Instantons from perturbation theory

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based on *Marco Serone, GS, Giovanni Villadoro* arxiv:1612.0XXXX

Motivation and summary of results

Perturbation series in QM and QFT usually divergent

- Borel resummation
- Transseries: $\mathcal{O}(\lambda) = \sum_{n\geq 0} c_n^{(0)} \lambda^n + \sum_i e^{-S_i/\lambda} \sum_{n\geq 0} c_n^{(i)} \lambda^n$

What I will show

→ Borel sum on a regular thimble is exact

→ trick in QM: complete results from the perturbative series

Lefschetz Thimble decomposition

$$Z(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} dx \, e^{-f(x)/\lambda}$$

f complex function

Lefschetz Thimble decomposition

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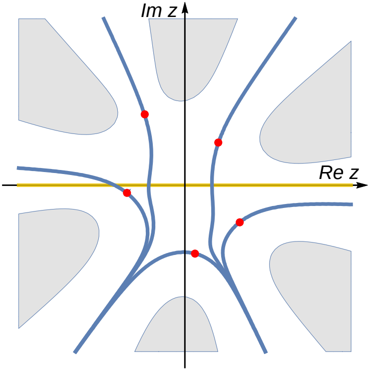
z_σ saddle points $f'(z_\sigma) = 0$
$f''(z_\sigma) \neq 0$

$\mathcal{J}_\sigma$ steepest descent paths (thimbles)



Lefschetz Thimble decomposition

$$Z(\lambda) = \sum_\sigma n_\sigma e^{-f(z_\sigma)/\lambda} \underbrace{\frac{1}{\sqrt{\lambda}} \int_{\mathcal{J}_\sigma} dz\, e^{-(f(z)-f(z_\sigma))/\lambda}}_{Z_\sigma(\lambda)}$$

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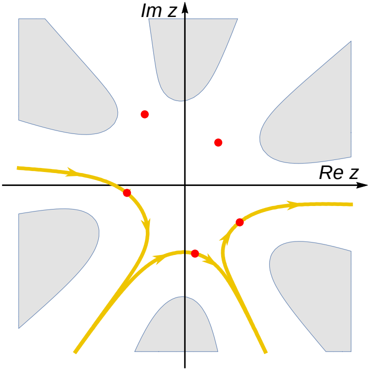

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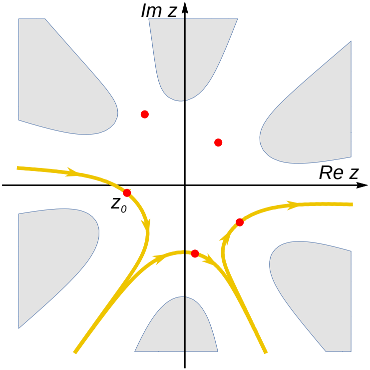
$\mathcal{J}_\sigma$ steepest descent paths (thimbles)

$$Z_0(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} dx\, g(x)\, e^{-f(x)/\lambda}$$

asymptotic series in λ



Borel summation

$$\sum_n c_n \lambda^n = \frac{1}{\lambda^{1+b}} \int_0^\infty dt\, e^{-t/\lambda} t^b \,\mathcal{B}(t)\,, \qquad \mathcal{B}(t) = \sum_n \frac{c_n t^n}{\Gamma(n+b+1)}$$

Borel summation – 1 dim integrals

$$\sum_n c_n \lambda^n = \frac{1}{\lambda^{1+b}} \int_0^\infty dt\, e^{-t/\lambda} t^b\, \mathcal{B}(t)\,, \qquad \mathcal{B}(t) = \sum_n \frac{c_n t^n}{\Gamma(n+b+1)}$$

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$$= \frac{1}{\sqrt{\lambda}} \int_0^\infty dt\, e^{-t/\lambda} t^{-1/2} \underbrace{\sum_\pm \frac{g(x_\pm(t))}{|f'(x_\pm(t)|} \sqrt{t}}_{\text{Borel function}}$$

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Borel summable if $\underbrace{f'(x(t)) \neq 0}_{\text{no Stokes line}}$

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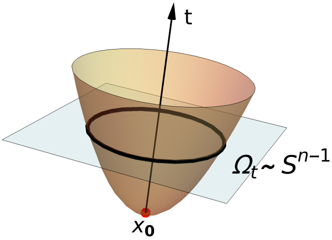
$$= \frac{1}{\sqrt{\lambda}} \int_0^\infty dt\, e^{-t/\lambda} t^{-1/2} \underbrace{\int_{\Omega_t} d\Sigma \frac{g(\mathbf{z}(t,\Sigma))}{|\nabla f(\mathbf{z}(t,\Sigma))|} \sqrt{t}}_{\text{Borel function}}$$



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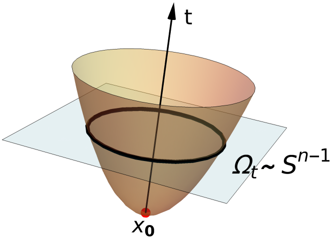

Borel summable if $\underbrace{\nabla f(\mathbf{x}) \neq 0}_{\text{no Stokes line}}$

Borel summation – path integral

$$Z(\lambda) = \int \mathcal{D}x(\tau)\, Q[x(\tau)] e^{-S[x(\tau)]/\lambda}$$

x_0 solution of eom: $S'[x_0(\tau)] = 0$

Borel summation – path integral

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Formally

$$Z(\lambda) = \int_0^\infty dt \int \mathcal{D}x(\tau)\, Q[x(\tau)]\, e^{-S[x(\tau)]/\lambda}\, \delta(t - S[x(\tau)])$$
$$= \int_0^\infty dt\, e^{-t/\lambda} \int \mathcal{D}\Sigma\, \frac{Q[x(t,\Sigma)]}{|S'[x(t,\Sigma)]|}$$

Expectation: Borel summable if $\underbrace{S'(x(t)) \neq 0}_{\text{no Stokes line}}$

Borel summation – path integral

$$Z(\lambda) = \int \mathcal{D}x(\tau)\, Q[x(\tau)] e^{-S[x(\tau)]/\lambda}$$

Remark: solutions to eom depend on boundary conditions (the observable)

$$\int_{\substack{x(0)=x_0 \\ x(T)=x_T}} \mathcal{D}x\, e^{-S[x]/\lambda} = \sum_n \psi_n(x_0)^* \psi_n(x_T)\, e^{-E_n T}$$

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$$S(x) = \int_0^\beta dt \left(\frac{\dot{x}^2}{2} + V(x) \right) \qquad V(x) \to \infty \text{ as } |x| \to \infty$$

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$V(x)$ has 1 minimum $\Rightarrow$ all observables Borel summable to exact result

example: Anharmonic Oscillator

$$V(x) = \frac{x^2}{2} + \frac{\lambda}{2}x^4$$

Borel sum of energy eigenvalues is exact

B. Simon and A. Dicke (1970)
S. Graffi, V. Grecchi and B. Simon(1970)

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Borel sum of **any observable** is exact

Checked numerically for wavefunctions

A simple observation

$$Z(\lambda) = \int \mathcal{D}x \, Q(x) \, \exp\left[-\frac{S(x)}{\lambda}\right]$$

Expansion in $\lambda \quad \Leftrightarrow \quad$ saddle points of $S(x)$

A simple observation

$$Z(\lambda) = \int \mathcal{D}x \; \exp\Big[-\frac{S(x) - \lambda \log Q(x)}{\lambda} \Big]$$

Expansion in λ $\quad\Leftrightarrow\quad$ saddle points of $S(x)$

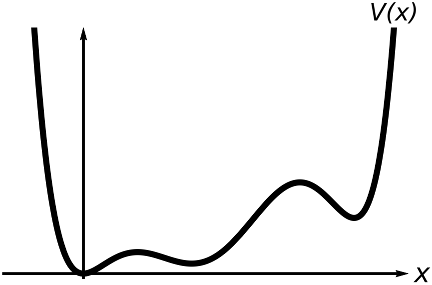
A simple observation

$$\hat{Z}(\lambda, \lambda_0) = \int \mathcal{D}x \; \exp\Big[-\frac{S(x) - \lambda_0 \log Q(x)}{\lambda}\Big]$$

Expansion in λ $\Leftrightarrow$ saddle points of $S_{\text{eff}}(x) = S(x) - \lambda_0 \log Q(x)$ (λ_0 fixed)

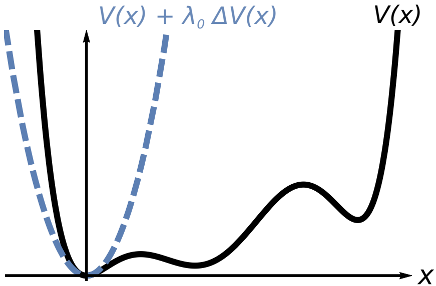
Exact Perturbation Theory (EPT)

$V(x)$
many minima



Exact Perturbation Theory (EPT)

$$\underset{\text{many minima}}{V(x)} \rightarrow \boxed{\underset{\text{one minimum}}{V(x) + \lambda_0 \Delta V(x)}} - \lambda \Delta V(x)$$



example: Pure Quartic Oscillator

$$V(x) = \frac{1}{2}x^4$$

Normal perturbation theory cannot be used: $V''(0) = 0$

WKB expansion not Borel resummable

R. Balian, G. Parisi and A. Voros, (1978)

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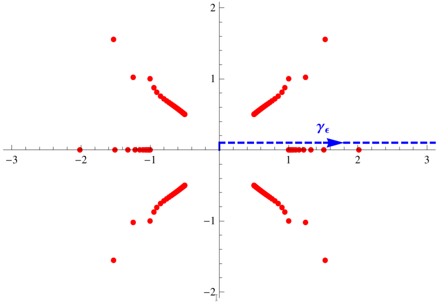
$$E_0^{\mathrm{WKB}} = 0.5302$$

using 320 orders
+ complex and real instantons



A. Grassi, M. Mariño and S. Zakany, (2015)

example: Pure Quartic Oscillator

$$\hat{V}(x;\lambda) = \boxed{\frac{\lambda}{2}x^4 + \frac{1}{2}x^2} - \frac{\lambda}{2}x^2$$

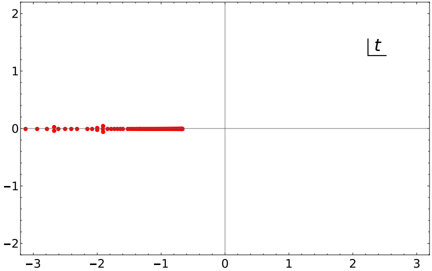
Normal perturbation theory cannot be used: $V''(0) = 0$

WKB expansion not Borel resummable

R. Balian, G. Parisi and A. Voros, (1978)

$E_0^{\mathrm{EPT}} = 0.53018104524209$

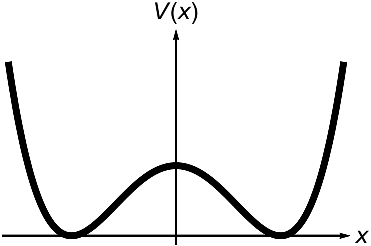
using 60 orders



BenderWu package – Tin Sulejmanpasic, Mithat Ünsal (2016)

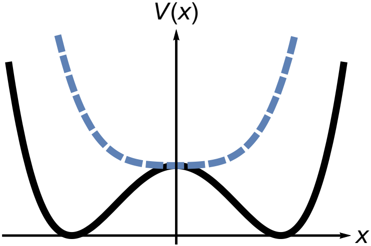
example: Symmetric Double Well

$$V(x;\lambda) = \frac{1}{32\lambda} - \frac{1}{4}x^2 + \frac{\lambda}{2}x^4$$



example: Symmetric Double Well

$$\hat{V}(x;\lambda,\lambda_0) = \boxed{\frac{1}{32\lambda} + \frac{\lambda_0}{2}x^2 + \frac{\lambda}{2}x^4} - \frac{\lambda}{2}\left(1+\frac{1}{2\lambda_0}\right)x^2$$



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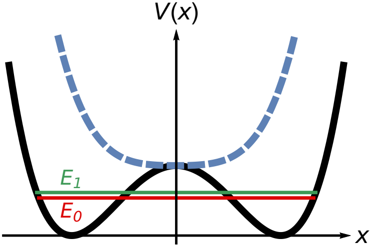
$$\lambda = 0.04$$

$$(E_1 - E_0)^{\mathrm{EPT}} = 0.060941915(6)$$

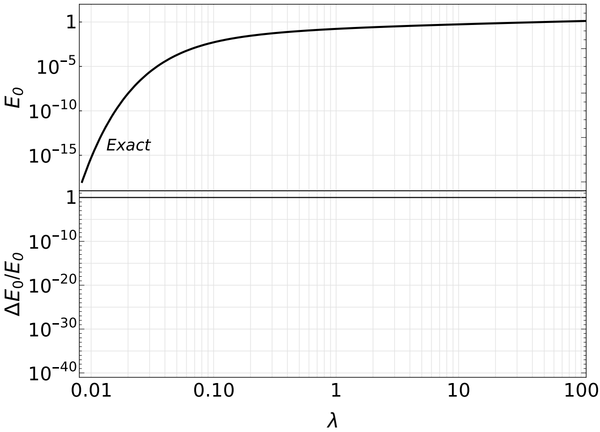
using 200 orders



example: SUSY Double Well

$$V(x;\lambda) = \frac{1}{32\lambda} - \frac{1}{4}x^2 + \frac{\lambda}{2}x^4 + \sqrt{\lambda}x$$

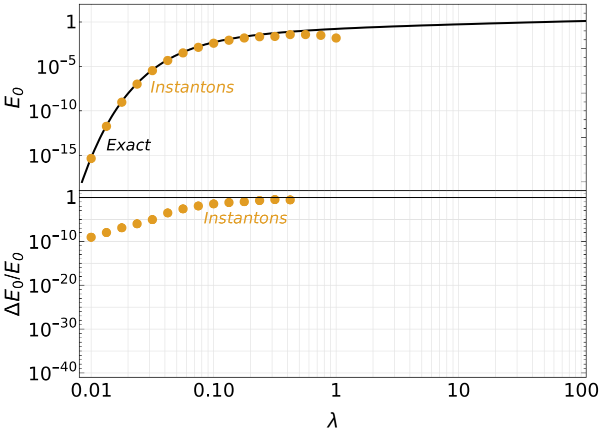
$E_0 > 0$ only non-perturbatively: $E_0 \sim e^{-1/\lambda}$ E. Witten, (1981)



example: SUSY Double Well

$$V(x;\lambda) = \frac{1}{32\lambda} - \frac{1}{4}x^2 + \frac{\lambda}{2}x^4 + \sqrt{\lambda}x$$

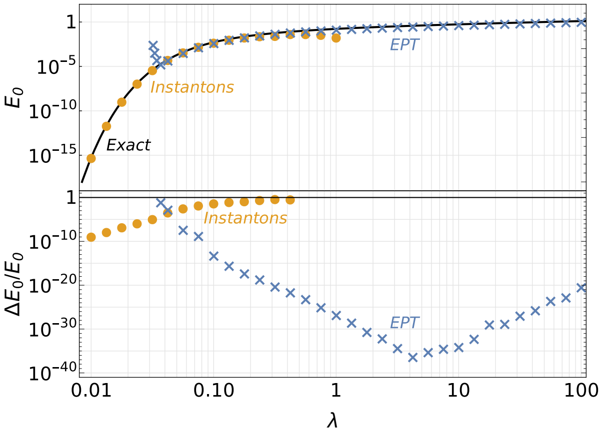
$E_0 > 0 \;\Leftarrow\;$ **instantons** U. D. Jentschura and J. Zinn-Justin, (2004)



example: SUSY Double Well

$$\hat{V}(x;\lambda,\lambda_0) = \boxed{\frac{1}{32\lambda} + \frac{\lambda_0}{2}x^2 + \frac{\lambda}{2}x^4} + \sqrt{\lambda}x - \frac{\lambda}{2}\left(1+\frac{1}{2\lambda_0}\right)x^2$$

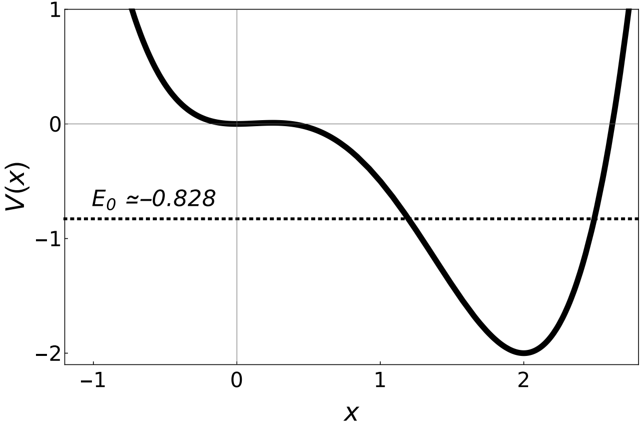
EPT 200 orders: $E_0 > 0$ perturbatively



example: False Vacuum

$$V(x;\lambda) = \frac{1}{2}x^2 + \frac{\lambda}{2}x^4 + \quad \frac{3}{2}\sqrt{\lambda}x^3$$

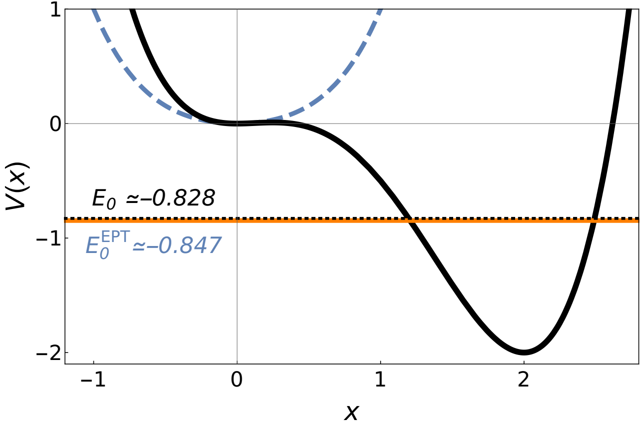
evaluated at $\lambda = 1$



$$\hat{V}(x;\lambda;\lambda_0) = \boxed{\frac{1}{2}x^2 + \frac{\lambda}{2}x^4} + \frac{\lambda}{\lambda_0}\frac{3}{2}\sqrt{\lambda}x^3$$

evaluated at $\lambda = 1$

EPT 280 orders: accuracy of 2%



Conclusions and prospects

Exact Perturbation Theory non-trivial results

- Borel resummable
- Gives the full answer
- Works very well at strong coupling

Moving to QFT

- EPT in $1+1$ dimensions?