On virtual turning points

— an important ingredient of the WKB theory of higher order ODEs

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1 Introduction — What are virtual turning points?

Exact WKB analysis for higher order ODEs

$$P\psi = \left[\left(\frac{d}{dx}\right)^m + a_1(x)\eta\left(\frac{d}{dx}\right)^{m-1} + \cdots + a_m(x)\eta^m \right] \psi = 0$$

($\eta > 0$: large parameter)

$$\psi = \exp\left(\eta \int^x \lambda \, dx\right) \sum_{n=0}^{\infty} \eta^{-(n+1/2)} \psi_n(x)$$: **WKB solution**

where $\lambda^m + a_1(x)\lambda^{m-1} + \cdots + a_m(x) = 0.$

Consider the Borel sum of ψ w.r.t. η, i.e.,

$$\int_{-s(x)}^{\infty} e^{-\eta y} \psi_B(x, y) \, dy$$

where $\psi_B(x, y)$ is the Borel transform of ψ and $s(x) = \int^x \lambda \, dx$.

“Stokes geometry” (or “Stokes graph”)

$$\left(\begin{array}{ll} x = a : \textbf{turning point} & \Longleftrightarrow \quad \exists\, j \neq k \ \text{ s.t. } \ \lambda_j(a) = \lambda_k(a) \\ \textbf{Stokes curve} & \Longleftrightarrow \quad \operatorname{Im} \eta \int_a^x (\lambda_j(x) - \lambda_k(x))\, dx = 0 \end{array} \right.$$

2nd order case :

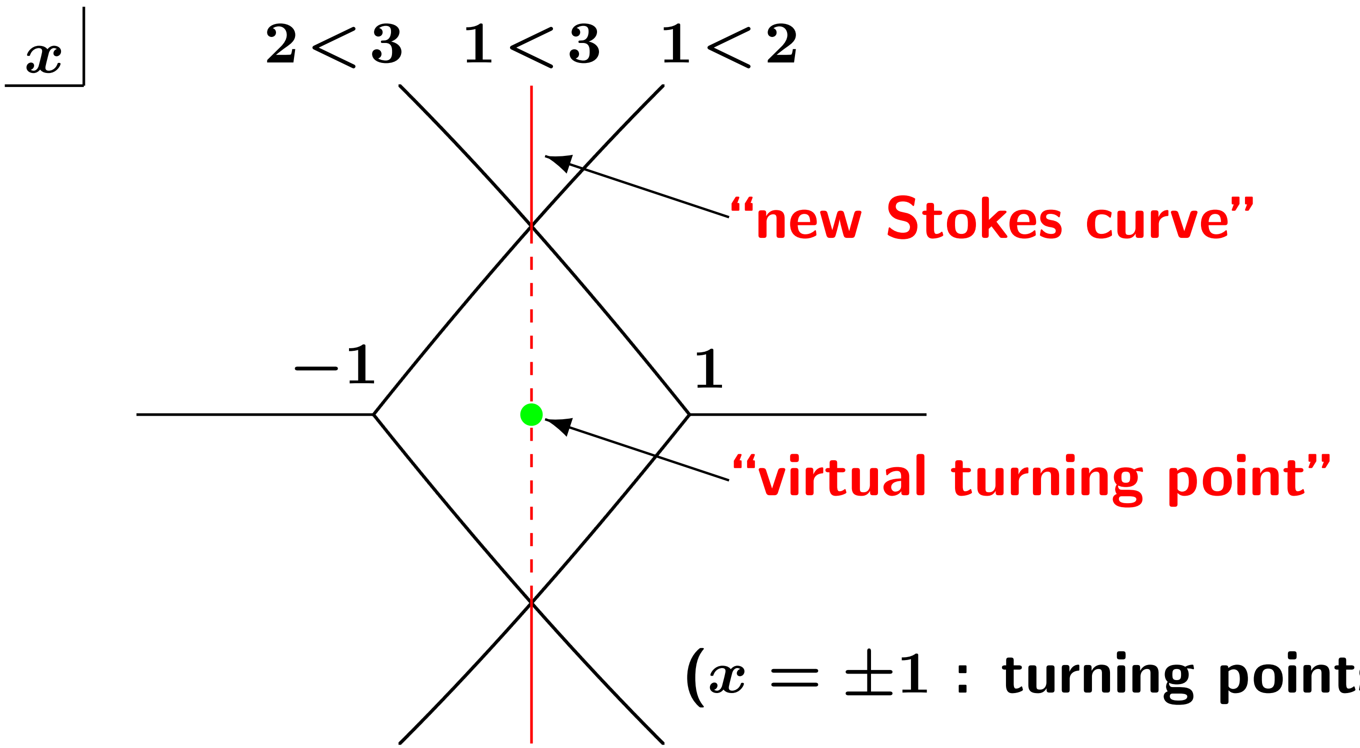
Borel summability of ψ breaks down only on Stokes curves.

higher order case :

Borel summability of ψ breaks down also on “new Stokes curves”.

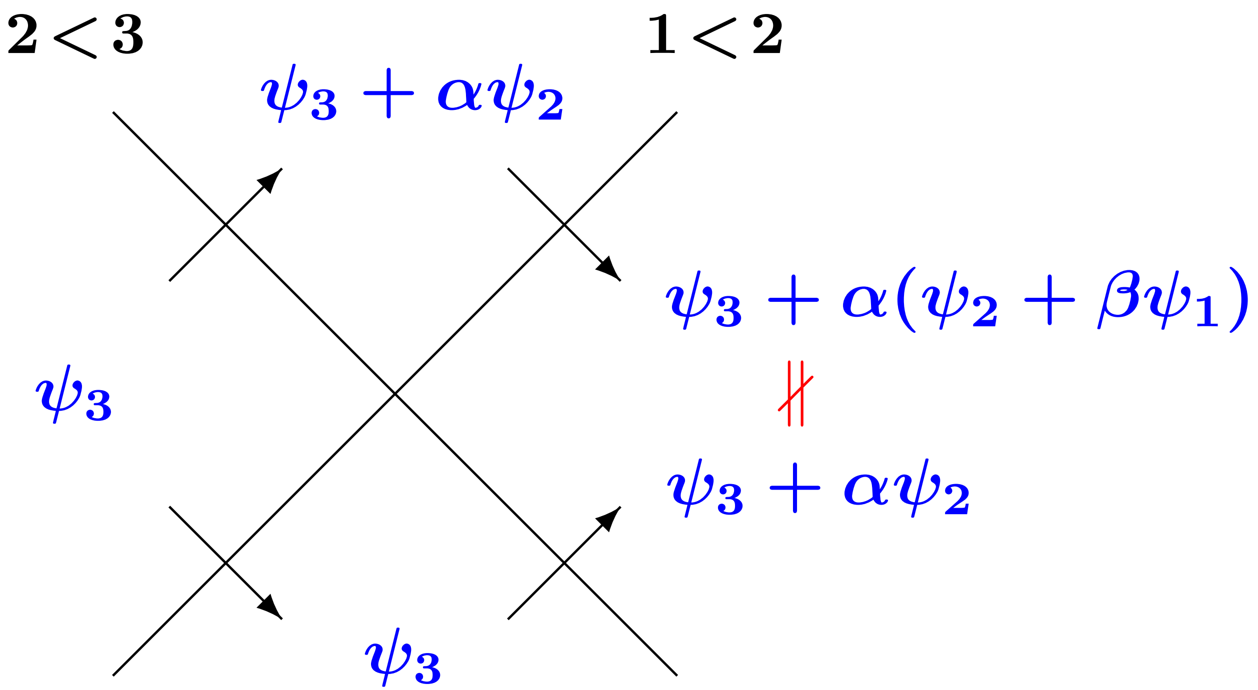
BNR equation (Berk-Nevins-Roberts [2], 1982)

$$\left[\left(\frac{d}{dx}\right)^3 + 3\eta^2\frac{d}{dx} + 2ix\eta^3\right]\psi = 0$$

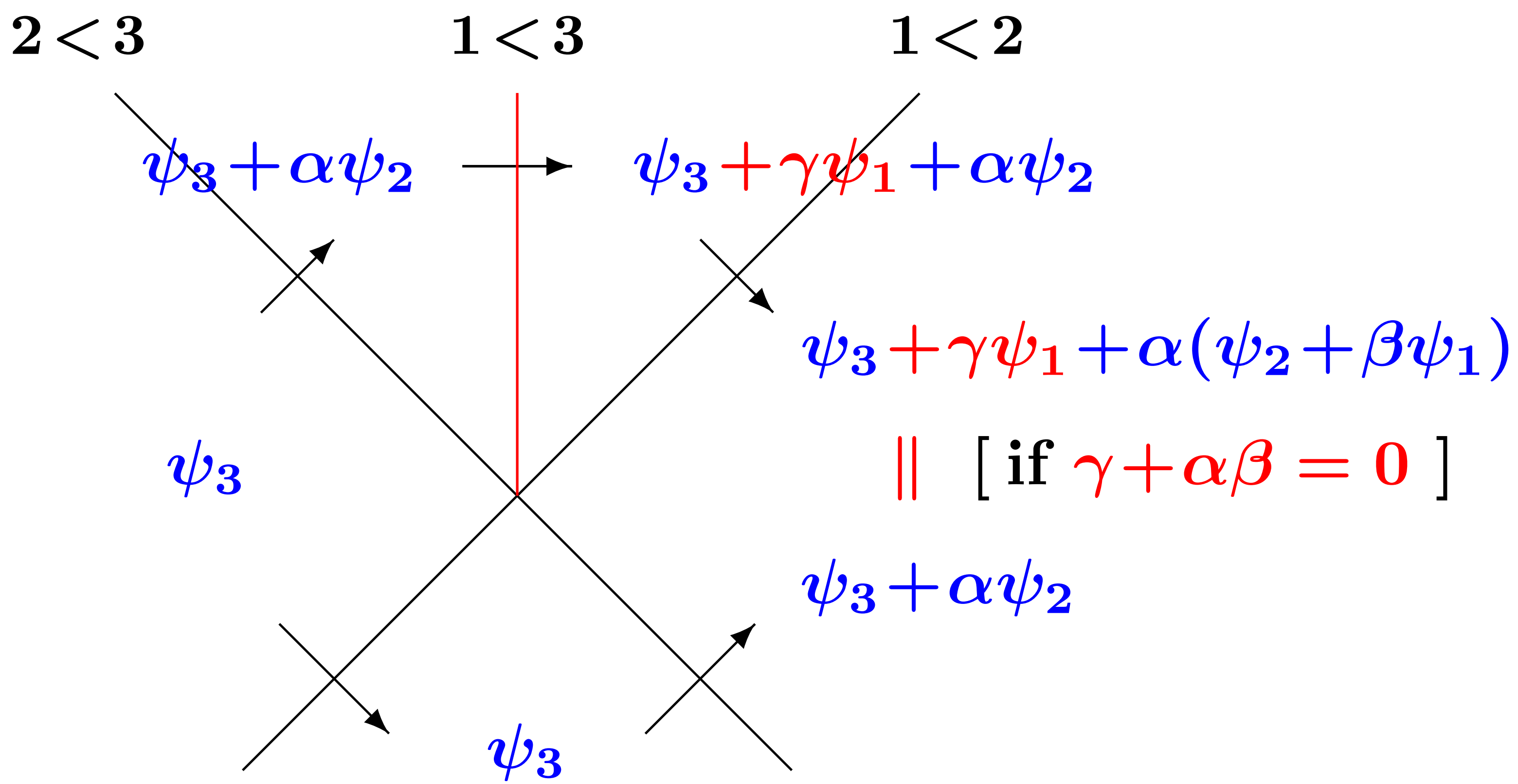


($x = \pm 1$: turning points)

In fact, unless a new Stokes curve is introduced, the following simple argument leads to a contradiction.



On the other hand, this contradiction disappears by introducing a new Stokes curve:



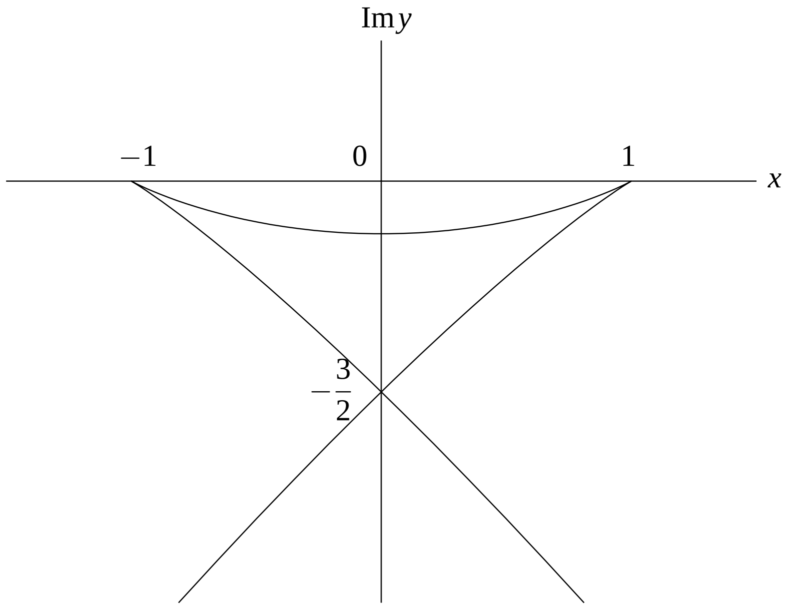
“Virtual turning point (VTP)”

$\Longleftrightarrow$ turning point where a new Stokes curve emanates

$\Longleftrightarrow$ the x-component of a point where two different singularities (more precisely, their projection to the base space) of the Borel transform $\psi_B(x,y)$ of ψ collide

$\Longleftrightarrow$ the x-component of a self-intersection point of the bicharacteristic curve of

$$P_B\psi_B = (\partial_x^m + a_1(x)\partial_y\partial_x^{m-1} + \cdots + a_m(x)\partial_y^m)\psi_B = 0$$



Reference

- Aoki-Kawai-Takei [1], 1994
- Honda-Kawai-Takei [3], 2015

cf. Gaiotto-Moore-Neitzke (*Ann. H. Poincaré*, 2013 & 2014) :
"spectral network"

It is not easy to handle VTPs as infinitely many VTPs appear in general. In this talk we discuss several topics concerning recent developments of the theory of VTPs such as

▶ VTPs of integral representations,

▶ triple crossing of Stokes curves,

▶ parametric Stokes phenomena associated with the degenerate Stokes graph peculiar to higher order ODEs.

2 s-VTPs & integral representations

(joint work with S. Hirose (Shibaura))

Consider

$$\psi = \int \exp(\eta S(x,t))\, dt \qquad (\eta : \text{large parameter}) \tag{1}$$

and a holonomic system of differential equations

$$P_j(x, \partial_x, \eta)\psi = 0 \quad (j = 1, \cdots, J) \tag{2}$$

that ψ satisfies.

Example 1 **(related to the singularity theory)**

- **type (A_n) :** $S(x,t) = t^{n+1} + x_{n-1}t^{n-1} + \cdots + x_1 t \quad (t \in \mathbb{C})$
- **type (D_n) :** $S(x,t) = t_1^2 t_2 + t_2^{n-1} + x_{n-1}t_1 + x_{n-2}t_2^{n-2} + \cdots + x_1 t_2 \quad (t = (t_1, t_2) \in \mathbb{C}^2)$

e.g.,

$$(A_2) \quad \psi = \int \exp(\eta(t^3 + x_1 t))\, dt \ : \ \textbf{Airy integral}$$

$$(3\partial_1^2 + x_1\eta^2)\psi = 0$$

$$(A_3) \quad \psi = \int \exp(\eta(t^4 + x_2 t^2 + x_1 t))\, dt \ : \ \textbf{Pearcy integral}$$

$$\begin{cases} (4\partial_1^3 + 2x_2\eta^2\partial_1 + x_1\eta^3)\psi = 0 \\ (\eta\partial_2 - \partial_1^2)\psi = 0 \end{cases}$$

Example 2 **(Shudo integral)**

$$S = \frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})^2 - \sum_{j=1}^{n-1} V(t_j), \quad V(t) = -\frac{1}{3}t^3 - ct$$

$$(t = (t_1, \cdots, t_{n-1}) \in \mathbb{C}^{n-1})$$

Assumptions

(A1) $S(x,t)$: polynomial

(A2) $\pi : \{ (x,t) \in \mathbb{C}^n_x \times \mathbb{C}^p_t \,|\, \mathrm{grad}_t S = (\frac{\partial S}{\partial t_1}, \ldots, \frac{\partial S}{\partial t_p}) = 0 \} \longrightarrow \mathbb{C}^n_x$ is a finite proper map, that is, for any point $\hat{x} \in \mathbb{C}^n_x$ there exist a neighborhood ω of $\hat{x}$ and continuous functions $t^{(j)}(x)$ $(j = 1, \ldots, J)$ on ω so that

$$\pi^{-1}(\omega) \subset \bigcup_{j=1}^{J} \{ (x,t) \,|\, t = t^{(j)}(x) \}$$

Definition

$x = x_0$: s-VTP $\iff$ There exist t and t' satisfying $t \neq t'$ and

$$\mathrm{grad}_t S(x_0, t) = \mathrm{grad}_t S(x_0, t')$$

$$S(x_0, t) = S(x_0, t')$$

Remark

1) Microlocal analysis tells us that the set of critical values

$$\{ (x,y) \,\big|\, \exists t \text{ s.t. } \mathrm{grad}_t S(x,t) = 0 \text{ and } y = S(x,t) \}$$

describes the singularities of the Borel transform $\psi_B(x,y)$ of a WKB solution ψ of (2). Thus s-VTPs are essentially equivalent to VTPs (although rigorous confirmation has not been obtained yet in full generality).

2) Stokes curves and Stokes surfaces emanating from s-VTPs are considered by some geometers (e.g., Arnold, Wright, etc.) under the name of "Maxwell set" or "Maxwell strata".

s-VTPs of the integral (1) of type (A_n)

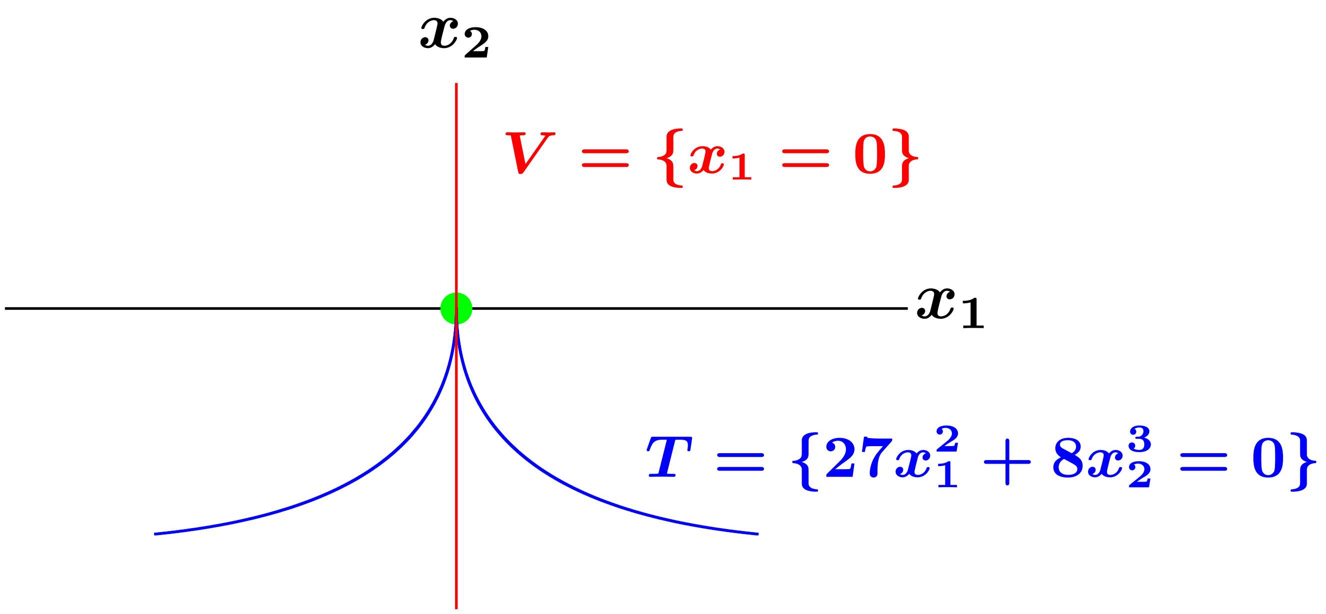
- **In the case of the integral (1) of type (A_n) the Stokes graph, i.e., the totality of Stokes curves and new Stokes curves, describes not only the regions for the Borel summability of WKB solutions of (2) but also the places where the configuration of steepest descent paths of (1) passing through saddle points changes.**

- **In particular, to understand the configuration of new Stokes curves of (1), VTPS or s-VTPs play an important role.**

- **In the current situation s-VTPs of the integral (1) can be neatly handled as there appear only finitely many s-VTPs.**

Example

$$(A_3): \quad \begin{cases} (4\partial_1^3 + 2x_2\eta^2\partial_1 + x_1\eta^3)\psi = 0 \\ (\eta\partial_2 - \partial_1^2)\psi = 0 \end{cases}$$



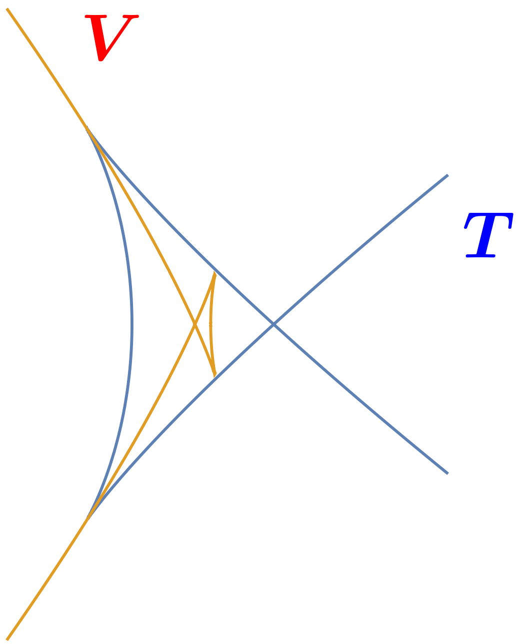
The origin is a cuspidal singularity of the set of turning points T.
The system of type (A_3) gives a normal form at such cuspidal singularities (Hirose, *Publ. RIMS*, 2014).

$$(A_4): \begin{cases} (5\partial_1^4 + 3x_3\eta^2\partial_1^2 + 2x_2\eta^3\partial_1 + x_1\eta^4)\psi = 0 \\ (\eta\partial_2 - \partial_1^2)\psi = 0 \\ (\eta^2\partial_3 - \partial_1^3)\psi = 0 \end{cases}$$

Let $x_3 = -1$ be fixed. Then we find

$$T = \{400x_1^3 - 135x_2^4 + 540x_1x_2^2x_3 - 360x_1^2x_3^2 - 27x_2^2x_3^3 + 81x_1x_3^4 = 0\}$$

$$V = \{1600x_1^3 + 135x_2^4 + 360x_1x_2^2x_3 - 1040x_1^2x_3^2 - 88x_2^2x_3^3 + 224x_1x_3^4 - 16x_3^6 = 0\}$$

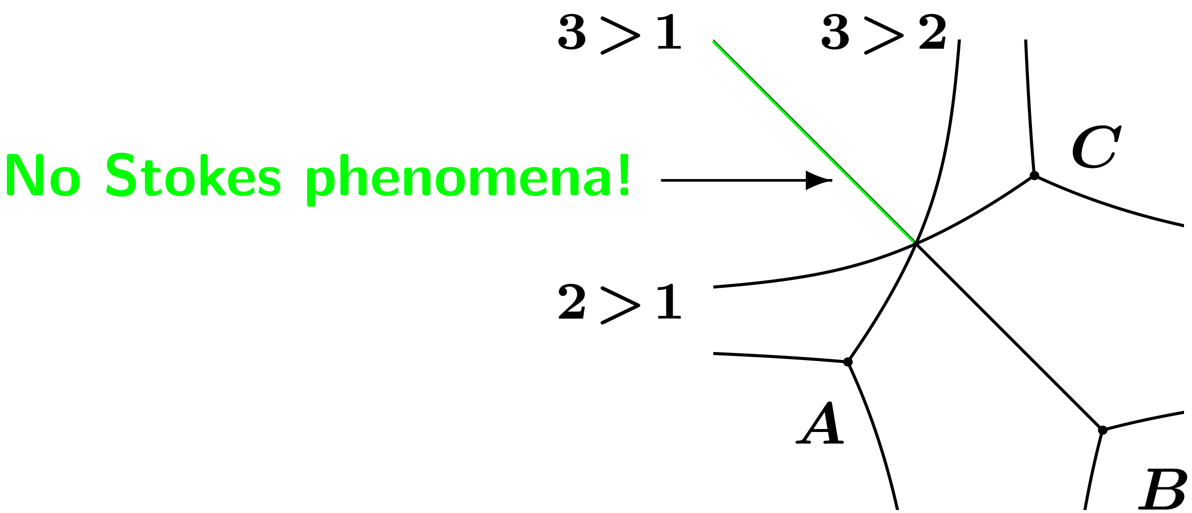


3 Triple crossing of Stokes curves

Consider the Pearcey system of type (A_3). Let x_1 be fixed, then we obtain the following 3rd order ODE:

$$\left(\partial_2^3 + x_2\eta\partial_2^2 + \frac{1}{4}x_2^2\eta^2\partial_2 + \frac{x_1^2}{16}\eta^3\right)\psi = 0$$

For this equation there exist no VTPs. Instead we find the following triple crossing of Stokes curves:

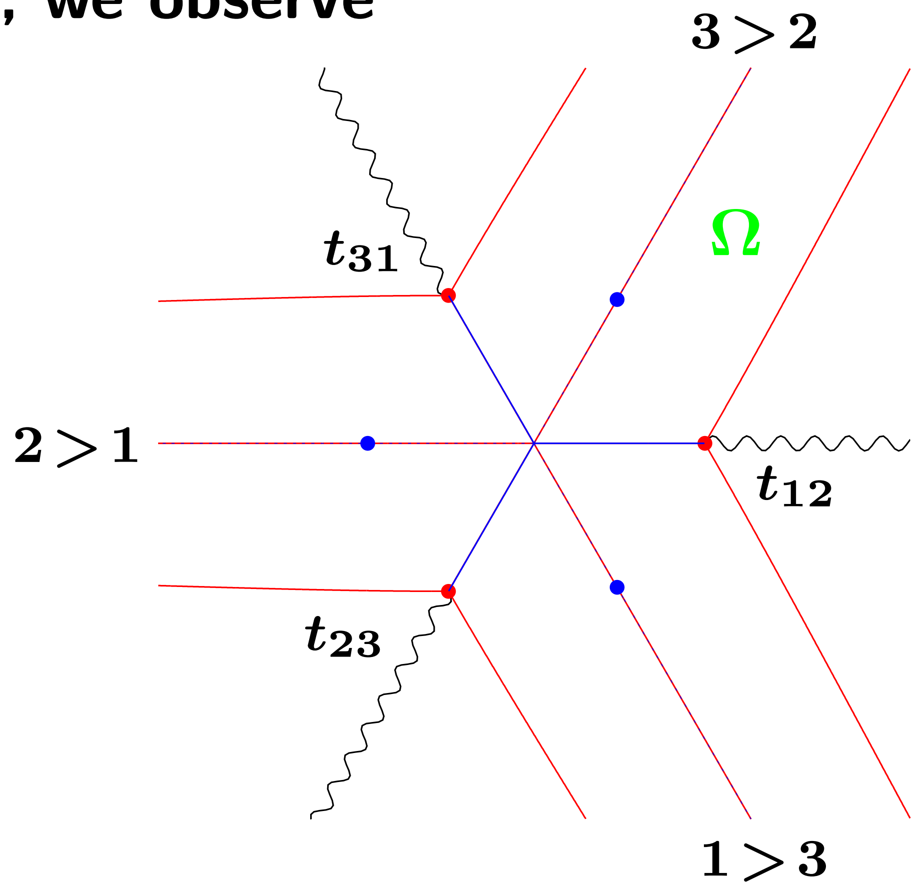


Such triple crossing and "vanishing of Stokes curves" are observed also for the system of type (A_4).

4 Parametric Stokes phenomena related to Stokes trees

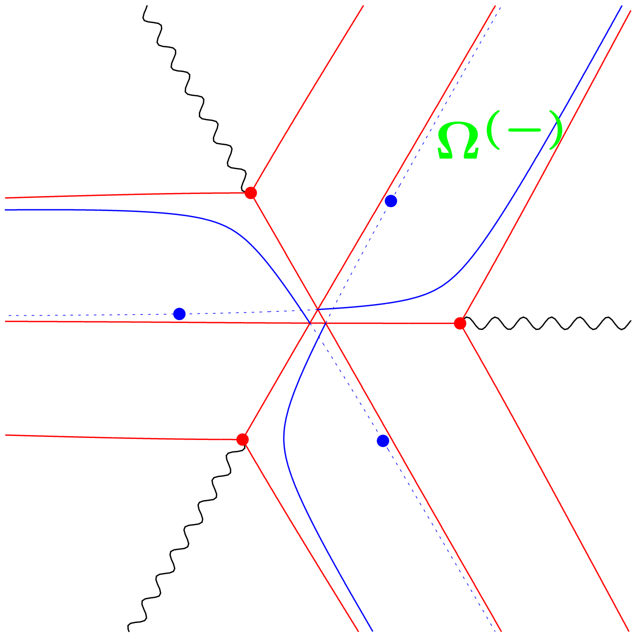
Example $$\left(\partial_x^3 + \frac{1}{3}x\eta^2\partial_x - \alpha\eta^3\right)\psi = 0 \tag{3}$$

When $\alpha \in i\mathbb{R}_-$, we observe

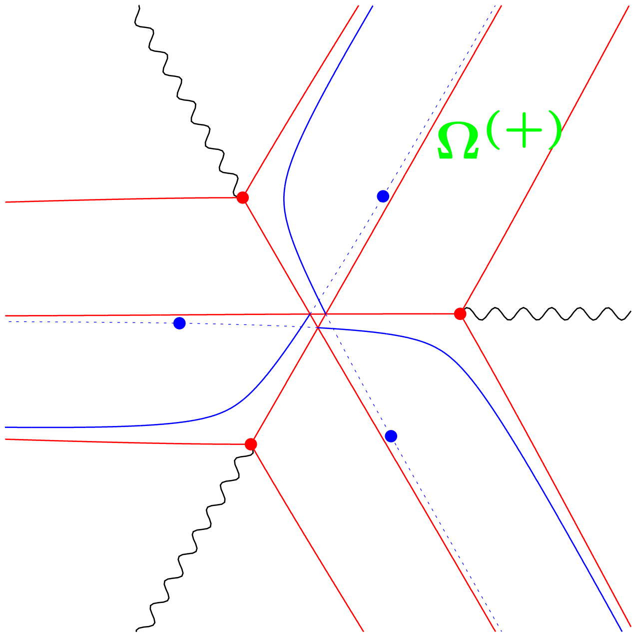


This is a typical degenerate configuration of Stokes curves peculiar to higher order ODEs.

S. Sasaki (Toronto) proves that this degenerate configuration induces the following parametric Stokes phenomenon for WKB solutions:



$(\mathrm{Re}\,\alpha < 0)$



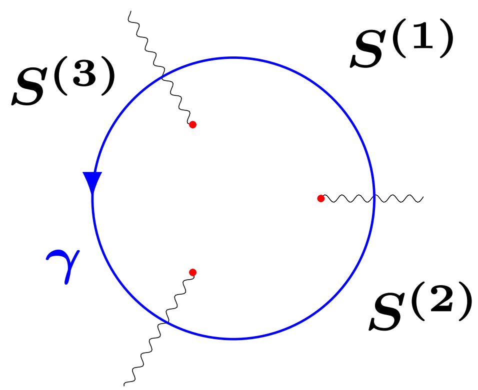
$(\mathrm{Re}\,\alpha > 0)$

Theorem (Sasaki, to appear in *RIMS Kôkyûroku Bessatsu*)

Let $\Psi_j^{(\pm)}$ be the Borel sum of WKB solutions ψ_j $(j = 1, 2, 3)$ in $\Omega^{(\pm)}$ for $\pm\mathrm{Re}\,\alpha > 0$, respectively. Then we have

$$\begin{cases} \Psi_1^{(-)} = \left[1 - \exp\left(\int_\gamma S\,dx\right)\right] \Psi_1^{(+)} \\ \Psi_2^{(-)} = \left[1 - \exp\left(\int_\gamma S\,dx\right)\right]^{-1} \Psi_2^{(+)} \\ \Psi_3^{(-)} = \Psi_3^{(+)} \end{cases}$$

where S denotes the formal solution of $(S^3 + 3SS' + S'') + (x/3)\eta^2 S - \alpha\eta^3 = 0$ associated with (3) and γ is a contour shown in the right.



Remark

1) Similar relations are also discussed by Gaiotto-Moore-Neitzke in a different context.

2) The above connection formula is related to the so-called "fixed singularities" of the Borel transform of WKB solutions, and hence can be expressed also in the form of resurgent relations for them. The fixed singularities are associated with periods of the algebraic curve $\lambda^m + a_1(x)\lambda^{m-1} + \cdots + a_m(x) = 0$.

3) The triple crossing of Stokes curves discussed here is the simplest example of "Stokes trees" introduced by N. Honda (*RIMS Kôkyûroku Bessatsu*, 2007). We can also discuss parametric Stokes phenomena for more general Stokes trees with many edges.

Summary

- In the case of higher order ODEs and holonomic systems of differential equations we need to introduce **virtual turning points (VTPs) and new Stokes curves** to describe the regions for the Borel summability of WKB solutions.

- **VTPs (or s-VTPs) can be neatly handled for the integral representations** and the holonomic systems that they satisfy.

- Typical degenerate configurations of Stokes curves peculiar to higher order ODEs are described by Stokes trees. **Connection formulas for parametric Stokes phenomena associated with Stokes trees are explicitly provided**.