# A String Theory Explanation for Quantum Chaos in the Hadronic Spectrum


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L. PZ and D. Reichmann, **JHEP 1304 (2013) 083, arXiv: 1209.5902**

P. Basu, D.Das, A. Ghosh, L PZ, **JHEP 1205 (2012) 077, arXiv:1201.5634**

P. Basu and LPZ, **Phys.Rev. D84 (2011) 046006**, arXiv:1105.2540

# Outline

- String theory and classical trajectories revisited.
- Chaos (classical) around holographic Regge Trajectories.
- What is quantum chaos in the spectrum of hadrons in QCD and other strongly coupled theories?
- The spectrum of highly excited Hadrons in holographic models.

# What has string theory been doing lately (last ten years)?

- String in curved spacetimes with RR fluxes.
- Semiclassical quantization: BMN, GKP (integrability) and Regge Trajectories.
- Prominent role of classical trajectories.

# Could QCD be a string theory?

- Regge trajectories are best explained by a string

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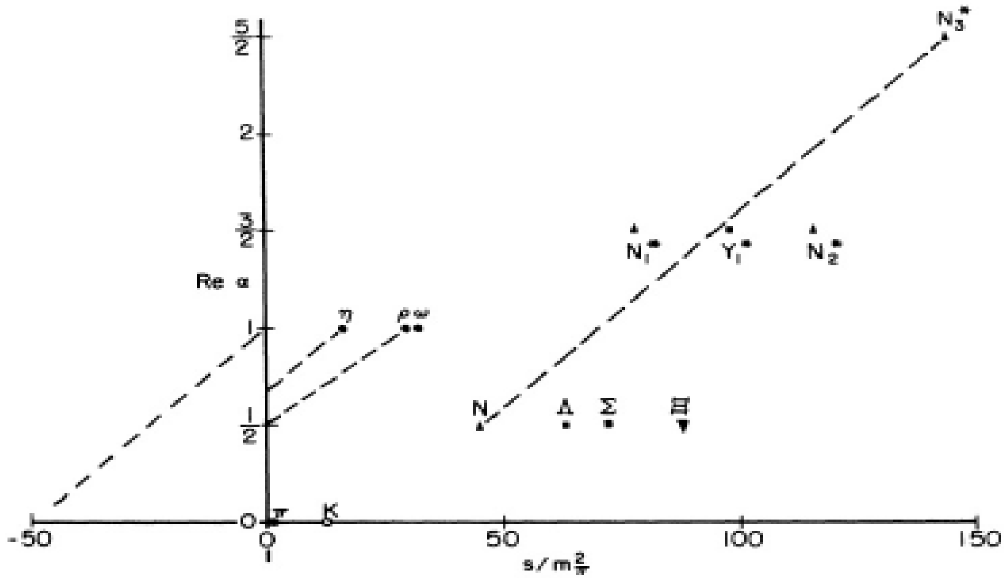
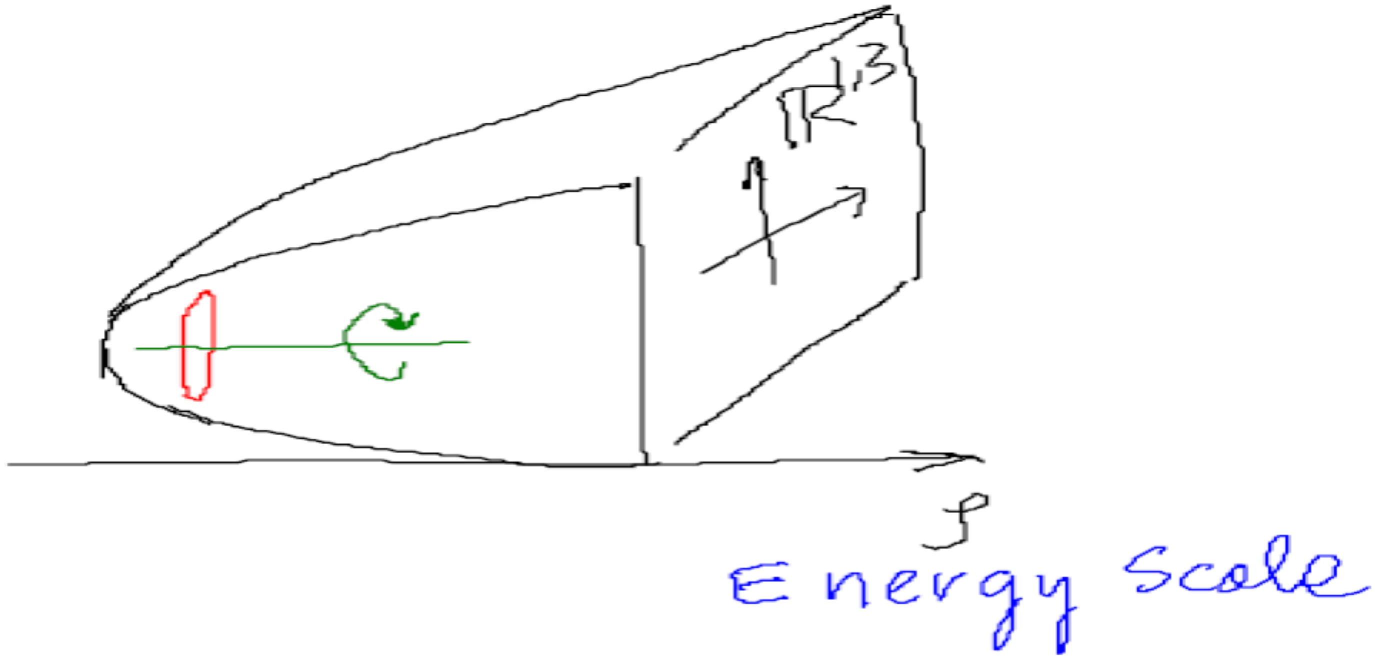



FIG. 1. The spin of particles of baryon number less than two, plotted against the square of their mass in units of $m_\pi^2$. In order to give a rough indication of slopes, the dashed lines connect pairs of points supposedly on the same trajectories, as explained in the text, but a strict linear behavior of the trajectories is not to be inferred.

$$J \sim M^2$$

# Holographic Regge Trajectories



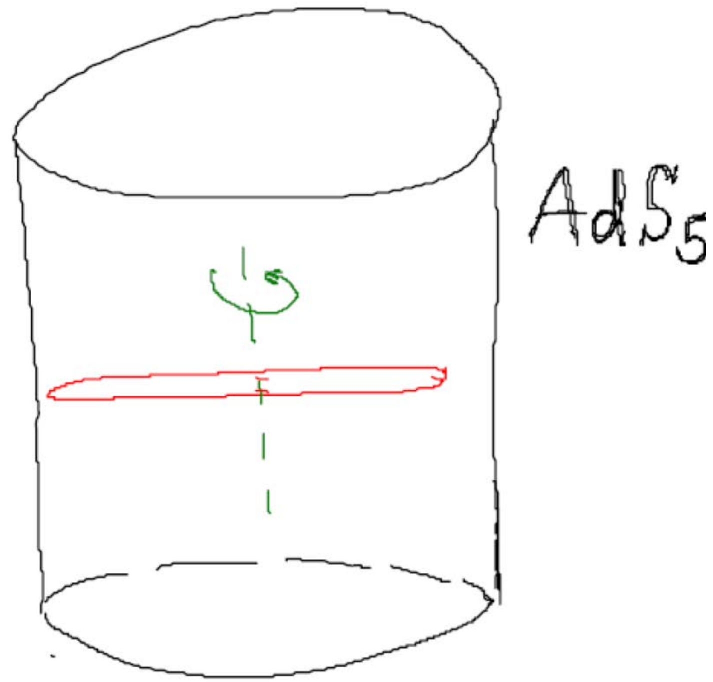

Holographic Regge Trajectory

# GKP Operators (twist-two in QCD)

$$\mathrm{Tr}\Phi^I \nabla_{(a_1} \ldots \nabla_{a_n)} \Phi^I$$

$$\Delta - S = (\sqrt{\lambda}/\pi) \ln S$$

# GKP String and Twist-Two Operators




GKP string dual to
Twist-two Operators

# Classical Chaos around Holographic Regge Trajectories

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\sqrt{-g}g^{ab}G_{MN}\partial_a X^M \partial_b X^N$$

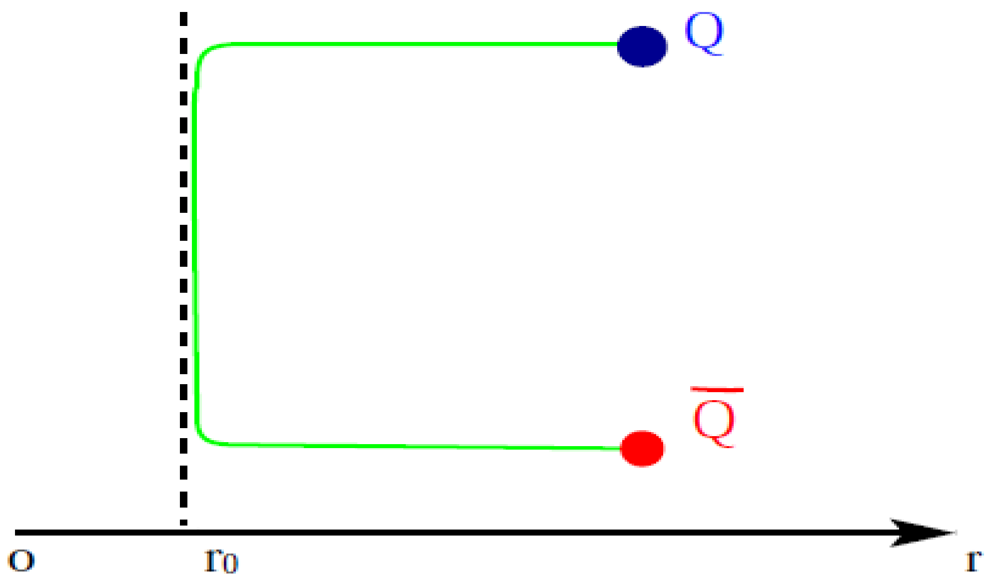
Virasoro constraints to get to the conformal gauge.

$$ds^2 = a^2(r)dx_\mu dx^\mu + b^2(r)dr^2 + c^2(r)d\Omega_d^2$$

# A generic property of holographic duals of confining theories

**Confinement == Wilson Area Law**

- End of Space.
- Wilson Loop shows confining behavior.




$$T_s = \frac{1}{2\pi\alpha'} g_{tt}(r_0)$$

- $g_{tt}(r_0) \neq 0$ and $g_{tt}(r_0)$ has a minimum

# Regge Trajectories in Holography

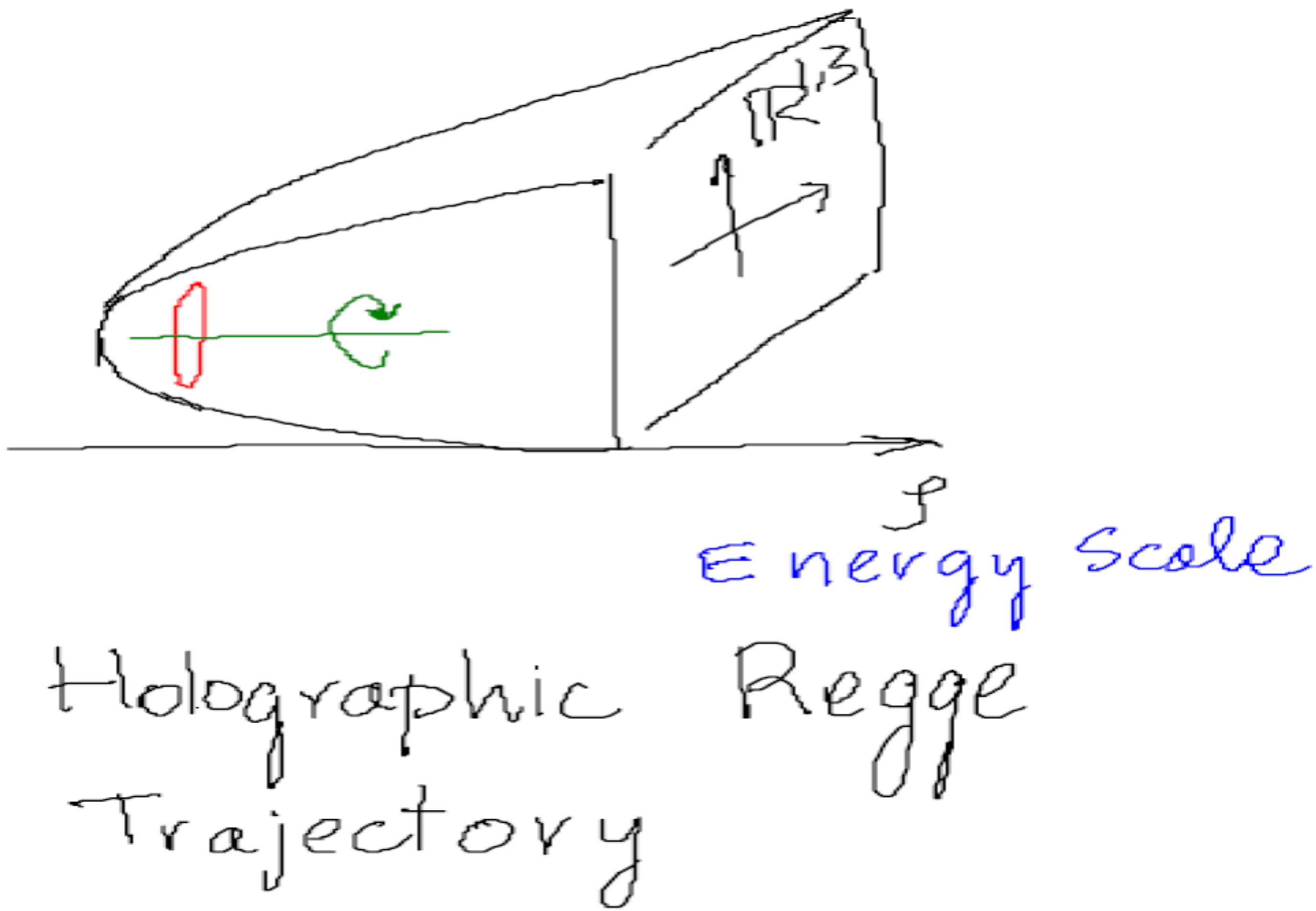
$$x^0 = e\,\tau, \qquad x^1 = e\cos\tau\,\sin\sigma, \qquad x^2 = e\sin\tau\,\sin\sigma.$$

$$E = 4\frac{e\,g_{00}(r_0)}{2\pi\alpha'}\int d\sigma = 2\pi g_{00}(r_0)T_s e, \qquad J = 4\frac{g_{00}(r_0)e^2}{2\pi\alpha'}\int \sin^2\sigma\, d\sigma = \pi g_{00}(r_0)T_s e^2$$

$$T_{s,\ eff} = g_{00}(r_0)/(2\pi\alpha')$$

$$J = \frac{1}{4\pi T_{s,\ eff}}E^2 \equiv \frac{1}{2}\alpha'_{eff}\ t.$$

# Holographic Regge Trajectories



# Winding Strings

$$t = t(\tau), \qquad r = r(\tau),$$
$$x_1 = R(\tau)\sin\alpha\sigma, \qquad x_2 = R(\tau)\cos\alpha\sigma.$$

Equations of motion

$$\frac{d}{d\tau}\left(b^2(r)\frac{d}{d\tau}r(\tau)\right) = \frac{E^2}{a^3(r)}\frac{d}{dr}a(r) + a(r)\frac{d}{dr}a(r)[\dot{R}^2 - \alpha^2 R^2] + b(r)\frac{d}{dr}b(r)(\frac{d}{d\tau}r)^2,$$

$$\frac{d}{d\tau}\left(a^2(r)\frac{d}{d\tau}R(\tau)\right) = -\alpha^2 a^2(r)R(\tau).$$

# Direct Analysis of Phase Space

- MN solution

$$\frac{d}{d\tau}\left(b^2(r)\frac{d}{d\tau}r(\tau)\right) = \frac{E^2}{a^3(r)}\frac{d}{dr}a(r) + a(r)\frac{d}{dr}a(r)[\dot{R}^2 - \alpha^2 R^2] + b(r)\frac{d}{dr}b(r)(\frac{d}{d\tau}r)^2,$$

$$\frac{d}{d\tau}\left(a^2(r)\frac{d}{d\tau}R(\tau)\right) = -\alpha^2 a^2(r) R(\tau).$$

# MN

$$ds^2 = e^{\phi}\left[\eta_{\mu\nu}dx^\mu dx^\nu + \alpha' g_s N\left(dr^2 + ds_5^2\right)\right],$$
$$ds_5^2 = e^{2g(r)}(e_1^2 + e_2^2) + \frac{1}{4}(e_3^2 + e_4^2 + e_5^2)),$$
$$e^{2\phi} = e^{-2\phi_0}\frac{\sinh 2r}{2e^{g(r)}},$$
$$e^{2g(r)} = r\coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4},$$

where $\mu, \nu = 0, 1, 2, 3$ and

$$e_1 = d\theta_1, \qquad e_2 = \sin\theta_1 d\phi_1,$$
$$e_3 = \cos\psi\, d\theta_2 + \sin\psi \sin\theta_2\, d\phi_2 - a(\tau)d\theta_1,$$
$$e_4 = -\sin\psi\, d\theta_2 + \cos\psi \sin\theta_2\, d\phi_2 - a(\tau)\sin\theta_1 d\phi_1,$$
$$e_5 = d\psi + \cos\theta_2\, d\phi_2 - \cos\theta_1 d\phi_1, \quad a(r) = \frac{r^2}{\sinh^2 r},$$

where $\mu = 0, 1, 2, 3$, we set the integration constant $e^{\phi_{D_0}} = \sqrt{g_s N}$ The 3-form is

$$H^{RR} = g_s N\left[-\frac{1}{4}(w^1 - A^1)\wedge(w^2 - A^2)\wedge(w^3 - A^3) + \frac{1}{4}\sum_a F^a \wedge (w^a - A^a)\right]$$

$$A = \frac{1}{2}\left[\sigma^1 a(r)d\theta_1 + \sigma^2 a(r)\sin\theta_1 d\phi_1 + \sigma^3 \cos\theta_1 d\phi_1\right] \qquad (8)$$

and the one-forms $w^a$ are given by:

$$w^1 + iw^2 = e^{-i\psi}(d\theta_2 + i\sin\theta_2 d\phi_2), \qquad w^3 = d\psi + \cos\theta_2 d\phi_2$$

# What we really need

$$a(r)^2 = e^{-\phi_0} \frac{\sqrt{\sinh(2r)/2}}{\left(r \coth 2r - \frac{r^2}{\sinh^2(2r)} - \frac{1}{4}\right)^{1/4}}, \qquad b(r)^2 = \alpha' g_s N a(r)^2$$

-

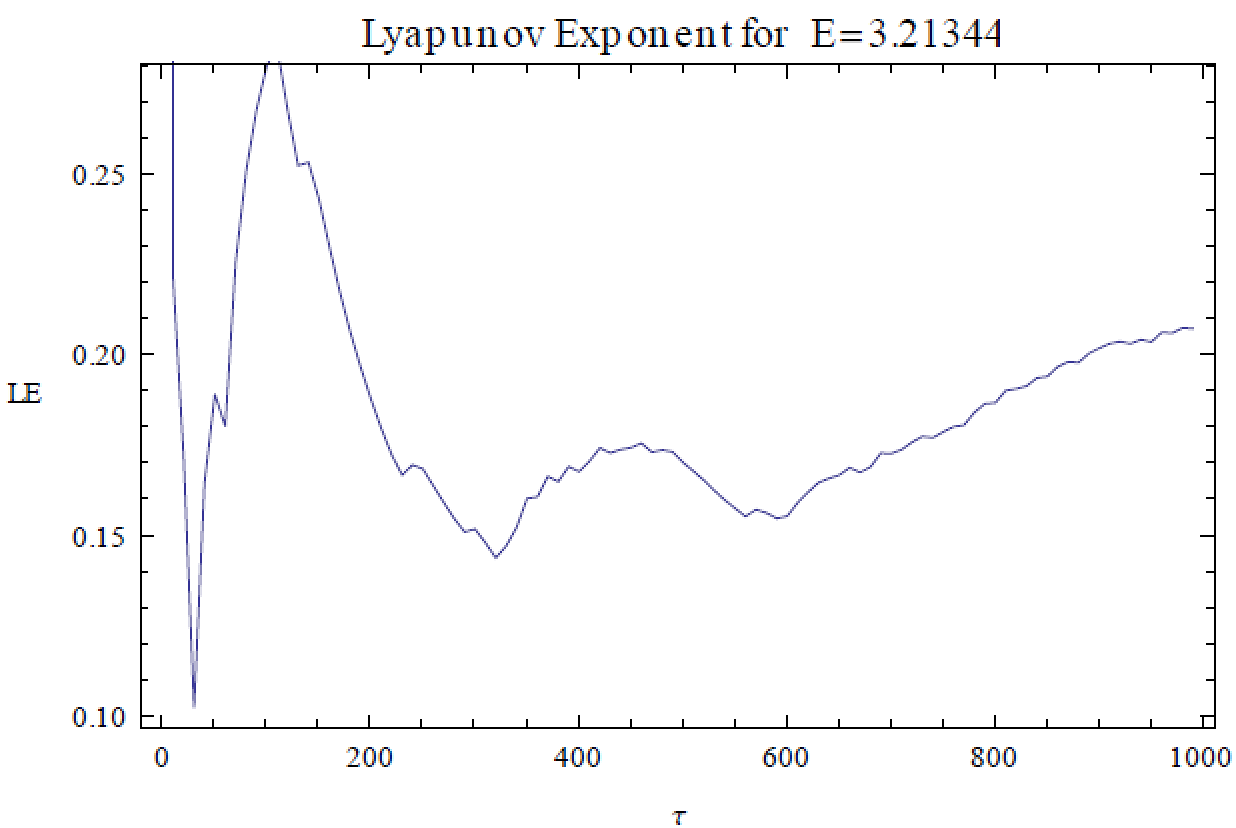
From particular solutions to a study of the full phase space.

# What is chaos?

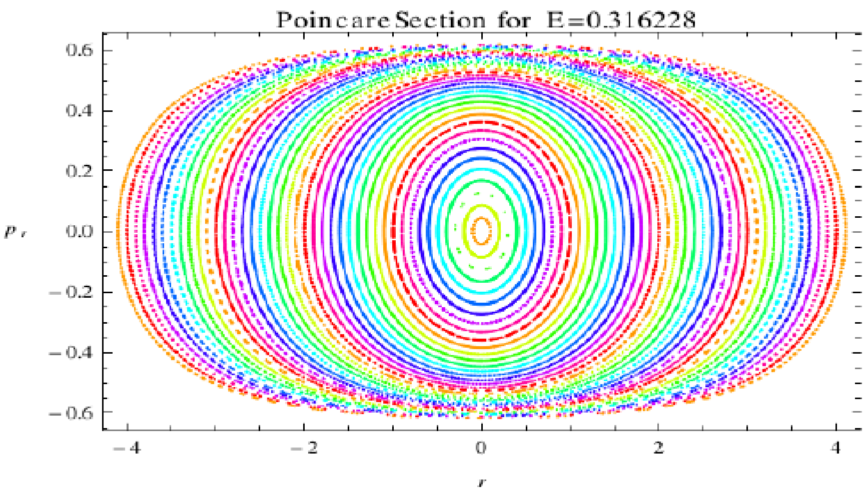
- Sensitivity to initial conditions.
- Largest Lyapunov exponent.
- Poincare sections and the Kolmogorov-Arnold-Moser theorem.
- Power spectrum.
- Fractal dimensions.

# Lyapunov Exponent

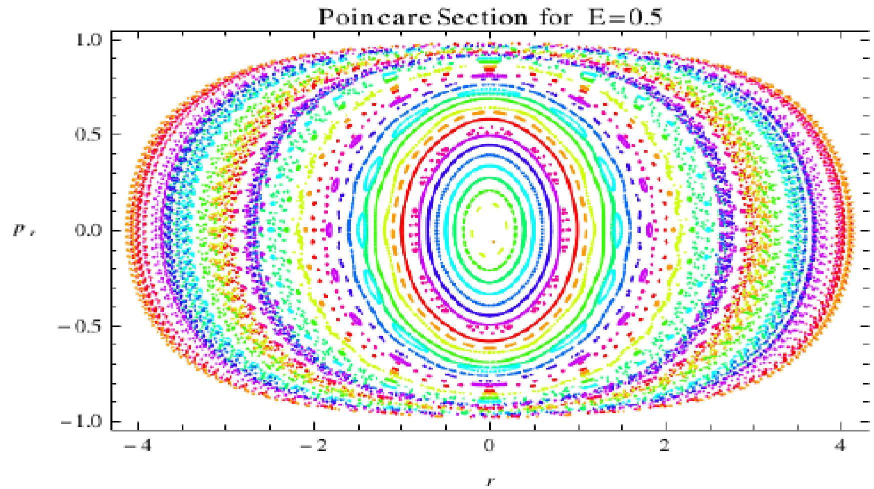
$$\lambda = \lim_{\tau\to\infty} \lim_{\Delta X_0 \to 0} \frac{1}{\tau} \ln \frac{\Delta X(X_0, \tau)}{\Delta X(X_0, 0)}$$



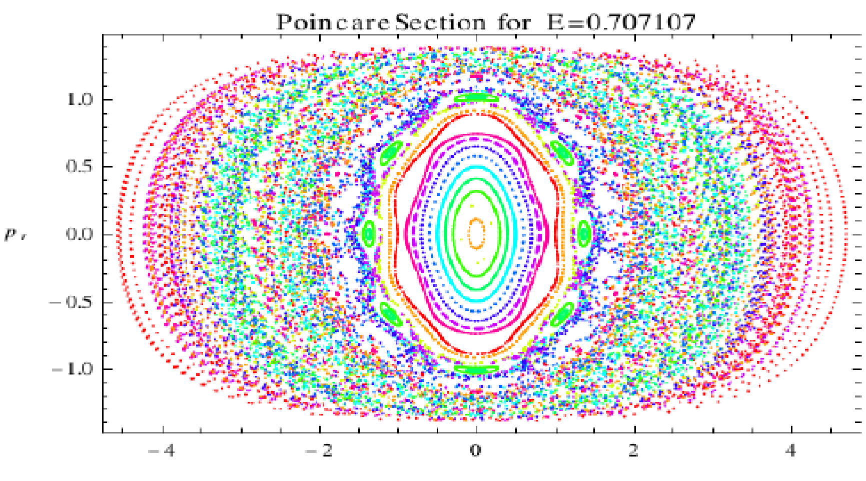
# Explicitly Chaotic: Poincare Sections



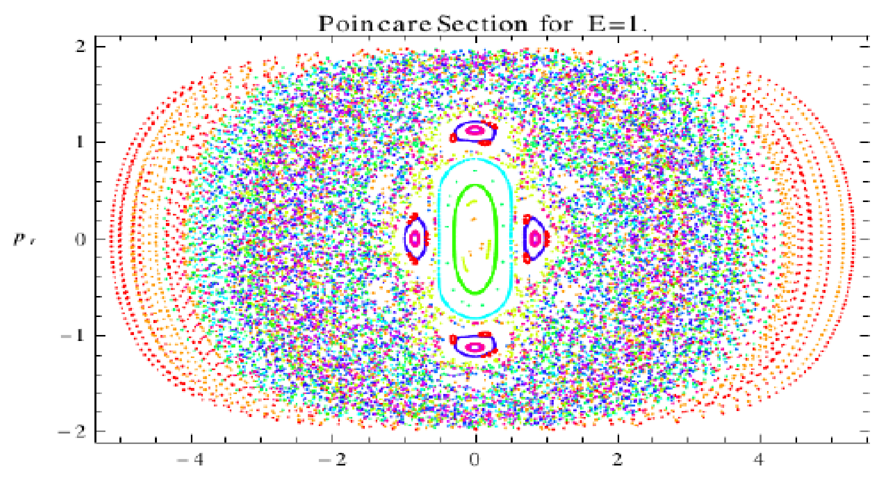

(a) $E = 0.316$




(b) $E = 0.5$




(c) $E = 0.71$




(d) $E = 1.0$

# Analytic Nonintegrability for Hamiltonian Systems (Virasoro)

- Given a system $\dot{\vec{x}} = \vec{f}(\vec{x})$

- Particular solution $\bar{x} = \bar{x}(t)$

- If the nonlinear system admits some first integrals so does the variational equation.
- Ziglin's theorems: Existence of a first integral of motion with the monodromy matrices around the straight line solution.
- Morales-Ruiz: Monodromy to the nature of the Galois group of the NVE.
- Kovacic's algorithm.

# Analytic Non-integrability for Regge Trajectories:

The straight line solution

Properties of confining metrics:

$$a(r) \approx a_0 - a_2(r - r_0)^2$$

$$r = r_0$$

$$\frac{d^2}{d\tau^2}R(\tau) + \alpha^2 R(\tau) = 0, \longrightarrow R(\tau) = A\sin(\alpha\tau + \phi_0).$$

# Normal Variational Equation

$$r = r_0 + \eta(\tau)$$

$$\ddot{\eta} + \frac{a_2 E^2}{2b_0^2 a_0^3}\left[1 + \frac{2\alpha^2 A^2 a_0^4}{E^2}\cos 2\alpha\tau\right]\eta = 0.$$

$$\ddot{y} = (a + b\sin t + c\cos t)y$$

$b \neq -c$ Matthieu -- Non-integrable

# Conclusions of the first part:

- Established classically chaotic behavior around the Regge trajectories (numerically).
- The Regge trajectory itself is an island of integrability in a phase space that is generically chaotic (typical dynamical system).
- Established non-integrability analytically using a generic property of confining backgrounds.
- What to do next? How is this important for hadrons?

# Quantum Chaos

- The study of quantum systems whose classical limit is chaotic (Einstein 1917): Spectrum

$$\oint_{C_i} \mathbf{p} \cdot d\mathbf{q} = n_i h \quad i = 1, \ldots, d,$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2\mu} \Delta \Psi \qquad \Delta \psi_n = -\frac{2\mu}{\hbar^2} E_n \psi_n$$

# Properties of Eigenvalues

$$N(\lambda) = \#\{\lambda_n \leq \lambda\}$$

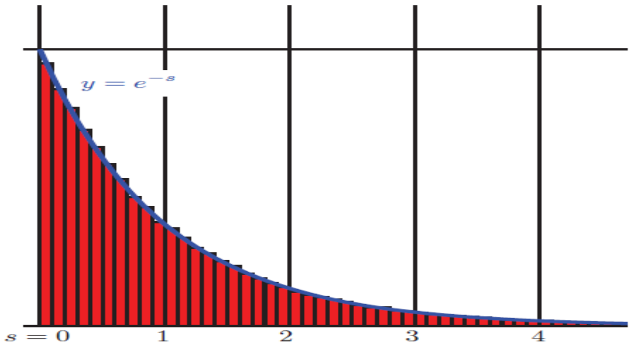
$$\lambda \to \infty, N(\lambda) \sim \mathrm{area}(\mathrm{Billiard})/(4\pi)\,\lambda.$$

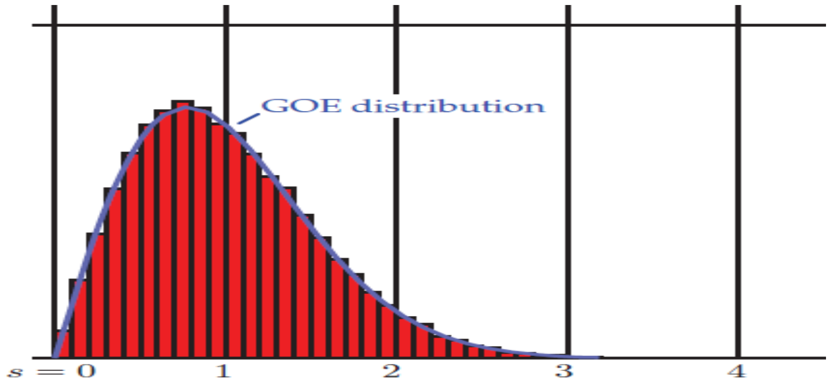
Level Spacing Distribution

$$\lambda_{n+1} - \lambda_n \qquad\qquad P(s)$$

# Quantum Chaos

•Classically Integrable Systems: Eigenvalue distribution coincides with a sequence of uncorrelated levels (Poisson ensemble) with the same spacing (Berry-Tabor)

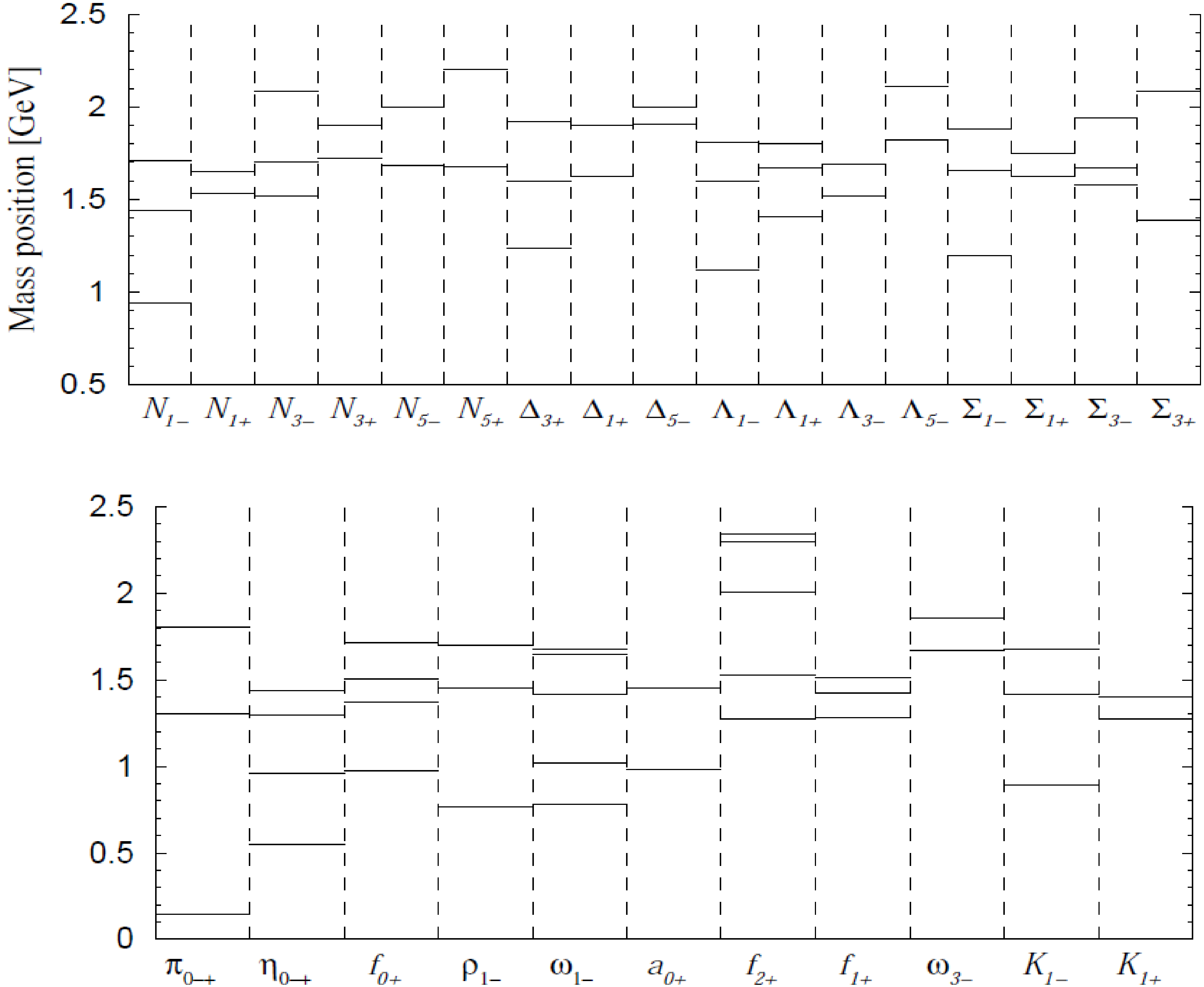


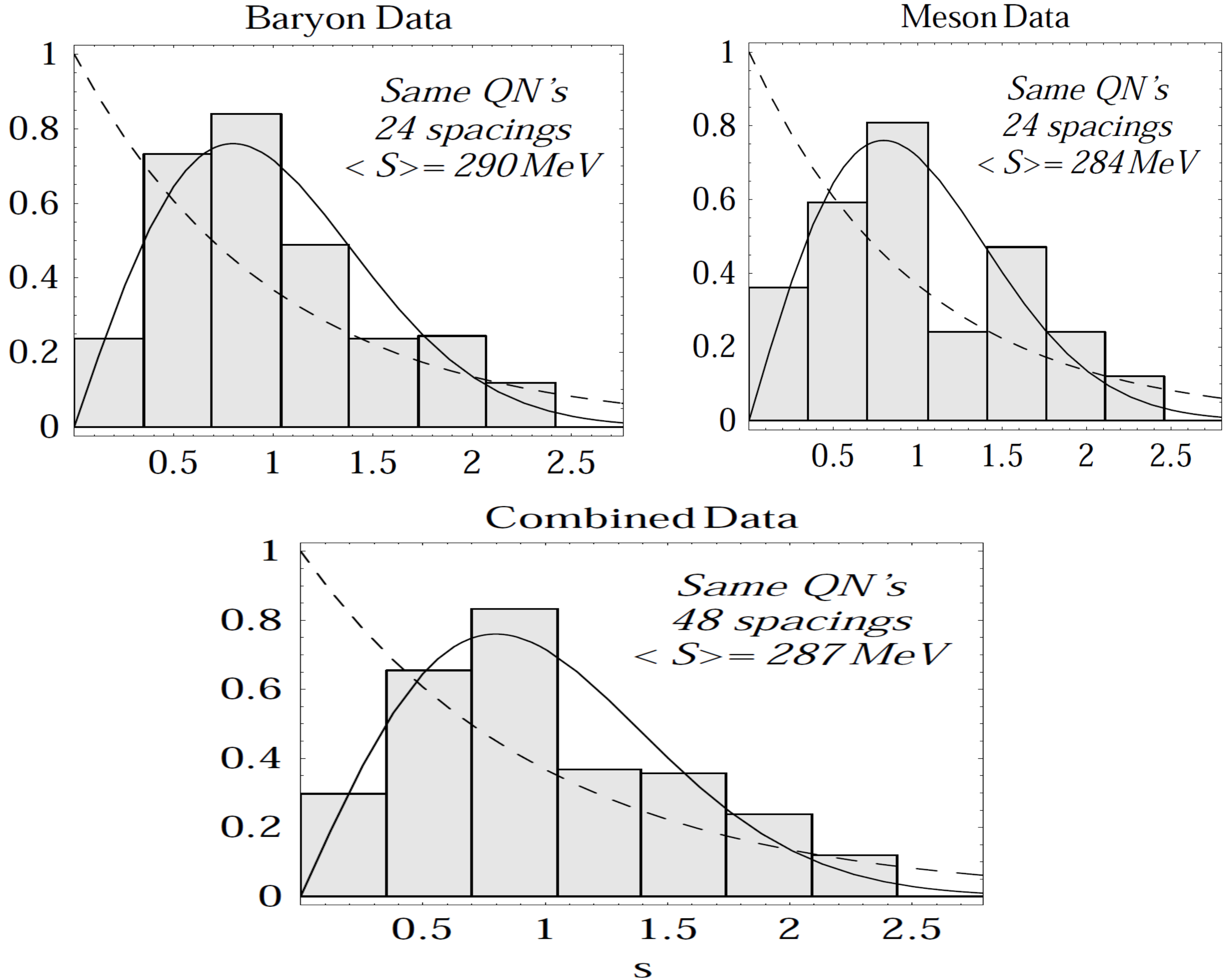

•Classically Chaotic Systems: Eigenvalue spacing distribution coincides with the corresponding quantity for eigenvalues of a suitable ensemble of random matrices (Bohigas, Giannoni and Schmit)

# Pascalutsa (2003)

- The experimentally measured mass spectrum of hadrons (N, Delta, Lambda, Sigma; and all the mesons up to f2(2340)) Particle Data Group Summary (2000).
- Conclusion: The nearest-neighbor mass-spacing distribution of the meson and baryon spectrum is described by the Wigner surmise corresponding to the GOE.



Mass position [GeV]
2.5
2
1.5
1
0.5
N1− N1+ N3− N3+ N5− N5+ Δ3+ Δ1+ Δ5− Λ1− Λ1+ Λ3− Λ5− Σ1− Σ1+ Σ3− Σ3+
2.5
2
1.5
1
0.5
0
π0−+ η0−+ f0+ ρ1− ω1− a0+ f2+ f1+ ω3− K1− K1+



Baryon Data
Same QN's
24 spacings
< S>= 290 MeV
Meson Data
Same QN's
24 spacings
< S>= 284 MeV
Combined Data
Same QN's
48 spacings
< S>= 287 MeV
0
0.2
0.4
0.6
0.8
1
0.5
1
1.5
2
2.5
s

# Lattice QCD (Markum)

- Lattice studies of QCD in the confinement and the deconfinement phases found similar results for eigenvalues of the Dirac operator, including with non-zero chemical potential and with supersymmetry (hep-lat/0505011, hep-lat/0402015)

# How to compute the string Spectrum?

$$(\mathcal{H} = (L_0 - a))\,|\Psi >= 0$$

$L_0$ is a Virasoro generator as Schrodinger

$|\Psi >$ as a sort of vertex operator

$$H \sim p^2 \qquad\qquad \Psi \sim e^{ikX}$$

mass-shell condition $(k^2 = 0)$

# Classical to Quantum via Minisuperspace

$$S = -\frac{1}{2\pi\alpha'}\int d\tau \int d\sigma\, G_{MN}\left(-\dot{X}^M\dot{X}^N + X'^M X'^N\right)$$

$$X^M = \left(x^n(\tau), X^9(\sigma)\right)$$

$$\mathcal{H} \;=\; \frac{1}{2}g^{mn}p_m p_n + \frac{1}{2}V(x),$$

$$g_{mn}(x) \;=\; \int_0^L d\sigma G_{MN}(x), \qquad V(x) = \int_0^L d\sigma G_{99}(x) X'^9 X'^9$$

$$-\Delta\Psi + V(x)\Psi = 0$$

$$ds^2 = A^2(r)(-dt^2 + dR^2 + R^2 d\varphi^2 + dz^2) + B^2(r)dr^2 + ds_5^2$$

$$\varphi(\sigma) = \alpha\,\sigma \quad X^9(\sigma) = \varphi(\sigma)$$

$$\left(-\partial_R^2 - l(r)\partial_r\left(m(r)\partial_r\right) + \omega^2 R^2 A(r)^4\right)\Psi = E^2\Psi$$

$$l(r) = 1/(A(r)B(r)\omega_5(r))$$

$$\omega = \alpha/(\pi\alpha')$$

# MN

$$ds^2 = e^{\phi}\left[\eta_{\mu\nu}dx^{\mu}dx^{\nu} + \alpha' g_s N\left(dr^2 + ds_5^2\right)\right],$$
$$ds_5^2 = e^{2g(r)}(e_1^2 + e_2^2) + \frac{1}{4}(e_3^2 + e_4^2 + e_5^2)),$$
$$e^{2\phi} = e^{-2\phi_0}\frac{\sinh 2r}{2e^{g(r)}},$$
$$e^{2g(r)} = r\coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4},$$

where $\mu, \nu = 0, 1, 2, 3$ and

$$e_1 = d\theta_1, \qquad e_2 = \sin\theta_1 d\phi_1,$$
$$e_3 = \cos\psi\, d\theta_2 + \sin\psi \sin\theta_2\, d\phi_2 - a(\tau)d\theta_1,$$
$$e_4 = -\sin\psi\, d\theta_2 + \cos\psi\sin\theta_2\, d\phi_2 - a(\tau)\sin\theta_1 d\phi_1,$$
$$e_5 = d\psi + \cos\theta_2\, d\phi_2 - \cos\theta_1 d\phi_1, \quad a(r) = \frac{r^2}{\sinh^2 r},$$

where $\mu = 0, 1, 2, 3$, we set the integration constant $e^{\phi_{D_0}} = \sqrt{g_s N}$ The 3-form is

$$H^{RR} = g_s N\left[-\frac{1}{4}(w^1 - A^1)\wedge(w^2 - A^2)\wedge(w^3 - A^3) + \frac{1}{4}\sum_a F^a \wedge (w^a - A^a)\right]$$

$$A = \frac{1}{2}\left[\sigma^1 a(r)d\theta_1 + \sigma^2 a(r)\sin\theta_1 d\phi_1 + \sigma^3\cos\theta_1 d\phi_1\right] \tag{8}$$

and the one-forms $w^a$ are given by:

$$w^1 + iw^2 = e^{-i\psi}(d\theta_2 + i\sin\theta_2 d\phi_2), \qquad w^3 = d\psi + \cos\theta_2 d\phi_2$$

# WQCD

$$ds^2 = \left(\frac{r}{L}\right)^{3/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(\frac{L}{r}\right)^{3/2} \frac{dr^2}{f(r)} + ds_5^2$$

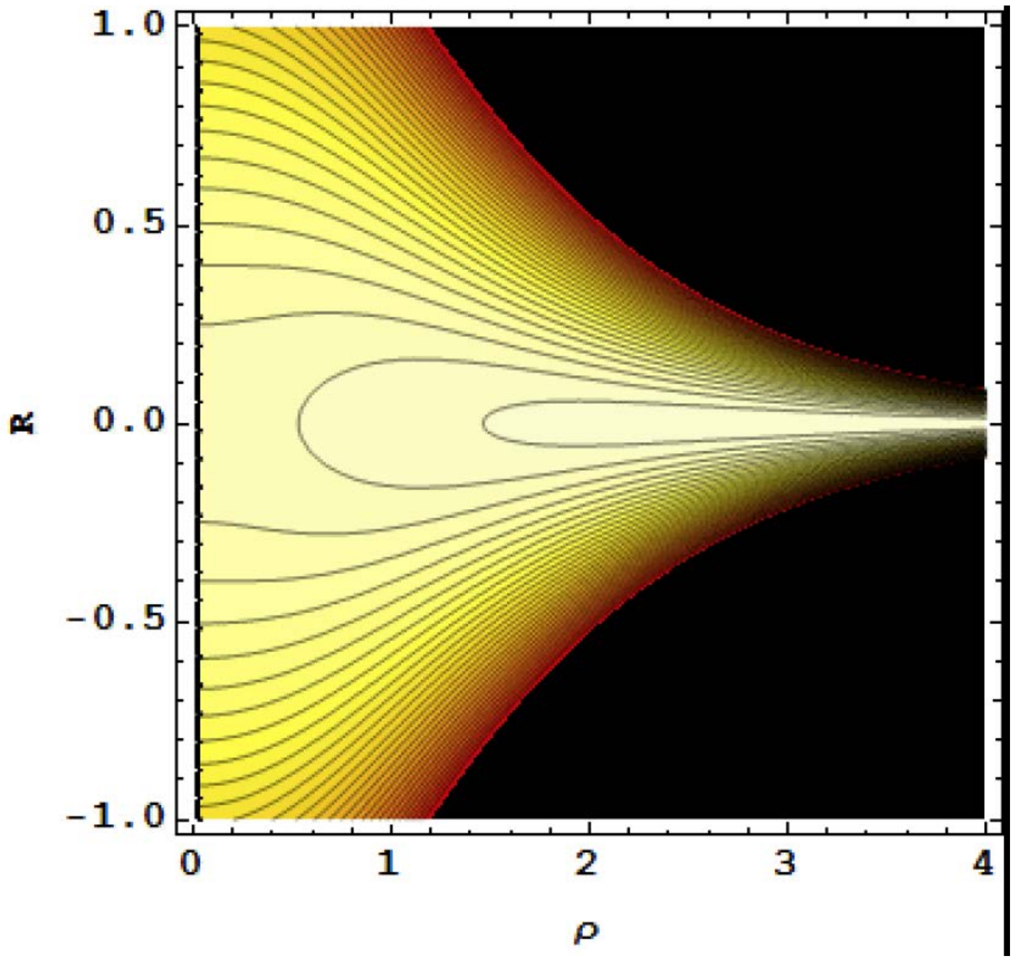
$$ds_5^2 = L^{3/2} r^{1/2} \left(\frac{4r}{9r_0} f(r) d\theta^2 + d\Omega_4^2\right),$$

$$f(r) = 1 - \frac{r_0^3}{r^3}, \qquad L = (\pi N g_s)^{\frac{1}{3}} \alpha'^{\frac{1}{2}},$$

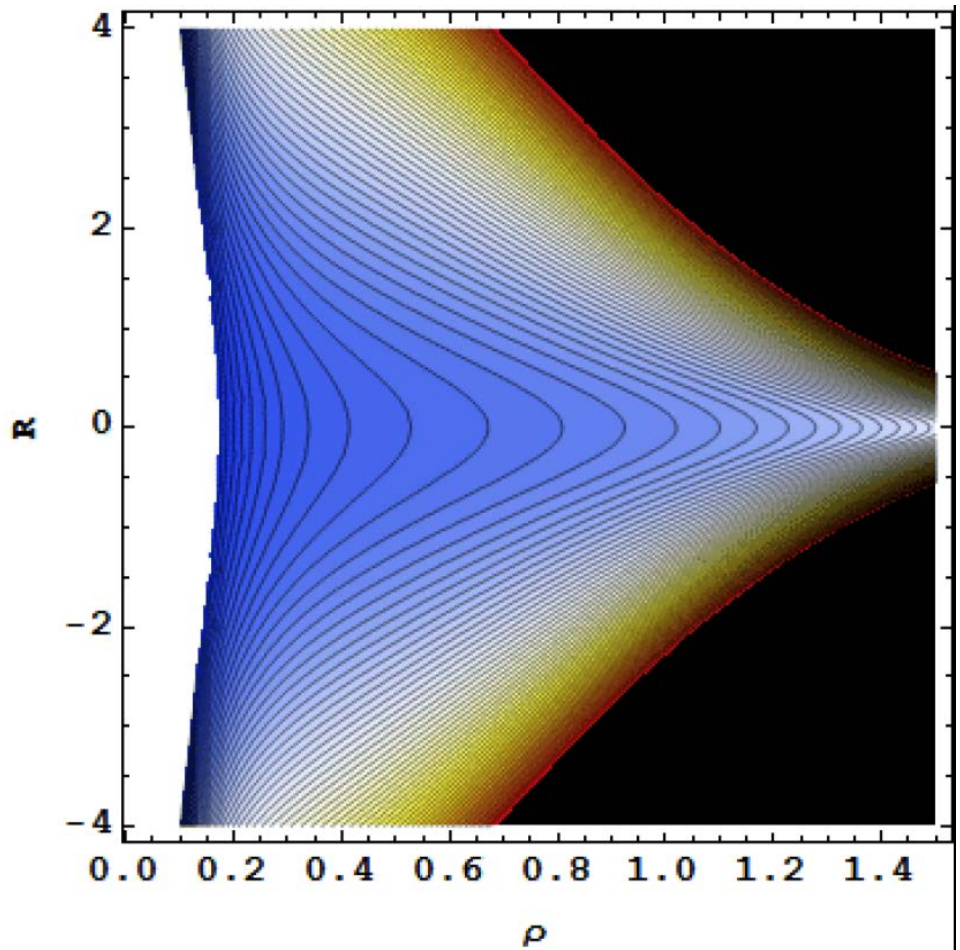
$$e^\Phi = g_s \left(\frac{r}{L}\right)^{3/4}.$$

$$F_4 = 3L^3 \omega_4$$

# MN Potential



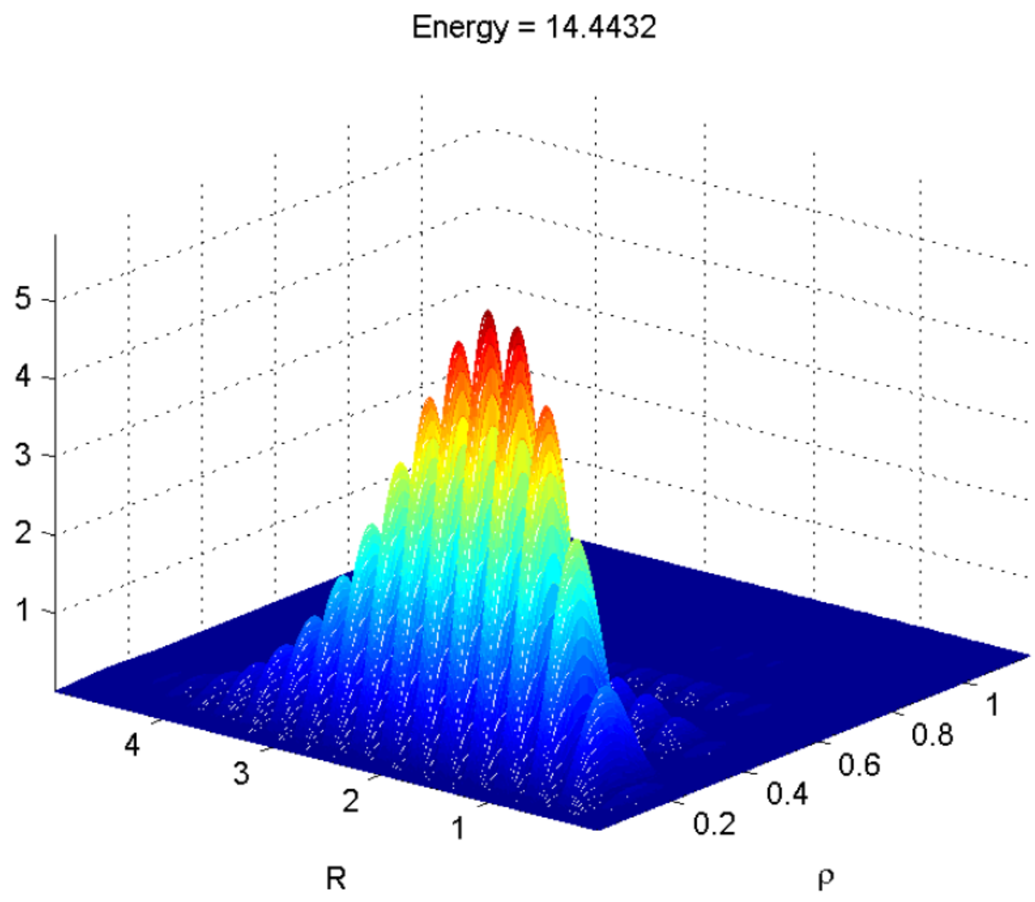
# WQCD Potential



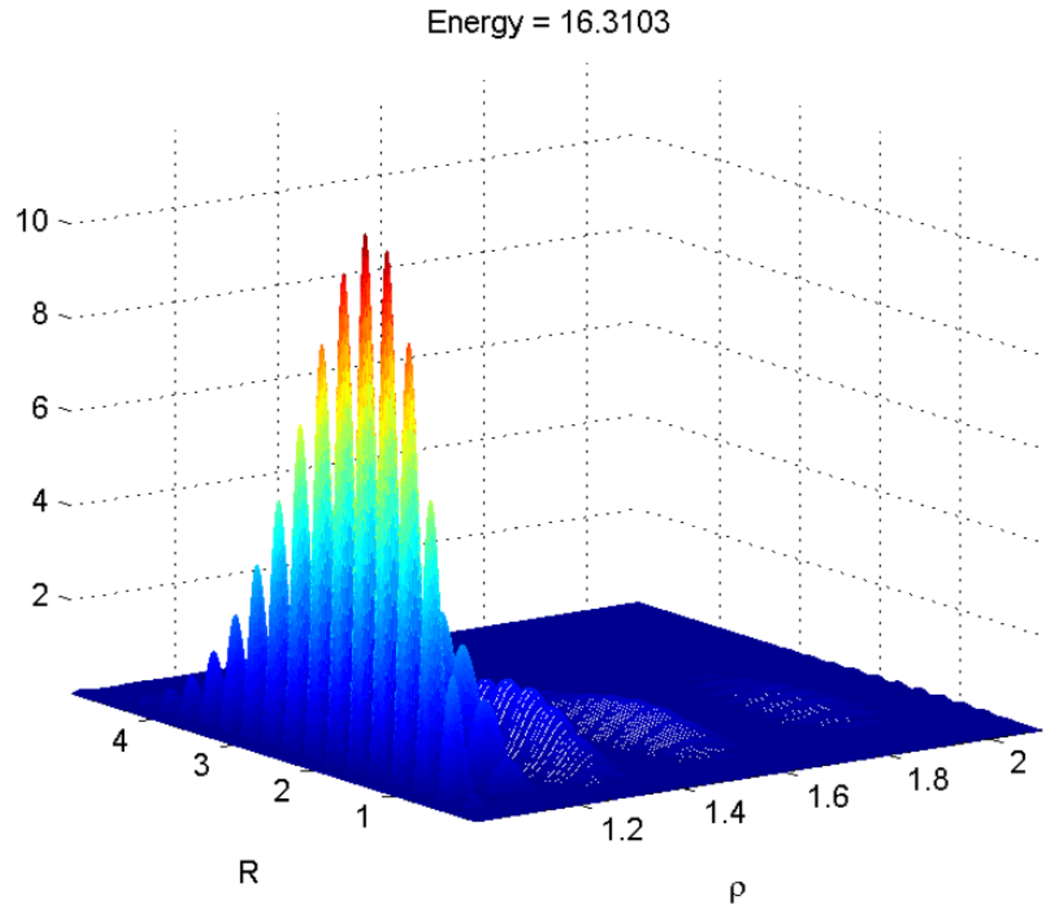
# Winding string sector

- The string can unwind.
- Not a clear separation of sectors.
- Can we show that the typical wave function is not localized close to $R=0$?

# MN Wave function



# WQCD Wave Function

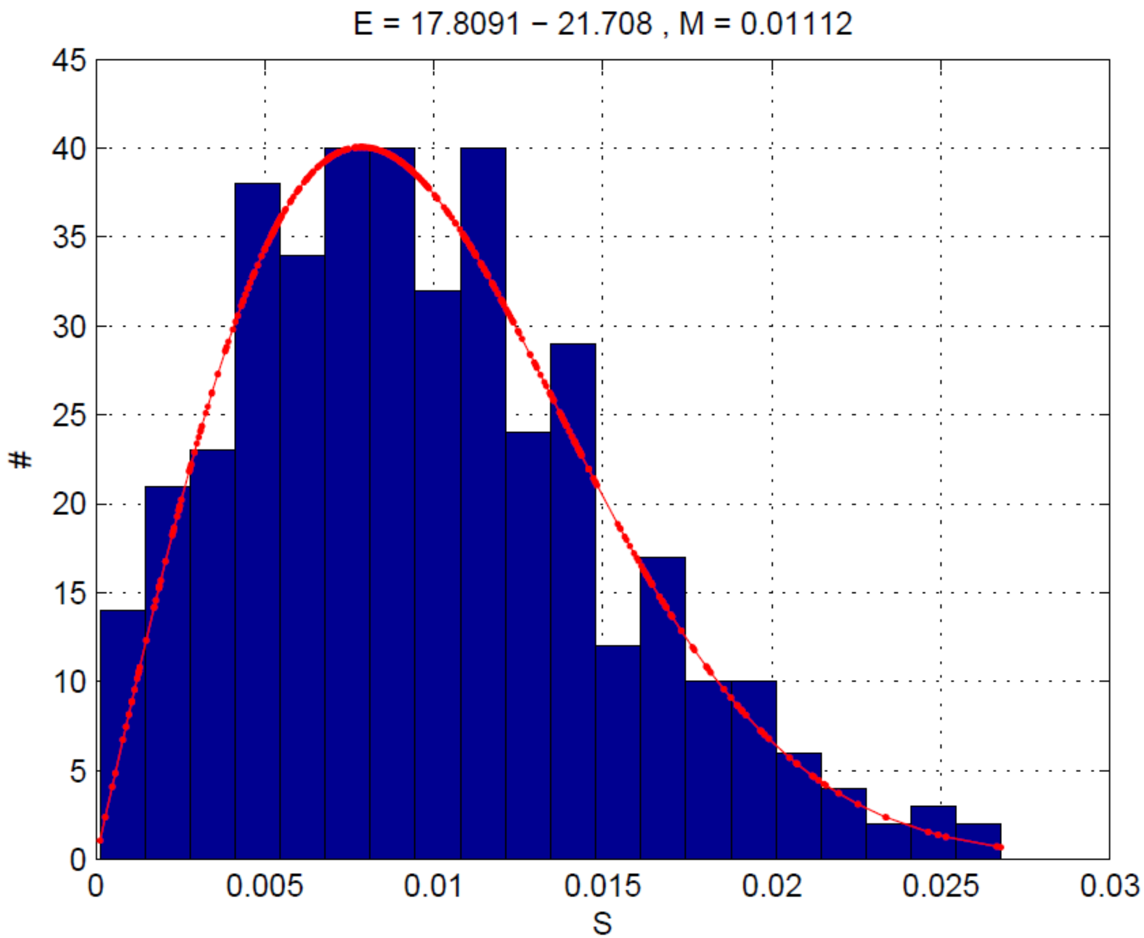


# Holographic Spectrum

- Choose an arbitrary energy range (above the range of small interactions) spanning about 400 energy levels.
- We calculate the energy difference between eigenvalues in the chosen range and plot them on a histograms, and compare them to Wigner distributions

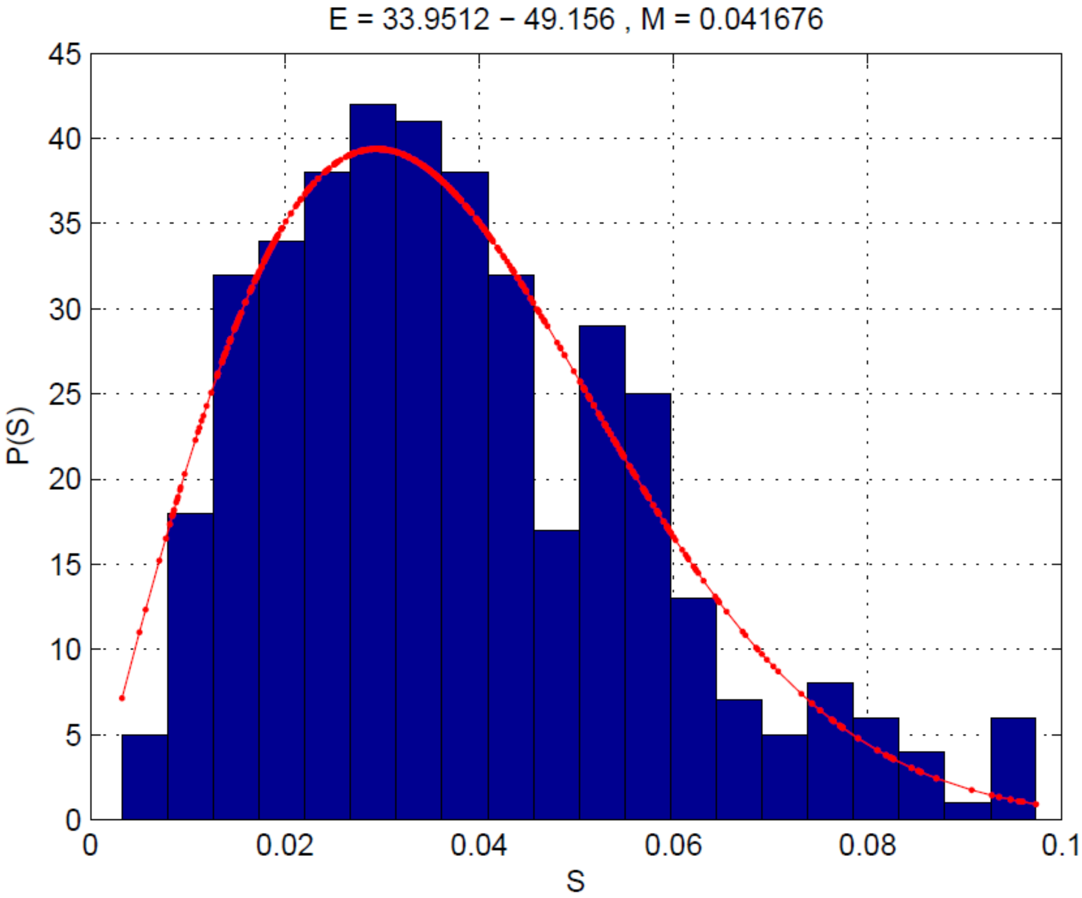
$$P(s) \sim s \exp(-(s/M)^2)$$

# MN



E = 17.8091 − 21.708 , M = 0.01112
45
40
35
30
25
20
15
10
5
0
#
0
0.005
0.01
0.015
0.02
0.025
0.03
S

# WQCD


E = 33.9512 − 49.156 , M = 0.041676
45
40
35
30
25
20
15
10
5
0
P(S)
0
0.02
0.04
0.06
0.08
0.1
S

# How good is the fit?

- The root mean square (RMS) between Wigner's distribution and data is below 10^(*–3) when the* distribution is normalized so the sum is one.
- This excellent matching to Wigner distribution proves our main claim that the spectrum of hadrons in the MN and WQCD theories shows a quantum chaotic eigenvalue distribution.

# Conclusions

- Found a spectrum of hadrons compatible with observations and with the Wigner's surmise.
- A setup for quantum chaos in the quantum/classical setup

## To do List

- Toward detailed structure: Mesons, Baryons.
- How exactly is Wigner's principle at work. (Block-diagonal matrices from the geometric confining property).
- Do we have Bjorken scaling in holographic models?
- Anderson Localization: insulator/metal and Poisson/Wigner.

# Bjorken Scaling?

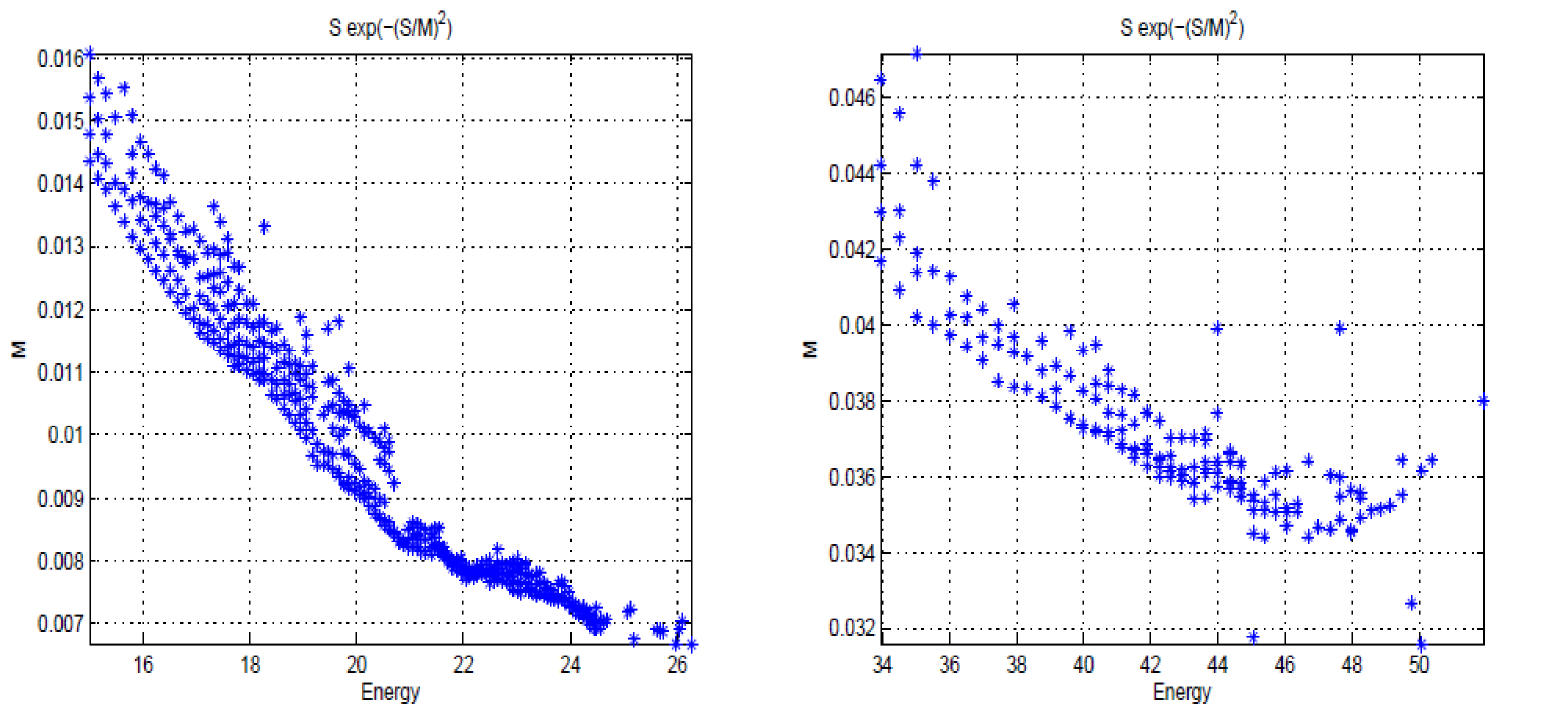



FIG. 5: Dependence of $M$ on the energy region for the MN background and for the WQCD background.