Equivalent dimensional reduction and Sasakian quiver gauge theories

based on joint work with O. Lechtenfeld, A. Popov, and R. Szabo
[arXiv:1601.05719]; [arXiv:1605.03521]

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Aims and Motivation

General Setup and Questions:

- dimensional reduction of higher-dimensional gauge theories on coset spaces: **Yang-Mills theory** on $M^d \times G/H \to$ Yang-Mills-Higgs theory on M^d
- coset G/H chosen with special geometric properties, here: **Sasaki-Einstein** and **3-Sasakian manifolds**
- systematic restrictions from imposing G-**equivariance** on gauge connection $\to$ fixes form of gauge connection, graphical encoding of the theory in terms of **quiver diagrams**
- study of gauge theories by **(generalized) instantons**
- Hermitian Yang-Mills (HYM) instantons on metric cones $C(G/H)$ and their moduli spaces

$\Longrightarrow$ study and characterize the gauge connection

In this talk: quiver gauge theory on $G/H = T^{1,1}$ (5-dim.) and $X_{1,1}$ (7-dim.).
(JG, Lechtenfeld, Popov, Szabo [arXiv:1601.05719]);(JG [arXiv:1605.03521].)

Sasakian quiver gauge theory as analogue of that for Kähler manifolds

1. Equivariant dimensional reduction

2. Generalized instantons

3. Sasakian quiver gauge theories
   - Quiver gauge theory on $T^{1,1}$
   - Quiver gauge theory on $X_{1,1}$

4. Summary

Quivers and respresentations

"Quiver = directed graph"

Formal: A quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ consists of *vertices* $\in \mathcal{Q}_0$ and *arrows* $\in \mathcal{Q}_1$, and the maps $t, h : \mathcal{Q}_1 \to \mathcal{Q}_0$ denote the starting (*tail*) and ending (*head*) point to a given arrow.

A *relation* on the quiver is a formal sum of arrows.

A *quiver representation* is given by assigning a vector spaces V_i to each vertex $v_i \in \mathcal{Q}_0$ and a linear map $\in \mathrm{Hom}(V_i, V_j)$ to each arrow from v_i to v_j. see e.g.(Derksen, Weyman 2005)

Therefore, quivers are a very useful tool for representations of algebras and applications from category theory.

Well-known applications: quiver varieties (Nakajima) and quiver gauge theory of a stack of D-branes.

Here: use quiver diagrams as diagrammatic tool for characterizing the form of the gauge connection caused by equivariance

Equivariant vector bundles and quiver diagrams

Basic Ideas: (Alvarez-Cónsul, García-Prada 2003), for review see e.g. (Dolan, Szabo 2010)
Consider a Hermitian vector bundle $\pi : \mathcal{E} \longrightarrow M^d \times G/H$ of rank k ($\Rightarrow$ structure group $U(k)$) with trivial G-action on M^d.

$\mathcal{E}$ is *G-equivariant* if the following diagram commutes

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\;G\,\curvearrowright\;} & \mathcal{E} \\ \pi\downarrow & & \downarrow\pi \\ M^d \times G/H & \xrightarrow{G\,\curvearrowright} & M^d \times G/H \end{array}$$

By induction $\mathcal{E} = G \times_H E$: G-equiv. bdls. $\mathcal{E} \to M^d \times G/H \quad \overset{1:1}{\longleftrightarrow} \quad H$-equiv. bdls. $E \to M^d$.

This requires a representation of H on fibres $\mathcal{E}_x \simeq \mathbb{C}^k$. Assume that it stems from irred. G-representation $\mathcal{D}_{|H} = \bigoplus_j \rho_j$, which yields *isotopical* decomposition $E_x = \bigoplus_j E_j$ and a breaking of the structure group $\mathrm{U}(k) \to \prod_j \mathrm{U}(k_j)$.

Quiver diagram: depict ρ_j's as vertices and maps $\in \mathrm{Hom}\left(\mathbb{C}^i, \mathbb{C}^j\right)$ given by G-action as arrow between vertices i and j ($\rightarrow$ representation of quiver).

Equivariant vector bundles: construction and examples

Construction procedure of the quiver diagram:

1. Choose an irreducible G-representation $\mathcal{D}$.
2. Construct the weight diagram.
3. "Collapse" it along generators of subalgebra $\mathfrak{h}$.

Consequence: If $\mathfrak{h}$ is a Cartan subalgebra, then the quiver diagram coincides with weight diagram.

Example 1: Kähler manifold $\mathbb{C}P^1 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$: $\mathbf{A_{m+1}}$**-Quiver**

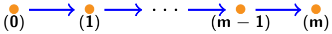
e.g. (Alvarez-Cónsul, García-Prada 2001), (Popov, Szabo 2006)

Representation of H inside G is then given by the generator

$$I_3 \; = \; \mathrm{diag}\left(m\, \mathbf{1}_{k_m}, (m-2)\, \mathbf{1}_{k_{m-1}} \dots, -m\, \mathbf{1}_{k_0}\right) \quad \text{on} \quad \mathbb{C}^k \; = \; \left(\mathbb{C}^{k_m}, \mathbb{C}^{k_{m-1}} \dots, \mathbb{C}^{k_0}\right)^{\mathrm{T}} . \qquad (1)$$

Equivariance condition yields quiver diagram (*"holomorphic chain"*) and gauge connection of the form

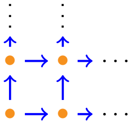
$$\mathcal{A} = \begin{pmatrix} \mathbf{1}_{k_m} \otimes a_m & \phi_{m-1} \otimes \bar{\Theta} & \mathbf{0} & \dots \\ -\phi^{\dagger}_{m-1} \otimes \Theta & \mathbf{1}_{k_{m-1}} \otimes a_{m-1} & \phi_{m-2} \otimes \bar{\Theta} & \dots \\ \mathbf{0} & -\phi^{\dagger}_{m-2} \otimes \Theta & \ddots & \dots \\ \vdots & \vdots & & \vdots \end{pmatrix}$$



with 1-forms $a_l, \Theta \in \mathfrak{su}(2)^*$, homomorphisms $\phi_l(x)$

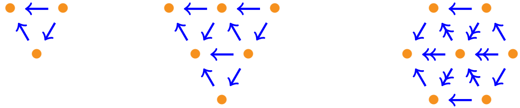
Equivariant vector bundles: examples (2)

Example 2: Kähler manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$:
$\mathbf{A_{m_1+1}} \otimes \mathbf{A_{m_2+1}}$**-Quiver**
(Lechtenfeld, Popov, Szabo 2008)
$\rightarrow$ yields a grading of the connection $\mathcal{A}$

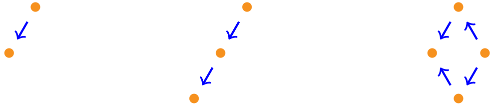


Further examples: Kähler manifolds $\mathbb{C}P^2 = \mathrm{SU}(3)/\mathrm{S}\left(\mathrm{U}(2) \times \mathrm{U}(1)\right)$ and $Q_3 = \mathrm{SU}(3)/\left(\mathrm{U}(1) \times \mathrm{U}(1)\right)$: (Lechtenfeld, Popov, Szabo 2008)

Q_3



$\mathbb{C}P^2$



Reminder: (classical) instantons on M^4

Consider Yang-Mills (YM) theory on four-dimensional manifold with gauge field $\mathcal{A}$, curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ and action

$$S_{YM} \propto \int_{M^4} \mathrm{Tr}\left(\mathcal{F} \wedge \star_4 \mathcal{F}\right). \qquad (2)$$

This yields the Yang-Mills equation

$$D \star_4 \mathcal{F} = 0. \qquad (3)$$

Hodge star $\star_4$ decomposes in two-forms ± 1-eigenspaces. Consider (anti-)self-dual connections ("instantons"),

$$\star_4 \mathcal{F} = \pm \mathcal{F} \qquad (4)$$

→ First-order condition yields solutions to second-order Yang-Mills equations due to Bianchi identity.

The action is bounded by the topological invariant (1st Pontryagin class)

$$\int_{M^4} \mathrm{Tr}\left(\mathcal{F} \wedge \mathcal{F}\right) \propto p_1 \in \mathbb{Z}. \qquad (5)$$

Self-dual connections are important for the characterization of four-manifolds. (Atiyah, Hitchin, Singer)

Generalized instantons: definition

Let $\mathcal{A}$ be a connection on an d-dimensional (Riemannian) manifold M^d and consider now the (generalized) instanton equation (Ward 1984), (Hull 1998), (Harland, Nölle 2012), ...

$$\star_d \mathcal{F} = -(\star_d Q) \wedge \mathcal{F}. \tag{6}$$

This implies

$$\underbrace{\mathrm{d} \star \mathcal{F} + A \wedge \star\mathcal{F} - (-1)^d \star \mathcal{F} \wedge A}_{\text{YM eq.}} + \underbrace{(\mathrm{d} \star Q) \wedge \mathcal{F}}_{\text{torsion}} = 0. \tag{7}$$

$\Longrightarrow$ YM equation (+torsion term) again follows from *first*-order equation.

The four-form Q depends on the chosen geometry, and the torsion term vanishes not only for special holonomy manifolds, but also for manifolds with real Killing spinors, $\nabla_X \psi = \alpha X \cdot \psi$.

Harland and Nölle give formula for those Q and define an instanton solution, called *canonical connection*, based on the geometry.

This definition implies another notation of instanton as

$$\mathcal{F} \cdot \epsilon = 0 \tag{8}$$

as *gaugino equation* in heterotic supergravity. $\Longrightarrow$ start from instanton solutions to construct full heterotic solutions.

Construction of instantons see (Ivanova, Popov 2012)

Consider a reductive homogeneous with generators I_μ,

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \mathfrak{m} =: \mathrm{span}\langle I_j \rangle \oplus \mathrm{span}\langle I_a \rangle. \tag{9}$$

Given an instanton connection (e.g. the characteristic one) $\Gamma = \Gamma^j I_j$, one studies connections of the form $\mathcal{A} = \Gamma^j I_j + X_\mu e^\mu$ whose curvature is

$$\mathcal{F}_{\mathcal{A}} = \mathcal{F}_\Gamma + \frac{1}{2}\Gamma^j \wedge e^\mu \left([I_j, X_\mu] - f^{\nu}_{j\mu} X_\nu \right) + \frac{1}{2} e^{\mu\nu} \left([X_\mu, X_\nu] + T^{\sigma}_{\mu\nu} X_\sigma \right) + \mathrm{d}X_\mu \wedge \mathrm{e}^\mu . \tag{10}$$

One imposes a condition on the $\mathrm{e}^{\mu\nu}$-part and the vanishing of the second term,

$$[I_j, X_a] = f^{b}_{ja} X_b \qquad \text{equivariance condition}. \tag{11}$$

The equivariance condition can also be motivated from differential geometry as invariance condition for a connection on a homogeneous space. (Kobayashi, Nomizu)

This condition is exactly that of equivariance for the vector bundles above. **Quiver gauge theory**, by choosing different representations and depicting the homomorphisms as arrows, allows a **refinement of the typical ansatz** (Harland, Nölle, Haupt, . . .)

$$X_j = \lambda_j(x)\, I_j. \tag{12}$$

Generalized instantons: overview

Geometric structures with Killing spinors one can consider for instantons: (huge number of articles ...)

- **nearly-Kähler manifolds:** $\mathrm{SU}(3)$ structure, examples: S^6, $S^3 \times S^3$, $\mathrm{SU}(3)/\mathrm{U}(1)^2$, $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$
- **nearly-parallel G_2 manifolds:** G_2 structure, examples: Aloff-Wallach spaces $X_{k,l} = \mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$, squashed $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, ...
- **Sasaki-Einstein manifolds:** $\mathrm{SU}(n)$ structures, examples: $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$, Stiefel manifolds $V(\mathbb{R}^{n+1}) = \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$, $\mathrm{SO}(2n)/\mathrm{SU}(n)$, $Q(1,1,1)$, ...
- **3-Sasakian manifolds:** $\mathrm{Sp}(n)$ structures, examples: $S^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(1)$, $G_2/\mathrm{Sp}(1)$, $X_{1,1}$, ...

Recall that real Killing spinors on $\left(M^d, g\right)$ lift to parallel spinors on the metric cone $C\left(M^d\right) = \left(\mathbb{R}^+ \times M, \tilde{g} = r^2 g + \mathrm{d}r^2\right)$ (used in Bär's classification). Thus, the cones have special holonomy and one can study instantons also on the metric cones. (Ivanova, Popov 2012)

Moreover, one can also consider instantons on cones with singularities, *sine-cones*.

Sasakian quiver gauge theories

Study of equivariant dimensional reduction and instantons on cosets G/H with Sasaki-Einstein structure.

Motivation:

- odd-dimensional counterpart of quiver gauge theories on Kähler (Einstein-) spaces
- interesting due to prominent role in AdS/CFT correspondence

Examples in the literature:

- quiver gauge theory on $S^3/\Gamma \cong \mathrm{SU}(2)/\Gamma$ (Lechtenfeld, Popov, Szabo 2014)
- quiver gauge theory on $S^5/\Gamma \cong (\mathrm{SU}(3)/\mathrm{SU}(2))\,/\Gamma$ (Lechtenfeld, Popov, Sperling, Szabo 2015)

First example here: $T^{1,1}$ (J.G., Lechtenfeld, Popov, Szabo 2016)
Spaces $T^{p,q}$ (Romans 1985) are a class of $\mathrm{U}(1)$ bundles over $S^2 \times S^2$,

$$T^{p,q} = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathrm{U}(1)_{p,q}} \quad \text{with} \quad \mathrm{U}(1)_{p,q} = \langle pI_3^{(1)} - qI_3^{(2)} \rangle. \tag{13}$$

Compactification on $\mathrm{AdS}_5 \times T^{1,1}$ is dual to $\mathcal{N} = 1$ SCFT (Klebanov, Witten 1998) ($\mathrm{AdS}_5 \times S^5$ dual to $\mathcal{N} = 4$ SYM).

The metric cone over $T^{1,1}$ is a Calabi-Yau 3-fold known as *conifold* in the literature.

Equivariance condition on $T^{1,1}$

Geometry: As Sasaki-Einstein 5-manifold, $T^{1,1}$ can be described as $\mathrm{SU}(2)$ structure with (Conti, Salamon 2007)

$$\eta = -e^5, \quad \omega^1 = e^{23} + e^{14}, \quad \omega^2 = e^{31} + e^{24}, \text{ and } \omega^3 = e^{12} + e^{34} \tag{14}$$

satisfying

$$\mathrm{d}\eta = 2\omega^3, \quad \mathrm{d}\omega^1 = -3\eta \wedge \omega^2, \text{ and } \mathrm{d}\omega^2 = 3\eta \wedge \omega^1, \tag{15}$$

where $\eta \longleftrightarrow \mathrm{U}(1)^{\perp}_{1,1}$ is the *contact form* and ω^3 is the *Kähler form* on $S^2 \times S^2$. The Ricci tensor is $\mathrm{Ric}^g = 4g$ with metric $g = \delta_{ij} e^i \otimes \mathrm{e}^j$.

Equivariant gauge connection: The *canonical connection* is $\Gamma = I_6 \otimes a$ with $I_6 = I_3^{(1)} - I_3^{(2)}$ (and a the dual 1-form). Its curvature $\mathcal{F}_\Gamma \propto e^{12} - e^{34}$ satisfies the instanton equation

$$\star\mathcal{F}_\Gamma = -(\star Q) \wedge \mathcal{F}_\Gamma \tag{16}$$

with $Q = e^{1234}$. The connection Γ takes values in $\mathfrak{h} = \langle I_6 \rangle$. We consider now connections on $M^d \times T^{1,1}$ written as

$$\mathcal{A} = A + \Gamma + \sum_{i=1}^{5} X_i \otimes e^i, \tag{17}$$

where A is a gauge connection on M^d and $X_i(x)$ homomorphisms ("Higgs fields").

Equivariance conditiond and quiver diagrams on $T^{1,1}$

This approach yields the equivariance conditions

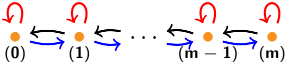
$$[I_6, \phi^{(1)}] = 2\phi^{(1)}, \quad [I_6, \phi^{(2)}] = -2\phi^{(2)}, \quad [I_6, \psi] = 0. \tag{18}$$

(with $\phi^{(1)} = 1/2(X_1 + \mathrm{i}\, X_2)$, $\phi^{(2)} = 1/2(X_3 + \mathrm{i}\, X_4)$ and $\psi = X_5$).

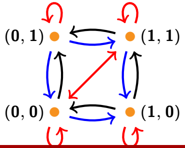
- weaker conditions than those of $\mathbb{C}P^1 \times \mathbb{C}P^1$ $\Rightarrow$ more arrows and "degeneracies"
- additional endomorphism ψ due to the $\mathrm{U}(1)$-factor (compared with Kähler case)

Quiver diagrams: denote by (m_1, m_2) the irreducible representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1}$, $\mathrm{U}(1)$-charge by $c_{j,\alpha} = 2(j - \alpha)$.

Example 1: $(m_1, m_2) = (\mathbf{m}, \mathbf{0})$ yields **"modified holomorphic chain"**



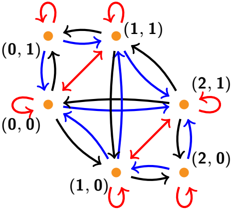
Example 2: representation $(\mathbf{1}, \mathbf{1})$



$$\mathcal{A} = \begin{pmatrix} a_{00}\mathbf{1}_{k_{00}} + \Psi_{00} & -\Phi^\dagger_{00,01} & \Phi_{10,00} & \Psi_{11,00} \\ \Phi_{00,01} & a_{01}\mathbf{1}_{k_{01}} + \Psi_{01} & 0 & \Phi_{11,01} \\ -\Phi^\dagger_{10,00} & 0 & a_{10}\mathbf{1}_{k_{10}} + \Psi_{10} & -\Phi^\dagger_{10,11} \\ -\Psi^\dagger_{11,00} & -\Phi^\dagger_{11,01} & \Phi_{10,11} & a_{11}\mathbf{1}_{k_{11}} + \Psi_{11} \end{pmatrix}$$

Quiver diagrams on $T^{1,1}$ (2)

Representation $(2,1)$:



Since the quiver diagram of $T^{1,1}$ depends only on *one* quantum number, $c_{j,\alpha}$, it is reasonable to combine vertices with the same $\mathrm{U}(1)_{1,1}$ charge (i.e. identify vertices along red arrows).

$\Rightarrow$ **$(\mathbf{m_1}, \mathbf{m_2})$ yields modified holomorphic chain of length $(\mathbf{m_1} + \mathbf{m_2} + \mathbf{1})$.**

One recovers $\mathbb{C}P^1 \times \mathbb{C}P^1$-result by fixing $X_5 \propto I_5$ and imposing equivariance also w.r.t. second Cartan generator I_5

$$[I_3^{(1)} + I_3^{(2)}, \phi^{(1)}] = 2\phi^{(1)} \quad \text{and} \quad [I_3^{(1)} + I_3^{(2)}, \phi^{(2)}] = 2\phi^{(2)}. \tag{19}$$

Hermitian Yang-Mills equations

The constructed equivariant gauge connection yields a Yang-Mills theory on $M^d \times T^{1,1}$ and, since the Higgs fields do not depend on $T^{1,1}$, one obtains the dimensional reduction to a Yang-Mills-Higgs theory on M^d:

$$\begin{aligned} \mathcal{L}_{\mathrm{YM}} &= -\frac{1}{2}\sqrt{\hat{g}}\,\mathrm{tr}_{k\times k}\Bigg[\frac{1}{2}F_{ab}F^{ab} + \sum_{\mu=1}^{5}(D_a X_\mu)(D^a X_\mu) + ([X_1,X_3])^2 + ([X_1,X_4])^2 \\ &+ ([X_2,X_3])^2 + ([X_2,X_4])^2 + ([X_1,X_2] - 2X_5 + \frac{3}{2}\,\mathrm{i}\,I_6)^2 + ([X_3,X_4] - 2X_5 - \frac{3}{2}\,\mathrm{i}\,I_6)^2 \\ &+ ([X_1,X_5] + \frac{3}{2}X_2)^2 + ([X_2,X_5] - \frac{3}{2}X_1)^2 + ([X_3,X_5] + \frac{3}{2}X_4)^2 + ([X_4,X_5] - \frac{3}{2}X_3)^2\Bigg]. \end{aligned} \tag{20}$$

Instanton solutions: We consider now the metric cone $\mathbb{R}^+ \times T^{1,1}$. The canonical connection Γ of $T^{1,1}$ lifts to an instanton on the cone (or, equivalently, the cylinder) and one obtains the same equivariance conditions.

Since the cone is a Calabi-Yau manifold with Kähler form Ω, one can use the Hermitian *Yang-Mills equations (HYM)* (also known as *Donaldson-Uhlenbeck-Yau equations*) (Donaldson 1985), (Uhlenbeck, Yau 1986)

$$\mathcal{F}^{(2,0)} = 0 = \mathcal{F}^{(0,2)} \quad \text{and} \quad \Omega \lrcorner \mathcal{F} = 0 \tag{21}$$

to evaluate the instanton equation. (Popov 2009)

Hermitian Yang-Mills and Nahm's equations

For the conifold, firstly, this yields the constraint

$$[\phi^{(1)}, \phi^{(2)}] = 0 \qquad \Rightarrow \text{commutativity of diagram} \tag{22}$$

Thus, one has to impose a *relation* on the quiver diagram. Secondly, one obtains the equations

(with $\tau = \ln r$, $s = 1/4\,\mathrm{e}^{-4\tau}$, $\phi^{(i)} = \mathrm{e}^{-3/2\tau} Z_i$ for $i = 1,2$ and $\phi^{(3)} = \mathrm{e}^{-4\tau} Z_3$)

$$\frac{\mathrm{d}}{\mathrm{d}s} Z_a = 2[Z_a, Z_3] \quad \text{for} \quad a = 1,2 \quad (\textit{"complex equations"}) \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(Z_3 + Z_3^\dagger\right) = 2\,(-s)^{-5/4}\left([Z_1, Z_1^\dagger] + [Z_2, Z_2^\dagger]\right) - 2[Z_3, Z_3^\dagger] \quad (\textit{"real equation"})$$

similar to the (original) Nahm equations on $\mathbb{C}^2$ (see e.g. Kronheimer 1984)

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = 2[\beta, \alpha] \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\alpha + \alpha^\dagger\right) = -2[\alpha, \alpha^\dagger] - 2[\beta, \beta^\dagger] \tag{24}$$

$\Rightarrow$ Techniques (moment maps, Kähler quotients, adjoint orbits) from the discussion of the original Nahm equations are applicable. (Donaldson 1984),(Kronheimer 1990)

Description of moduli space see (Donaldson 1984),(Kronheimer 1990), (Sperling 2015)

The form of these HYM equations for connections based on the canonical connection of a Sasaki-Einstein manifold depends only on its dimension (which determines scaling and number of fields) and allows a general description. (Popov, Ivanova 2012),(Sperling 2015)

Denote moduli space of complex equations as $\mathbb{A}_{1,1}$; it is it is invariant under the gauge transformation

$$Z_a \mapsto g Z_a g^{-1} \quad \text{for} \quad a = 1, 2 \quad \text{and} \quad Z_3 \mapsto g Z_3 g^{-1} + \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}s} g^{-1}, \tag{25}$$

for $g(s) \in \mathcal{G} \subset \mathrm{GL}(\mathbb{C}, k)$ such that constraints (equivariance conditions) are satisfied. The real equation can be considered as moment map $\mu : \mathbb{A}_{1,1} \to \mathrm{Lie}\,(\mathcal{G})$

$$\mu(Z, Z^\dagger) \;=\; \frac{\mathrm{d}}{\mathrm{d}s}\left(Z + Z^\dagger\right) - 2\,(-s)^{(-5/4)}\left([Z_1, Z_1^\dagger] + [Z_2, Z_2^\dagger]\right) + 2[Z_3, Z_3^\dagger], \tag{26}$$

and, thus, the moduli space $\mathcal{M}$ as Kähler quotient

$$\mathcal{M} \;=\; \mu^{-1}(0)\,/\mathcal{G} \tag{27}$$

Recall: Original Nahm equations admit hyper-Kähler structure.

Description of moduli space (2)

see (Donaldson 1984),(Kronheimer 1990), (Sperling 2015)

Apply gauge transformation such that

$$Z_a = g^{-1} U_a g \text{ for } a = 1, 2 \quad \text{and} \quad Z_3 = -\frac{1}{2} g^{-1} \frac{\mathrm{d}g}{\mathrm{d}s} \tag{28}$$

with constant matrices U_1 and U_2. Any matrices with $[U_1, U_2] = 0$ solve the complex equations in that gauge.

The real equation can be interpreted as equation of motion of a suitable Lagrangian.

Imposing the boundary conditions

$$\lim_{s \to \infty} Z_\mu(s) = g_0 T_\mu g_0^{-1} \tag{29}$$

uniquely determines the solutions.

Thus, the moduli space of framed HYM instantons can be also described as adjoint orbit

$$\mathcal{M} = \mathcal{M}(T_\mu) \tag{30}$$

of constant matrices at the boundary $s \to \infty$

Quiver gauge theory on the Aloff-Wallach space $X_{1,1}$

As **second example** of a quiver gauge theories, we study the **Aloff-Wallach space** $X_{1,1}$
These spaces are defined as (Aloff, Wallach 1975)

$$X_{k,l} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,l}} \quad \text{with} \quad \mathrm{U}(1)_{k,l} \ni h = \left(\mathrm{e}^{\,\mathrm{i}\,(k+l)}, \mathrm{e}^{\,-\,\mathrm{i}\,l}, \mathrm{e}^{\,-\,\mathrm{i}\,k} \right). \tag{31}$$

and admit G_2 structures. (instantons on these spaces studied e.g. by Haupt)
The space $X_{1,1}$ admits a Sasaki-Einstein and even a 3-Sasaki structure, which is an $\mathrm{SO}(3)$ bundle over the quaternionic space $\mathbb{C}P^2$. The structure is described by

$$\begin{aligned}
\mathrm{d}e^1 &= \sqrt{3}\,\mathrm{e}^{82} - e^{72} - e^{35} - e^{46}, \quad \mathrm{d}e^2 = -\sqrt{3}e^{81} + e^{71} - e^{36} + e^{45} \\
\mathrm{d}e^3 &= -\sqrt{3}\,\mathrm{e}^{84} - e^{74} + e^{15} + e^{26}, \quad \mathrm{d}e^4 = \sqrt{3}e^{83} + e^{73} - e^{16} - e^{25} \\
\mathrm{d}e^5 &= 2e^{67} - 2e^{13} + e^{24}, \quad \mathrm{d}e^6 = 2e^{75} - 2e^{14} - 2e^{23} \\
\mathrm{d}e^7 &= 2e^{56} + 2e^{12} + e^{34}.
\end{aligned} \tag{32}$$

This can be either seen as Sasaki-Einstein structure with contact form $\eta = e^7$ or as 3-Sasaki-structure, i.e.

$$\mathrm{d}\eta^\alpha = \epsilon_{\alpha\beta\gamma}\eta^{\beta\gamma} + 2\omega^\alpha \qquad \mathrm{d}\omega^\alpha = 2\epsilon_{\alpha\beta\gamma}\eta^\beta \wedge \omega^\gamma \tag{33}$$

with $\eta^\alpha = e^5, e^6, e^7$. On the metric cone one has the Kähler form and the closed top-degree form (making it Calabi-Yau) as well as additionally another closed form (implying hyper-Kähler).

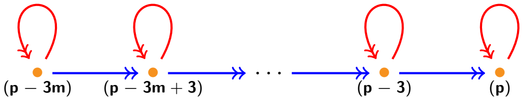
Equivariance condition on $X_{1,1}$

We consider the connection $\Gamma = I_8 \otimes \mathrm{e}^8$ (with I_8 the $\mathrm{U}(1)_{1,1}$ generator), whose curvature $\mathcal{F}_\Gamma \propto -e^{12} + e^{34}$ satisfies the Sasaki-Einstein instanton equation with $Q = e^{1234} + e^{1256} + e^{3456}$.

The connection $\mathcal{A} = A + \Gamma + \phi^{(\alpha)} \otimes \overline{\theta}_{(\bar\alpha)} + \mathrm{c.c} + X_7 \otimes \mathrm{e}^7$ requires the equivariance conditions (Haupt, Ivanova, Lechtenfeld, Popov 2011)

$$[\hat{I}_8, \phi^{(1)}] = 3\phi^{(1)}, \quad [\hat{I}_8, \phi^{(2)}] = -3\phi^{(2)}, \quad \text{and} \quad [\hat{I}_8, \phi^{(3)}] = 0 = [\hat{I}_8, X_7]. \tag{34}$$

and yields

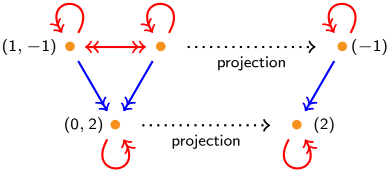


(35)

$$X_a e^a = \begin{pmatrix} \Psi_p & \Phi_{p-3} & 0 & \cdots & 0 \\ -\Phi^\dagger_{p-3} & \Psi_{p-3} & \Phi_{p-6} & \cdots & \vdots \\ 0 & -\Phi^\dagger_{p-6} & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \Psi_{p-3} & \Phi_{p-3m} \\ 0 & \cdots & 0 & -\Phi^\dagger_{p-3m} & \Psi_{p-3m} \end{pmatrix}, \tag{36}$$

Quiver diagrams on $X_{1,1}$

Fundamental representation:



(37)

The **other representations** are modified holomorphic chains as well. As for $T^{1,1}$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$, the reduction to the underlying Kähler manifold Q_3 is done by setting $X_7 \propto I_7$.

Instantons on the metric cone: HYM equation leads to

$$[W_1, W_2] = 2W_3, \quad [W_1, W_3] = 0 = [W_2, W_3] \tag{38}$$

$$\frac{W_j}{\mathrm{d}s} = 2[W_j, Z] \;\; j = 1,2,3 \;\; \text{and} \;\; 0 = \mu := \frac{(Z + Z^\dagger)}{\mathrm{d}s} + 2\lambda_j(s)[W_j, W_j^\dagger] + [Z, Z^\dagger].$$

with $(\phi_{1,2} = \mathrm{e}^{-\tau} W_{1,2}),\ \phi_3 = \mathrm{e}^{-2\tau} W_3,\ X_7 = \mathrm{e}^{-6\tau} Z,\ s = -1/6\,\mathrm{e}\,-6\tau, \tau = \ln(r)$.

Instantons on metric cone over $X_{1,1}$

Equations have similar form as for the five-dimensional examples and allow similar techniques for discussion. **However**, the scaling coefficients $\lambda_j(s)$ and also the commutation relations differ from the uniform, symmetric behavior of all fields!

Reason: The connection $\Gamma = I_8 \otimes e^8$, valued in $\mathfrak{h}$, we started with is *a* Sasaki-Einstein instanton, but *not* the characteric connection of the Sasaki-Einstein structure. Rather, it is the characteristic connection of the *3-Sasakian* geometry. The corresponding instanton equation uses

$$Q_{3S} = e^{1234} \quad \text{in contrast to} \quad Q_{SE} = e^{1234} + e^{1256} + e^{3456} \tag{39}$$

for the instanton curvature $\mathcal{F}_\Gamma \propto -e^{12} + e^{34}$.

$\Longrightarrow$ **Current/future work:** Compare the quiver gauge theories arising from the rich

geometric structures of 3-Sasakian coset spaces.

Summary and Outlook

- Imposing G-equivariance on vector bundles over coset spaces yields systemactic restrictions of the gauge connection which can be depicted as quiver diagrams.
- The underlying condition of equivariance naturally occurs when constructing generalized instantons on homogeneous spaces.
- The endomorphisms encoded in the quivers allow more general studies of instantons than the usual approach, setting $X_a = f_a(x)\, I_a$.
- Sasakian quiver gauge theory is the odd-dimensional counterpart of that on Kähler manifolds. The main features are the weaker equivariance conditions and the loop caused by the additional $\mathrm{U}(1)$.
- We discussed the Hermitian Yang-Mills equation and techniques for the description of the instanton moduli spaces on the metric cone.
- Current/future work: compare Sasakian and 3-Sasakian cases and construct explicit solutions.

Thank you for your attention!

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