function of *f*=0 sector**

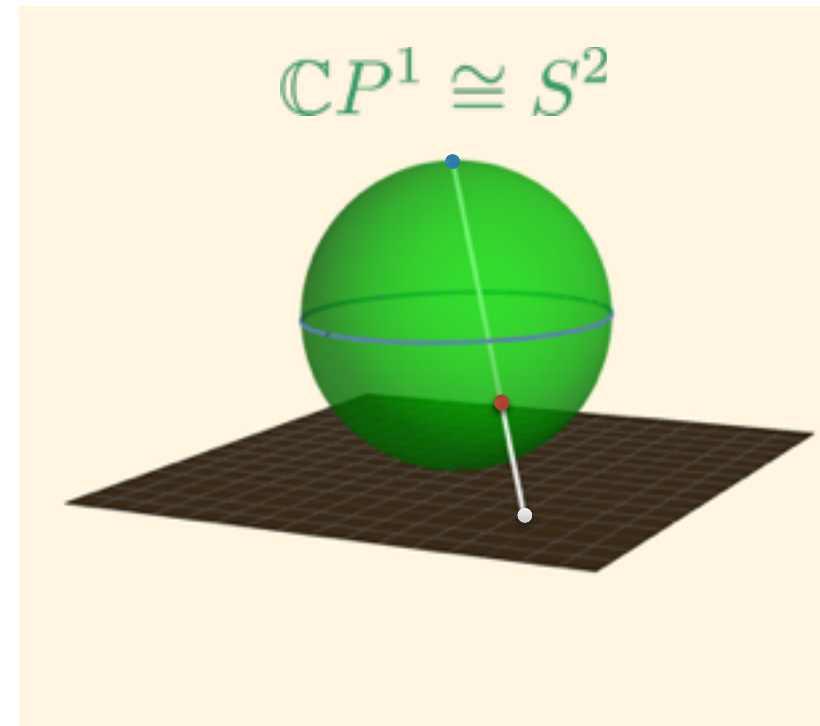
$$Z_{f=0} = \int \mathcal{D}\varphi \, \exp\left[- \int d\tau (L + V_f) \right]$$

induced potential

CP^1 Quantum Mechanics

- **Euclidean effective action**

$$S_E = \frac{1}{g^2}\int d\tau \left[\frac{|\dot{\varphi}|}{(1+|\varphi|^2)^2} + V(|\varphi|)\right]$$

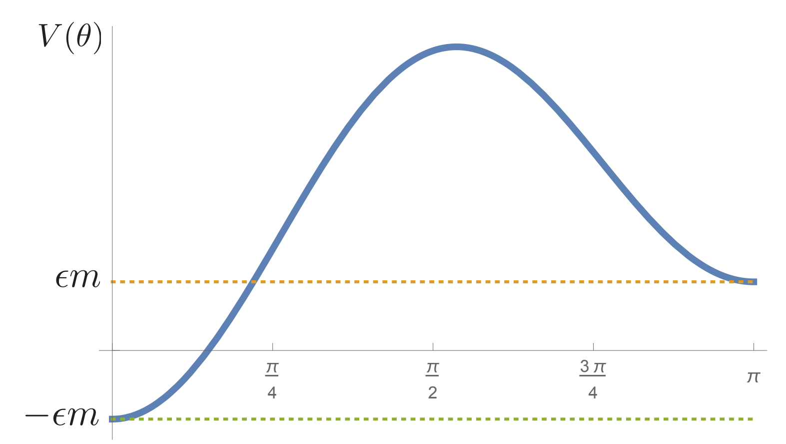


- **Induced potential**

$$V(|\varphi|) = \frac{m^2|\varphi|^2}{(1+|\varphi|^2)^2} - \lambda\frac{1-|\varphi|^2}{1+|\varphi|^2}$$

twisted b.c. **fermion**

$$\lambda = \epsilon m g^2$$

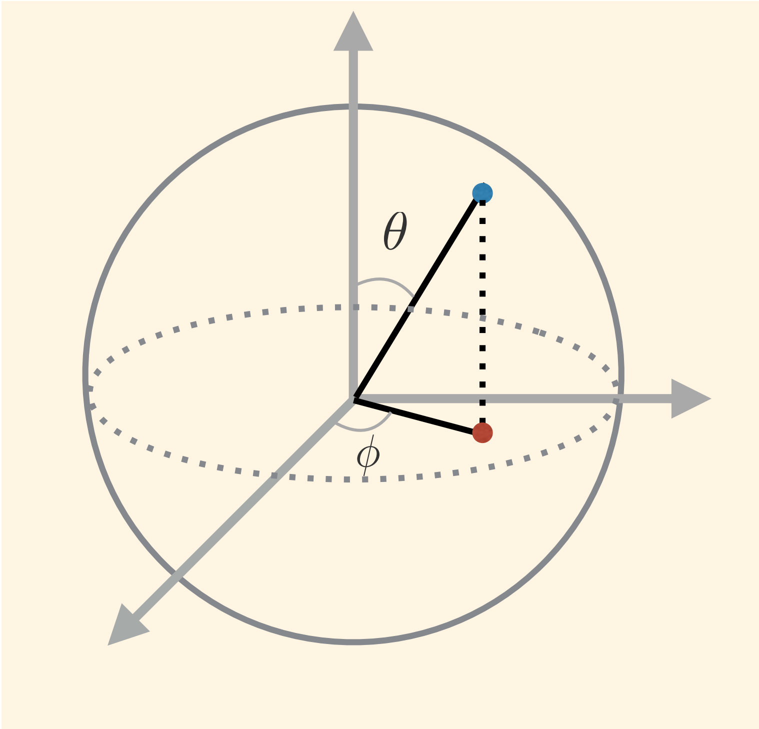


$$\theta = 2\arctan|\varphi|$$

Properties of Potential

In spherical coordinate

$$V = \frac{m^2}{4}\sin\theta - \epsilon m g^2 \cos\theta$$



ε : number of fermion d.o.f.

$\epsilon = 1$ **supersymmetric**

$\epsilon = 0$ **bosonic**

Exact Results

Exact ground state energy in CP1

- **CP1 Hamiltonian**

[Fujimori, Kamata, TM, Nitta, Sakai(16)]

$$H = -g^2(1+\varphi\bar{\varphi})^2 \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\bar{\varphi}} + V(\varphi\bar{\varphi})$$

- $\epsilon = 1$ **SUSY case**

$$\Psi_0 = \exp\left(\frac{m}{2g^2}\frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}}\right) \Rightarrow H\Psi = 0$$

Exact wave function

Zero Ground State Energy
Witten Index $\neq 0$

Exact ground state energy in CP1

- **CP1 Hamiltonian**

[Fujimori, Kamata, TM, Nitta, Sakai(16)]

$$H = -g^2(1+\varphi\bar{\varphi})^2 \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\bar{\varphi}} + V(\varphi\bar{\varphi})$$

- $\epsilon \approx 1$ **near-SUSY case**

$$\delta H = H - H_{\epsilon=1} = -\delta\epsilon m \frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}} \quad \Rightarrow \quad E = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle} + \frac{\langle \delta\psi|\delta H|\delta\psi\rangle}{\langle 0|0\rangle} + \mathcal{O}(\delta\epsilon^3)$$

Hamiltonian $\quad \delta\epsilon = \epsilon - 1$

Nonzero Ground State Energy

$\boxed{\text{Exact ground state energy in CP1}}$

- **Near-SUSY Energy**

$$E \;=\; E^{(1)}\delta\epsilon \;+\; E^{(2)}\delta\epsilon^2 \;+\; \mathcal{O}(\delta\epsilon^3)$$

$$\Rightarrow \quad E^{(1)} = -m\langle\frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}}\rangle_{\epsilon=1}$$

$$= \boxed{g^2 \;-\; m\coth\frac{m}{g^2}}$$

$$\Rightarrow \quad E^{(1)} \;=\; E^{(1)}_{\text{pert}} + E^{(1)}_{\text{bion}}$$

Perturbative part **Saddle-point part**

Exact ground state energy in CP1

$$E^{(1)} = g^2 - m \coth \frac{m}{g^2} = E^{(1)}_{\rm pert} + E^{(1)}_{\rm bion}$$

- **Perturbative part**

$$E^{(1)}_{\rm pert} = -m + g^2$$

finite order unlike SG

exact agreement with the perturbative calculation

- **Saddle-point part**

$$E^{(1)}_{\rm bion} = -2m \sum_{k=1}^{\infty} e^{-\frac{2km}{g^2}} = -2me^{-\frac{2m}{g^2}} + \mathcal{O}(e^{-\frac{4m}{g^2}})$$

single bions   multi bions

no Im ambiguity unlike SG

consistent with real and complex bion contributions?

Exact ground state energy in CP1

$$E^{(1)} = g^2 - m \coth \frac{m}{g^2} = E^{(1)}_{\text{pert}} + E^{(1)}_{\text{bion}}$$

- **Perturbative part**

$$E^{(1)}_{\text{pert}} = -m + g^2$$

cf.)inspired by Sulejmanpasic, Unsal (16)

$$A_l \sim -\frac{1}{2^{l-1}} \frac{\Gamma(l+2(1-\epsilon))}{\Gamma(1-\epsilon)^2}$$

exact agreement with the perturbative calculation

- **Saddle-point part**

$$E^{(1)}_{\text{bion}} = -2m \sum_{k=1}^{\infty} e^{-\frac{2km}{g^2}} = -2m e^{-\frac{2m}{g^2}} + \mathcal{O}(e^{-\frac{4m}{g^2}})$$

single bions **multi bions**

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consistent with real and complex bion contributions?

Exact ground state energy in CP1

$$E^{(1)} = g^2 - m \coth \frac{m}{g^2} = E^{(1)}_{\text{pert}} + E^{(1)}_{\text{bion}}$$

- **Perturbative part**

cf.)inspired by Sulejmanpasic, Unsal (16)

$$\mathrm{Im}\, \mathcal{S}_{\pm} E_{\text{pert}} = \mp \frac{2\pi m}{\Gamma(1-\epsilon)^2} \left(\frac{g^2}{2m} \right)^{2(\epsilon-1)} e^{-\frac{2m}{g^2}}$$

$$E^{(1)}_{\text{pert}} = -m + g^2$$

exact agreement with the perturbative calculation

- **Saddle-point part**

$$E^{(1)}_{\text{bion}} = -2m \sum_{k=1}^{\infty} e^{-\frac{2km}{g^2}} = \underbrace{-2m e^{-\frac{2m}{g^2}}}_{\text{single bions}} + \underbrace{\mathcal{O}(e^{-\frac{4m}{g^2}})}_{\text{multi bions}}$$

no Im ambiguity unlike SG

consistent with real and complex bion contributions?

Real and complex saddle results

Saddle point equation

- **Euclidean action**

$$S_E = \frac{1}{g^2}\int d\tau\left[\frac{|\dot{\varphi}|^2}{(1+|\varphi|^2)^2}+\frac{m^2|\varphi|^2}{(1+|\varphi|^2)^2}-\lambda\frac{1-|\varphi|^2}{1+|\varphi|^2}\right]$$

Euclidean e.o.m

$$\frac{\delta S_E}{\delta\varphi} = -\partial_\tau^2\varphi + \frac{\partial V}{\partial\bar{\varphi}} = 0$$

- symmetry : time and phase shift ⇨ **conservation law**

$$\frac{1}{g^2}\frac{\partial_\tau\varphi\partial_\tau\bar{\varphi}}{(1+\varphi\bar{\varphi})^2} - V(\varphi\bar{\varphi}) \;=\; \epsilon m \;=\; E|_{\varphi=0}$$

Solution of E.O.M.

solution

$$\varphi = \sqrt{\frac{\omega^2}{\lambda}} \frac{e^{i\phi_0}}{i \sinh \omega(\tau - \tau_0)}$$

$$\omega^2 = m^2 + \lambda$$

τ_0 : position $\quad \phi_0$: phase $\quad$ are moduli parameters

· kink-antikink pair

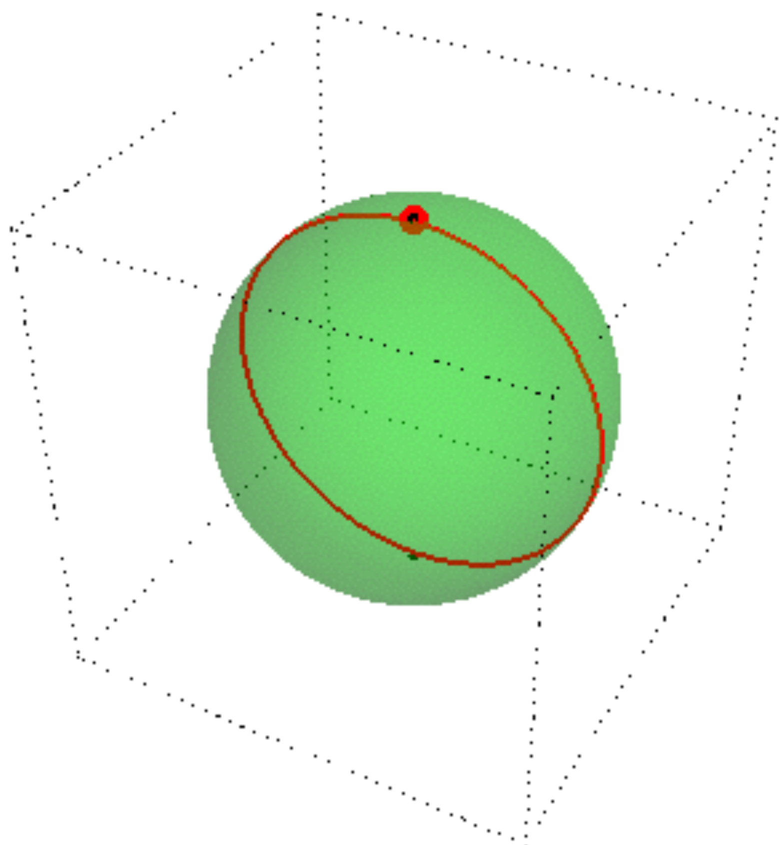
$\left(\phi_\pm = \phi_0 \mp \frac{\pi}{2}\right)$

$$\varphi^{-1} \propto \underbrace{e^{\omega(\tau - \tau_+)}}_{\text{kink}} - \underbrace{e^{-\omega(\tau - \tau_-)}}_{\text{antikink}}$$

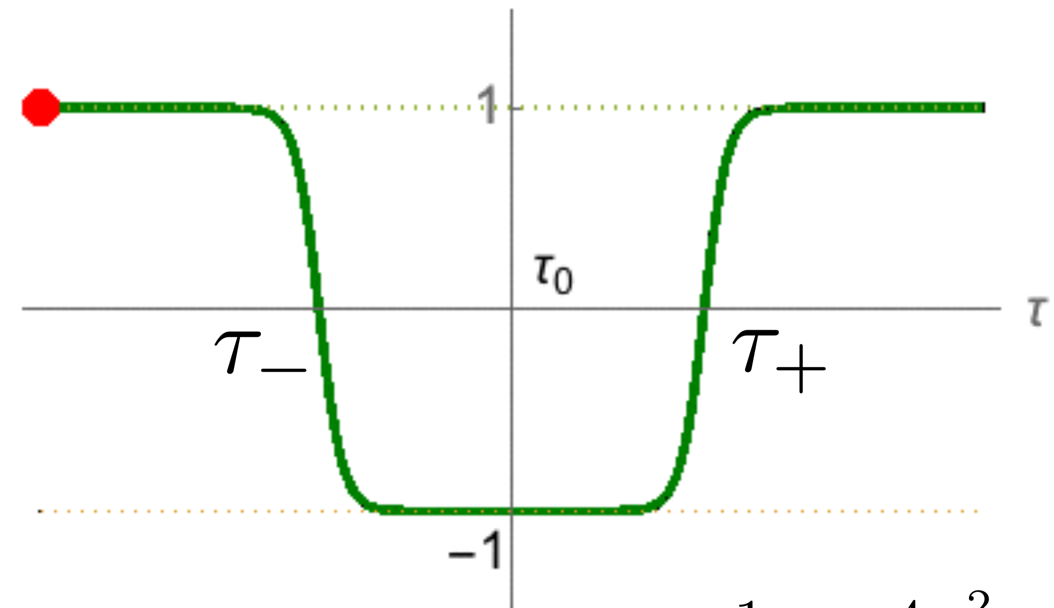
$$\tau_+ - \tau_- = \frac{1}{\omega} \log \frac{4\omega^2}{\lambda}$$

relative distance (stabilized)

real bion solution



$$\frac{|\varphi|^2}{1+|\varphi|^2}$$ **: kink profile**



$$\tau_+ - \tau_- = \frac{1}{\omega}\log\frac{4\omega^2}{\lambda}$$

$$\varphi = \sqrt{\frac{\omega^2}{\lambda}}\frac{e^{i\phi_0}}{i\sinh\omega(\tau-\tau_0)}$$

“real” bion : saddle point on original integration contour

contribution of real bion

$$\exp[-S_{\mathrm{rb}}] = \exp\left[-\frac{2\omega}{g^2}\left(1+\frac{\lambda}{m\omega}\log\frac{\omega-m}{\omega+m}\right)\right]$$

- **does not vanish in the supersymmetric case** $\lambda = mg^2$

$\Downarrow$

There should be other saddle points which cancel the real bion contribution

Complexification

· real and imaginary parts of φ ⇨ **complex**

$$\varphi = \varphi_{\mathrm{R}} + i\varphi_{\mathrm{I}} \qquad \bar{\varphi} = \varphi_{\mathrm{R}} - i\varphi_{\mathrm{I}} \ \to \ \tilde{\varphi}$$

$$\mathbb{C}P^1 \cong \frac{SU(2)}{U(1)} \ \to \ \frac{SU(2)^{\mathbb{C}}}{U(1)^{\mathbb{C}}} \cong T^*\mathbb{C}P^1$$

complexification of CP^1

· Analytically continued holomorphic action

⇨ $S[\varphi, \bar{\varphi}] \ \to \ S[\varphi, \tilde{\varphi}]$ **holomorphic**

Complex bion solution

solution

$$\tau_0 \;\to\; \tilde{\tau}_0 = \tau_0 + \frac{1}{\omega}\frac{\pi i}{2}$$

- The action is invariant under time and phase transformation with complexified parameters
- A solution distinct from real bion is obtained by complexified shift giving a jump of the action

Complex bion solution

solution

$$\varphi = \sqrt{\frac{\omega^2}{\lambda}} \frac{e^{i\phi_0}}{\cosh \omega(\tau - \tau_0)} \qquad \tilde{\varphi} = -\varphi^*$$

- **kink-antikink pair**
  $\left(\phi_\pm = \phi_0 - \frac{\pi}{2} \right)$

$$\varphi^{-1} \propto \underbrace{e^{\omega(\tau - \tau_+)}}_{\text{kink}} + \underbrace{e^{-\omega(\tau - \tau_-)}}_{\text{antikink}}$$

$$\tau_+ - \tau_- = \frac{1}{\omega} \left(\log \frac{4\omega^2}{\omega^2 - m^2} + \pi i \right)$$ **: "complex relative distance"**

Complex bion solution

solution

$$\varphi = \sqrt{\frac{\omega^2}{\lambda}} \frac{e^{i\phi_0}}{\cosh \omega(\tau - \tau_0)} \qquad \tilde{\varphi} = -\varphi^*$$

- **Lagrangian** $$L = -4m\epsilon \left[\frac{\omega^2 \sinh \omega(\tau - \tau_0)}{\omega^2 - (\omega^2 - m^2)\cosh^2 \omega(\tau - \tau_0)} \right]^2$$

$\Rightarrow$ $$\tau_{\text{pole}}^{\pm} = \tau_0 \pm \frac{1}{\omega} \text{arccosh} \sqrt{\frac{\omega^2}{\omega^2 - m^2}}$$ **two poles**

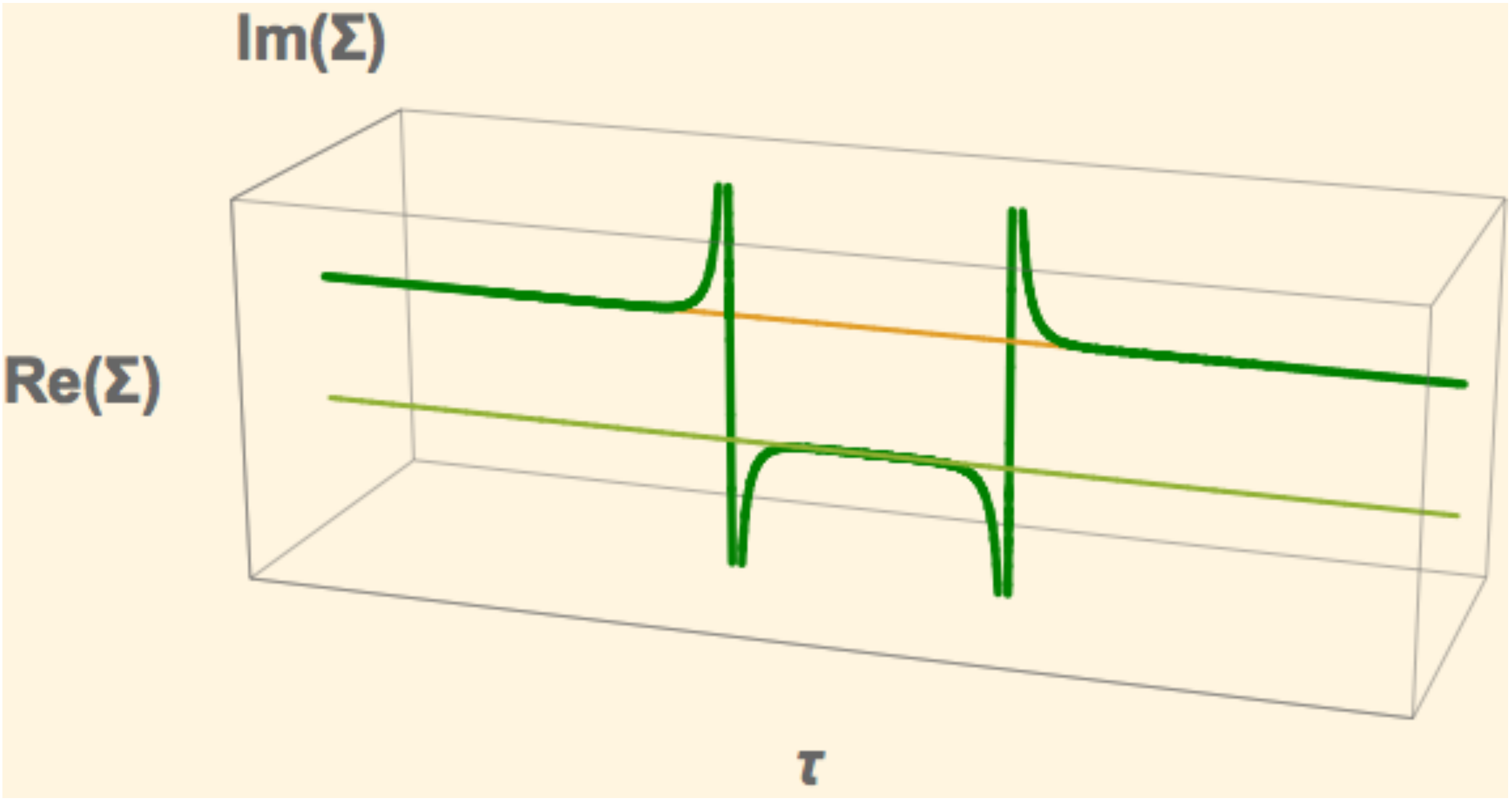
$\Rightarrow$ **Singular solution**

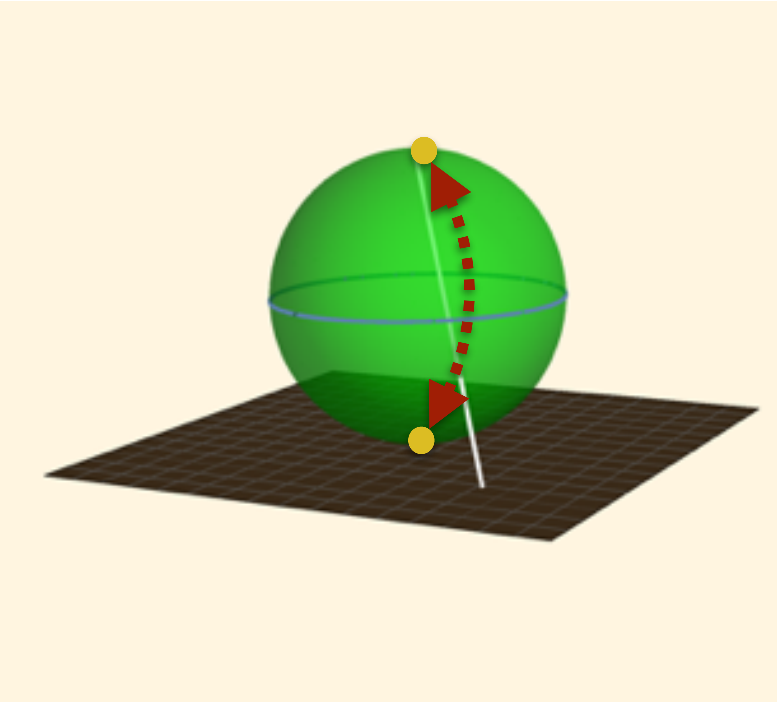
Kink profile of bion

height Σ

$$\frac{|\varphi|^2}{1+|\varphi|^2} \quad \Rightarrow \quad \frac{\varphi\tilde{\varphi}}{1+\varphi\tilde{\varphi}}$$

complexification





complex bion

Kink profile of bion

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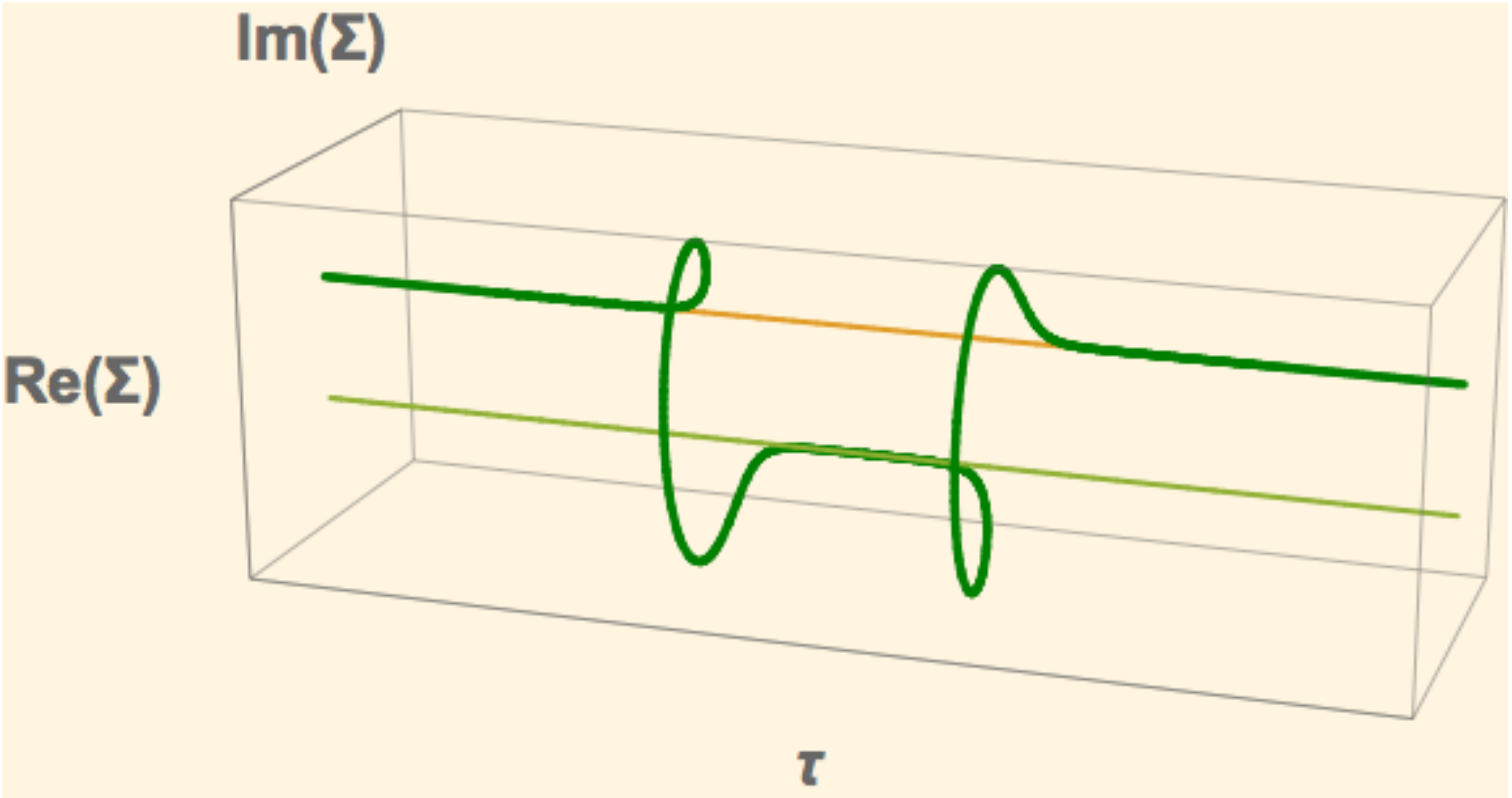
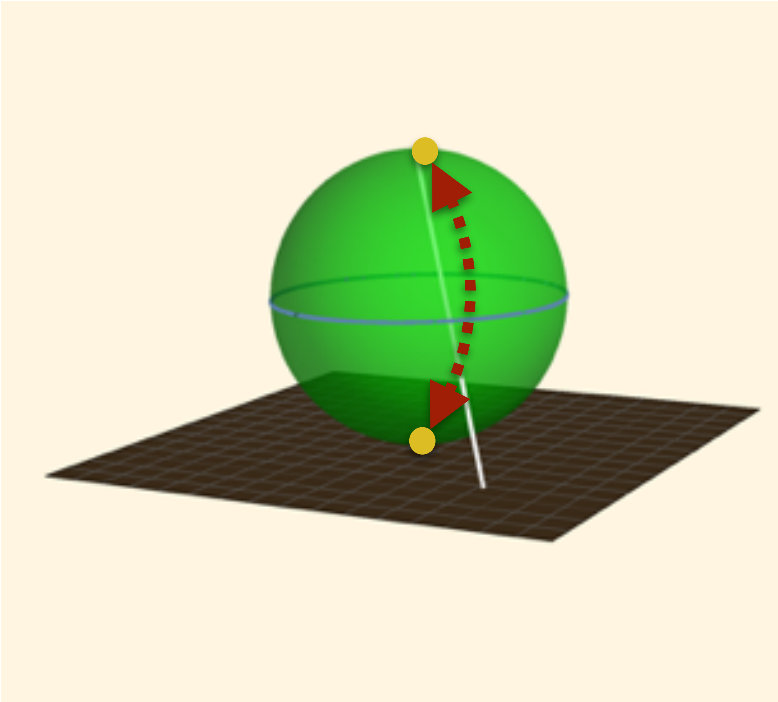
$$\frac{|\varphi|^2}{1+|\varphi|^2} \;\Rightarrow\; \frac{\varphi\tilde{\varphi}}{1+\varphi\tilde{\varphi}}$$

complexification



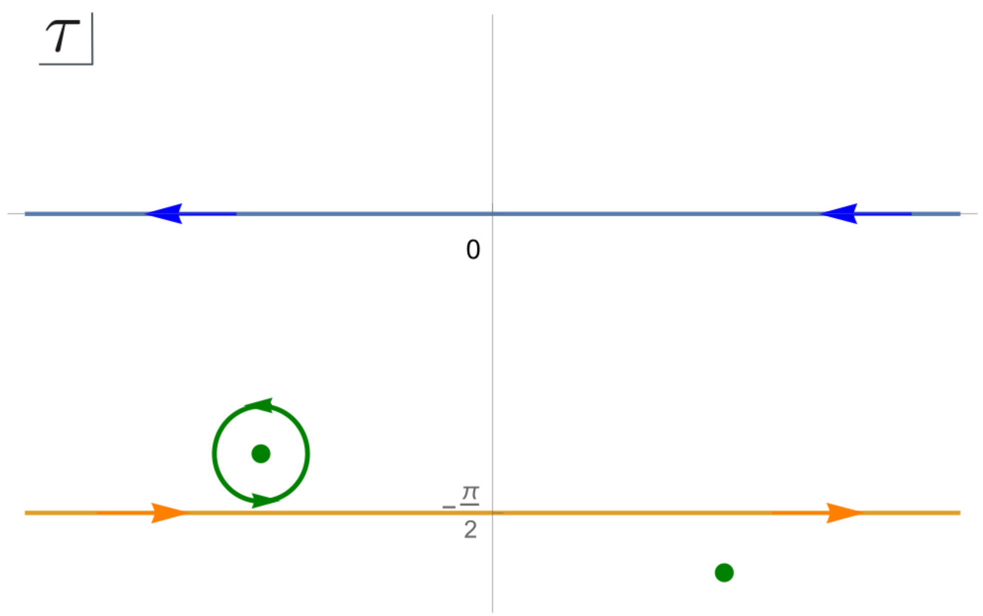


$\arg[g^2] \neq 0$ ⇨ **regularized complex bion**

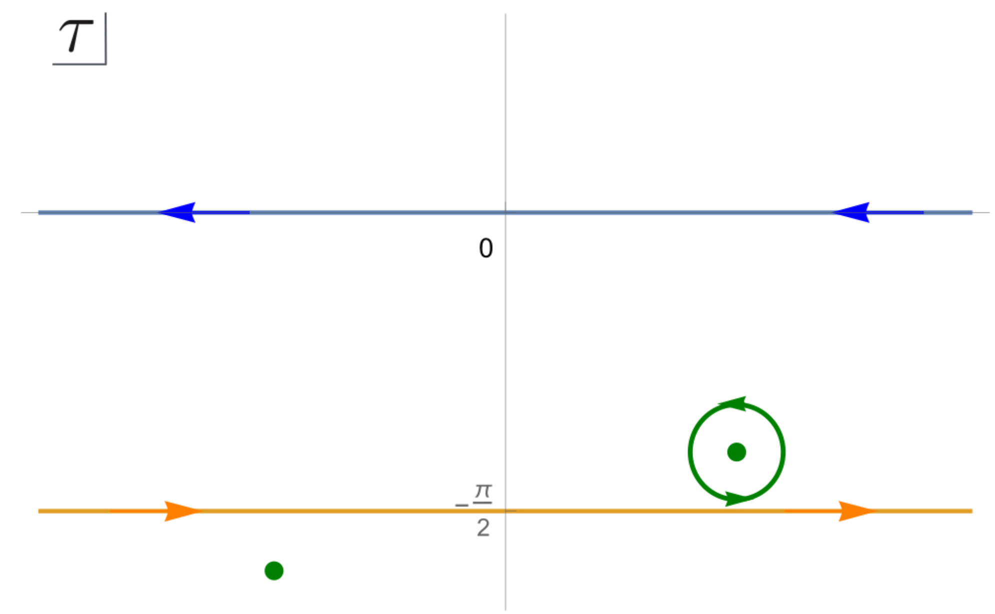
Contribution of complex bion

cf.)SG case by Behtash, Sulejmanpasic, Schaefer, Unsal (15)

$$S_{\mathrm{cb}} - S_{\mathrm{rb}} = \oint d\tau L_E \Rightarrow \boxed{S_{\mathrm{cb}} = S_{\mathrm{rb}} \pm 2\pi i\epsilon}$$



$\theta = \arg g^2 > 0$



$\theta = \arg g^2 < 0$

indicates contribution of complex bion has imaginary ambiguity depending on arg g^2

Fluctuations around saddle points

Quadratic fluctuations around saddle points

$$S = S_{\rm sol} + \int d\tau \delta\Phi^T \Delta \delta\Phi + \cdots$$

Gaussian Integral $\delta\Phi = \begin{pmatrix} \delta\varphi \\ \delta\tilde{\varphi} \end{pmatrix}$

Determinant of differential operator Δ

$$\frac{Z_1}{Z_0} = \beta \frac{16 i \omega^4}{g^2 \lambda} [e^{-S_{\rm rb}} - e^{-S_{\rm cb}}] + \mathcal{O}(\beta^0)$$

β : inverse temperature

Leading non-perturbative correction

- Gaussian integration

one loop determinant

$$\frac{Z_1}{Z_0} = \beta \frac{16 i \omega^4}{g^2 \lambda} [e^{-S_{\rm rb}} - e^{-S_{\rm cb}}] + \mathcal{O}(\beta^0)$$

- correction to ground state energy

$$E_{\rm bion} = - \lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0}$$

$$\boxed{E_{\rm bion} = i(1 - e^{\pm 2\pi \epsilon i}) \frac{16 \omega^4}{g^2 \lambda} \left(\frac{\omega + m}{\omega - m} \right)^{2\epsilon} \exp \left[- \frac{2\omega}{g^2} \right]} \qquad \epsilon = \frac{\lambda}{m g^2}$$

- asymptotic form in the limit $g^2 \to 0$ with fixed λ

SUSY case

$$E_{\text{bion}} = i(1 - e^{\pm 2\pi\epsilon i})\frac{16\omega^4}{g^2\lambda}\left(\frac{\omega + m}{\omega - m}\right)^{2\epsilon} \exp\left[-\frac{2\omega}{g^2}\right]$$

- supersymmetric case $\epsilon = 1$

$$\Rightarrow \quad E_{\text{bion}} = 0$$

- cancelation of real and complex bion contributions

- consistent with the exact result

We are just lucky.....

$g^2 \to 0$ with fixed λ

v.s.

$g^2 \to 0$ with fixed ϵ

near SUSY case

$$E_{\text{bion}} = i(1 - e^{\pm 2\pi\epsilon i}) \frac{16\omega^4}{g^2\lambda} \left(\frac{\omega + m}{\omega - m}\right)^{2\epsilon} \exp\left[-\frac{2\omega}{g^2}\right]$$

- near supersymmetric case $\epsilon \approx 1$

$$E \approx 0 + A(\omega, m, g) e^{-\frac{2m}{g^2}} \delta\epsilon + \cdots$$

⇨ incompatible with the exact result

- $g^2 \to 0$ with fixed $\epsilon = \dfrac{\lambda}{mg^2}$ ⇨ nearly flat directions appear

$$\tau_+ - \tau_- \approx \frac{1}{m} \log \frac{2m^2}{\lambda} \to \infty \quad (\lambda \to 0)$$

any superposition of position and phase gets massless

near SUSY case

$$E_{\text{bion}} = i(1 - e^{\pm 2\pi\epsilon i})\frac{16\omega^4}{g^2\lambda}\left(\frac{\omega+m}{\omega-m}\right)^{2\epsilon}\exp\left[-\frac{2\omega}{g^2}\right]$$

- near supersymmetric case $\epsilon \approx 1$

$$E \approx 0 + A(\omega, m, g)e^{-\frac{2m}{g^2}}\delta\epsilon + \cdots$$

$\Rightarrow$ incompatible with the exact result

- $g^2 \to 0$ with fixed $\epsilon = \dfrac{\lambda}{mg^2}$ $\Rightarrow$ nearly flat directions appear

Gaussian approximation is not valid

Quasi-Moduli (Thimble) Integral

- nearly flat directions : quasi-moduli parameters

relative kink distance τ and phase ϕ

no other quasi-moduli numerically checked

contribution from real and complex bion

$$\frac{Z_1}{Z_0} \approx \int d\tau_0 d\phi_0\, d\tau_r d\phi_r \sqrt{\det\left(\frac{\mathcal{G}}{2\pi}\right)\det\left(\frac{\mathcal{G}'}{2\pi}\right)\frac{\det\ \Delta_0}{\det''\Delta_{k\bar{k}}}}\exp\left(-V_{\text{eff}}\right)$$

effective action on complexified quasi-moduli space TM, Sakai, Nitta (14)

$$S_{\text{eff}} \approx -\frac{4m}{g^2}\cos\phi\, e^{-m\tau} + 2\epsilon m\tau$$

(for well-separated kinks)

Quasi-Moduli (Thimble) Integral

- nearly flat directions : quasi-moduli parameters

relative kink distance τ and phase ϕ

no other quasi-moduli numerically checked

complexified quasi-moduli integral

$$Z_{\mathrm{q.m.}} = \int d\tau d\phi \, \exp\left[-S_{\mathrm{eff}}\right]$$

effective action on complexified quasi-moduli space

$$S_{\mathrm{eff}} \approx -\frac{4m}{g^2}\cos\phi \, e^{-m\tau} + 2\epsilon m\tau$$ (for well-separated kinks)

Lefschetz Thimble Method

- decomposition of integration contour

$$\mathcal{C}_{\mathbb{R}} = \sum_{\sigma} n_\sigma \mathcal{J}_\sigma \qquad \sigma \text{ : set of saddle points}$$

thimble	$\mathcal{J}_\sigma$	: upward flow	$\dfrac{d\varphi}{dt} = \overline{\dfrac{\partial S_{\text{eff}}}{\partial \varphi}}$
dual thimble	$\mathcal{K}_\sigma$	: downward flow	flow equation

$$\langle \mathcal{K}_\sigma, \mathcal{J}_{\sigma'} \rangle = \delta_{\sigma\sigma'}$$

intersection pairing

⇨

$$n_\sigma = \langle \mathcal{C}_{\mathbb{R}}, \mathcal{K}_\sigma \rangle$$

intersection number

Quasi-Moduli Integral

· application of Lefschetz thimble method $\theta = \arg[g^2]$

saddle points

$$\tau_\sigma \;=\; \frac{1}{m}\log\frac{2m}{\epsilon g^2} + \frac{i}{m}(\sigma\pi - \theta), \qquad \phi_\sigma \;=\; -(\sigma - 1)\pi \pmod{2\pi}$$

$\sigma = 0$: real bion $\qquad \sigma = \pm 1$: complex bion

solution of flow eq.

$$\tau_i \;=\; \frac{1}{m}\log\left[\frac{2m}{\epsilon g^2}\frac{\sin(a_i - b_i e^{-\epsilon m t} - \theta)}{b_i e^{-\epsilon m t}}\right] - \frac{i}{m}(a_i - b_i e^{-\epsilon m t})$$

Thimble $\mathcal{J}_\sigma$ and Dual Thimble $\mathcal{K}_\sigma$

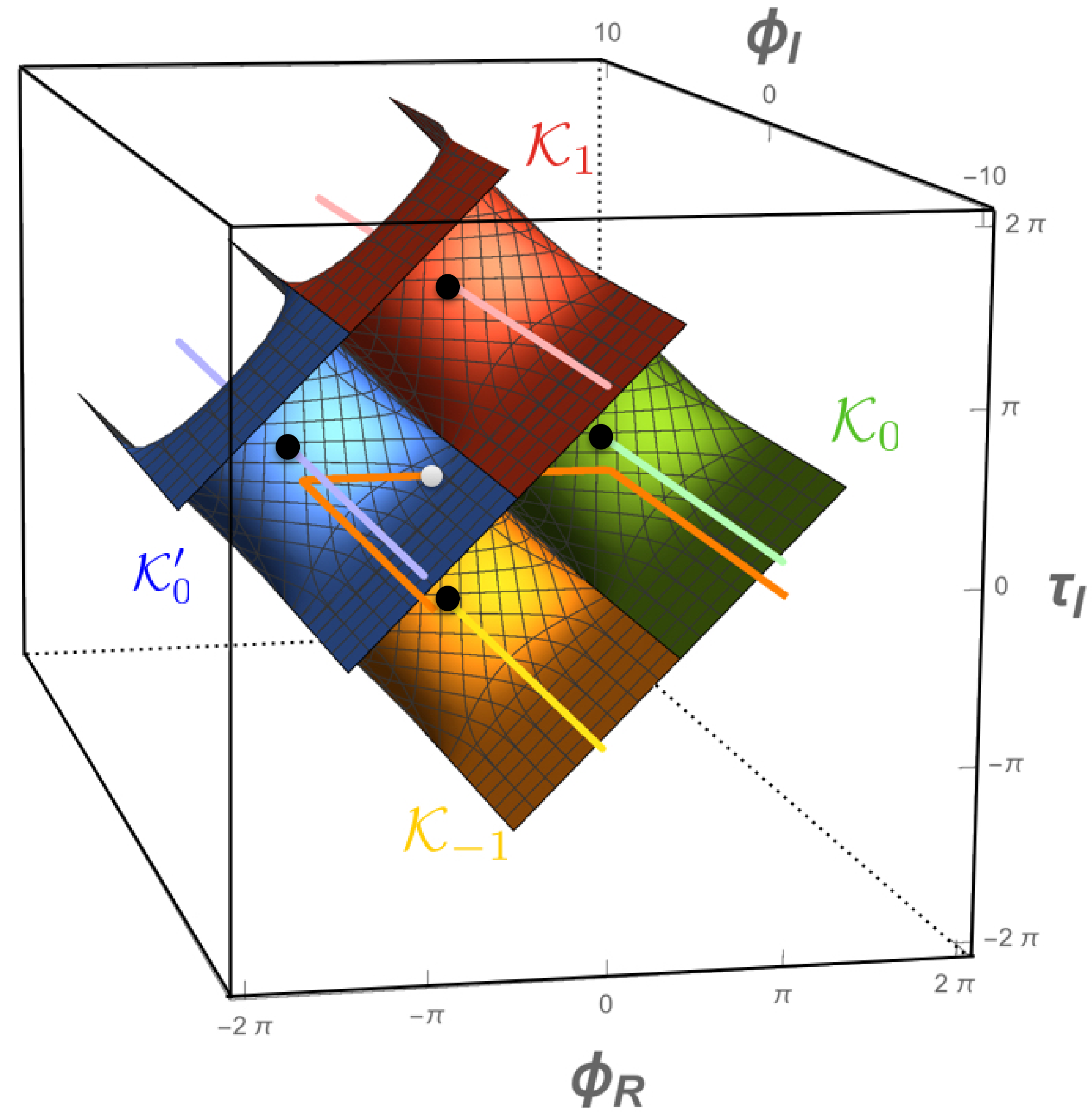
- Thimbles are surfaces in 4d space $\quad (\tau, \phi) \in \mathbb{C}^2$

thimble J_σ

$$m\tau_I = \sigma\pi - \theta, \qquad \phi_R = -(\sigma - 1)\pi$$

dual thimble K_σ

$$\begin{aligned} m\tau_R - \phi_I &= \log\left[\frac{2m}{\epsilon g^2}\frac{\sin(m\tau_I + \phi_R + a_{+\sigma})}{m\tau_I + \phi_R + a_{+\sigma}}\right] \\ m\tau_R + \phi_I &= \log\left[\frac{2m}{\epsilon g^2}\frac{\sin(m\tau_I - \phi_R + a_{-\sigma})}{m\tau_I - \phi_R + a_{-\sigma}}\right] \end{aligned}$$



3d projection $\quad \theta = \arg[g^2] < 0$

Quasi Moduli Integral

$$Z_{\mathrm{q.m.}} = \sum_{\sigma} n_\sigma Z_\sigma$$

- integral along J_σ

$$Z_\sigma = \int_{\mathbb{R}} d\tau' \int_{i\mathbb{R}} d\phi' \, e^{-V} = \frac{i}{2m} \left(\frac{g^2 e^{i\theta}}{2m} \right)^{2\epsilon} e^{-2\pi i \epsilon \sigma} \, \Gamma(\epsilon)^2$$

- intersection number of original contour and K_σ

$$(n_{-1}, n_0, n_1) = \begin{cases} (-1, 1, 0) & \text{for } \theta = -0 \\ (0, -1, 1) & \text{for } \theta = +0 \end{cases}$$

Stokes phenomenon $\Rightarrow$ **ambiguity**

Saddle-point Contribution

contribution to ground state energy

$$E_{\rm bion} = -2m \left(\frac{g^2}{2m}\right)^{2(\epsilon-1)} \frac{\sin\epsilon\pi}{\pi}\Gamma\left(\epsilon\right)^2 e^{-\frac{2m}{g^2}} \times \begin{cases} e^{\pi i\epsilon} & \text{for } \theta = -0 \\ e^{-\pi i\epsilon} & \text{for } \theta = +0 \end{cases}.$$

exactly zero at ε=1(SUSY)
exactly cancels the perturbative imaginary ambiguity

cf.)CPN

$$E_{\rm bion} = -\sum_{i=1}^{N-1} 2m_i \left(\frac{g^2}{2m_i}\right)^{2(\epsilon'-1)} \frac{\sin\epsilon'\pi}{\pi}\Gamma\left(\epsilon'\right)^2 e^{-\frac{2m_i}{g^2}} \times \begin{cases} e^{\pi i\epsilon} & \text{for } \theta = -0 \\ e^{-\pi i\epsilon} & \text{for } \theta = +0 \end{cases}$$

$$\epsilon' = 1 + \tfrac{1}{2}(\epsilon-1)N$$

Saddle-point Contribution

contribution to ground state energy

$$E_{\rm bion} = -2m \left(\frac{g^2}{2m}\right)^{2(\epsilon-1)} \frac{\sin \epsilon\pi}{\pi} \Gamma(\epsilon)^2 e^{-\frac{2m}{g^2}} \times \begin{cases} e^{\pi i \epsilon} & \text{for } \theta = -0 \\ e^{-\pi i \epsilon} & \text{for } \theta = +0 \end{cases} .$$

$$= \boxed{-2m e^{-\frac{2m}{g^2}}} \delta\epsilon + \mathcal{O}(\delta\epsilon^2)$$

precise agreement with exact result!

$$E_{\rm bion}^{(1)} = -2m \sum_{k=1}^{\infty} e^{-\frac{2m}{g^2}} = \boxed{-2m e^{-\frac{2m}{g^2}}} + \mathcal{O}(e^{-\frac{4m}{g^2}})$$

Comparison and Resurgence

Exact ground state energy in CP1

$$E^{(1)} = g^2 - m \coth \frac{m}{g^2} = E^{(1)}_{\rm pert} + E^{(1)}_{\rm bion}$$

- **Perturbative part**

$$E^{(1)}_{\rm pert} = -m + g^2$$

finite order unlike SG

exact agreement with the perturbative calculation

- **Saddle-point part**

$$E^{(1)}_{\rm bion} = -2m \sum_{k=1}^{\infty} e^{-\frac{2m}{g^2}} = \underbrace{-2m e^{-\frac{2m}{g^2}}}_{\text{single bions}} + \underbrace{\mathcal{O}(e^{-\frac{4m}{g^2}})}_{\text{multi bions}}$$

no Im ambiguity unlike SG

exact agreement with real and complex bion contributions!

Exact ground state energy in CP1

$$E^{(1)} = g^2 - m \coth \frac{m}{g^2} = E^{(1)}_{\mathrm{pert}} + E^{(1)}_{\mathrm{bion}}$$

- **Perturbative part**

$$E^{(1)}_{\mathrm{pert}} = -m + g^2$$

finite order unlike SG

exact agreement with the perturbative calculation

- **Saddle-point part**

$$E^{(1)}_{\mathrm{bion}} = -2m \sum_{k=1}^{\infty} e^{-\frac{2m}{g^2}} = -2me^{-\frac{2m}{g^2}} + \mathcal{O}(e^{-\frac{4m}{g^2}})$$

multi bions

can be exactly obtained from multi-bion quasi-moduli integral
will be announced in a forthcoming paper

Complete Resurgence Structure

- **Exact result as expansion of δε**

$$
\begin{aligned}
E = \quad & \delta\epsilon \left[g^2 - m \coth \frac{m}{g^2} \right] \\
& + \delta\epsilon^2 \left[g^2 - m \frac{\coth \frac{m}{g^2}}{\sinh^2 \frac{m}{g^2}} \left(\frac{\mathrm{Ei}(\frac{2m}{g^2}) + \mathrm{Ei}(-\frac{2m}{g^2})}{2} \right.\right. \\
& \qquad \left.\left. - \gamma - \log \frac{2m}{g^2} \right) \right] + \mathcal{O}(\delta\epsilon^3) \\
& = \delta\epsilon\, E^{(1)} + \delta\epsilon^2\, E^{(2)} + \mathcal{O}(\delta\epsilon^3)\,,
\end{aligned}
$$

Richer resurgence (cancellation) structure including multi-bion saddle contributions: See our forthcoming paper

Summary

1 Non-perturbative contribution from real and complex bion solutions in CP^N quantum mechanics

2 SUSY exact results are reproduced

3 Near SUSY result is exactly reproduced from perturbative and saddle-point contributions.

Forthcoming paper contains

- Perturbative results based on Bender-Wu recursion relation
- A number of exact Multi-bion solutions
- Rich and full resurgence structure at δε^2 order

What we can do further

- Upgrade the solutions to 2D CPN sigma model

- Other exactly solvable models (near-SUSY, QES)

cf.) Kozcaz, Sulejmanpasic, Tanizaki, Unsal (16)

- Extension to Multi-variable QM