On degeneration limits of Virasoro conformal blocks

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Aiming to obtain new special functions, P. Painlevé classified 2nd order nonlinear ordinary differential equations

$$R\left(z, w(z), \frac{dw(z)}{dz}, \frac{d^2w(z)}{dz^2}\right) = 0$$

whose movable singular points are pole only and obtained new six equations in 1900, which are called Painlevé equations.

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Hence, Weierstrass $\wp(z)$ function is a solution to (1.1). Weierstrass $\sigma(z)$ function is defined by

$$\wp(z) = -(\log \sigma(z))''.$$

$\mathrm{P_I}$ and its tau function

The first Painlevé equation is

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For any solution $\lambda(z)$, we define $\tau(z)$ by

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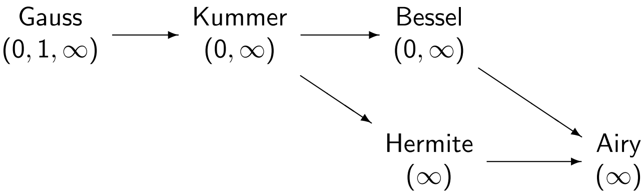
$$\frac{d^3 H}{dz^3} + 6\left(\frac{dH}{dz}\right)^2 + z = 0.$$

Degeneration scheme



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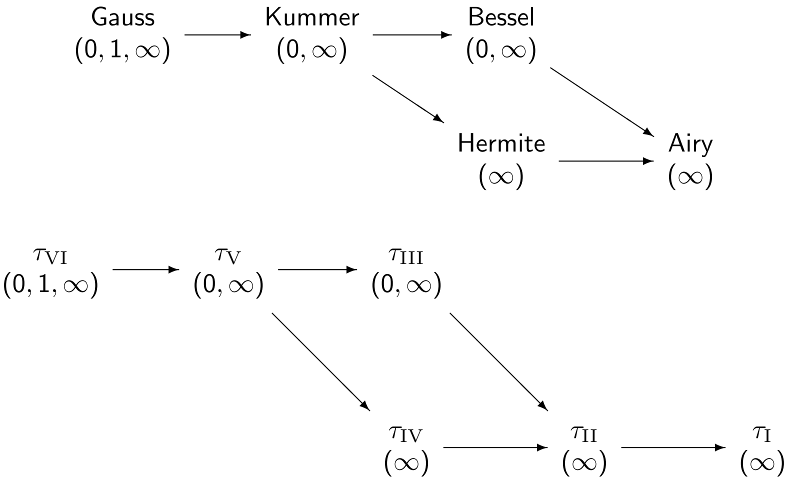


Explicit series expansion of $\tau_{\mathrm{VI}}(t)$

In 2012, Gamayun, Iorgov and Lisovyy conjectured an expansion formula P_{VI} tau function in terms of regular conformal blocks:

$$\tau_{\mathrm{VI}}(t) = \sum_{n\in\mathbb{Z}} s^n C\begin{pmatrix} \theta_1, \theta_t \\ \theta_\infty, \sigma+n, \theta_0 \end{pmatrix} \mathcal{F}\begin{pmatrix} \theta_1, \theta_t \\ \theta_\infty, \sigma+n, \theta_0 \end{pmatrix}; t\Bigg),$$

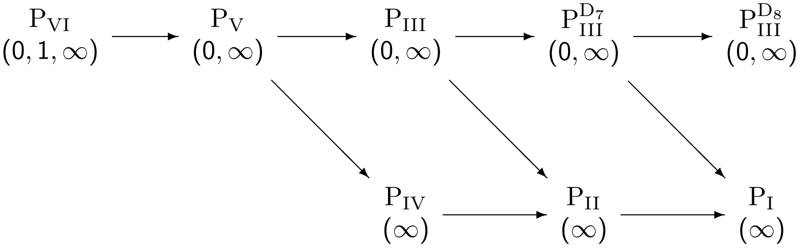
where $s, \sigma \in \mathbb{C}$, $\mathcal{F}(\theta, \sigma; t) = t^{\sigma^2-\theta_t^2-\theta_0^2}(1+O(t))$ is the 4-pt Virasoro conformal block with $c = 1$, and

$$C(\theta, \sigma) = \frac{\prod_{\epsilon,\epsilon'=\pm} G(1+\theta_t+\epsilon\theta_0+\epsilon'\sigma)G(1+\theta_1+\epsilon\theta_\infty+\epsilon'\sigma)}{\prod_{\epsilon=\pm} G(1+2\epsilon\sigma)},$$

where $G(z)$ is the Barnes G-function such that $G(z+1) = \Gamma(z)G(z)$.

Degeneration

Series expansions of the tau functions at $t = 0$, namely *a regular singular point* of the first line of the following degeneration scheme



were obtained in [Gamayun, Iorgov, Lisovyy, 2013] by taking the scaling limits. Irregular conformal blocks used in these series expansions are obtained by certain confluence limits from the four point conformal block and all explicit.

It had been known that

- irregular conformal blocks as degenerations of (regular) conformal blocks or pairings of irregular vectors. [Gaiotto, 2009]
- (intertwining) commutation relations between Virasoro algebra and vertex operators, in other words, operator product expansions(OPE).
- special cases by free field realizations, for example, a rank r vertex operator realized by the free field $\varphi(z)$: [N-Sun, 2010]

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Another approach

Provide irregular versions of vertex operator directly, then define irregular conformal blocks as expectation values of new vertex operators.

Module

For $r \in \mathbb{Z}_{\geq 1}$, define a module $M_\Lambda^{[r]}$ as a representation of Vir with irregular vector $|\Lambda\rangle$ such that

$$L_n|\Lambda\rangle = \Lambda_n|\Lambda\rangle \quad (n = r, r+1, \ldots, 2r),$$

with $\Lambda = (\Lambda_r, \ldots, \Lambda_{2r})$ and $M_\Lambda^{[r]}$ is spanned by linearly independent vectors of the form

$$L_{i_1} \cdots L_{i_k}|\Lambda\rangle \quad (i_1 \leq \cdots \leq i_k < r).$$

Vertex operator

Define a vertex operator

$$\Phi^{\Delta}_{\Lambda',\Lambda}(z) : M^{[r]}_{\Lambda} \to M^{[r]}_{\Lambda'}$$

by

$$[L_n, \Phi^{\Delta}_{\Lambda',\Lambda}(z)] = z^n \left(z \frac{d}{dz} + (n+1)\Delta \right) \Phi^{\Delta}_{\Lambda',\Lambda}(z),$$

$$\Phi^{\Delta}_{\Lambda',\Lambda}(z)|\Lambda\rangle = z^{\alpha} \exp\left(\sum_{n=1}^{r} \frac{\beta_n}{z^n} \right) \sum_{n=0}^{\infty} v_n z^n,$$

where $\alpha, \beta_n \in \mathbb{C}$, $v_n \in M^{[r]}_{\Lambda'}$ and $v_0 = |\Lambda'\rangle$.

Theorem (N, 2015)

If $\Lambda_{2r} \neq 0$, then the vertex operator $\Phi^{\Delta}_{\Lambda',\Lambda}(z)$: $M^{[r]}_{\Lambda} \to M^{[r]}_{\Lambda'}$ exists and is uniquely determined by the given parameters Λ, Δ, β_r with $\alpha = -(r+1)\Delta + \tilde{\alpha}(\beta_r, \Lambda)$, $\beta_n = \beta_n(\beta_r, \Lambda)$ $(n = 1, \ldots, r-1)$ and

$$\Lambda'_n = \Lambda_n - \delta_{n,r} r \beta_r \quad (n = r, \ldots, 2r).$$

Moreover, the coefficients v_n are polynomials of Λ, β_r, Δ and Λ_{2r}^{-1}.

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We remark that $M^{[r]}_{\Lambda}$ is irreducible if and only if $\Lambda_{2r-1} \neq 0$ or $\Lambda_{2r} \neq 0$ [Lu, Guo and Zhao, 2011], [Felińska, Jaskólski and Kosztolowicz, 2012].

Pairing

A bilinear pairing $\langle\cdot\rangle$: $M_\Delta^* \times M_\Lambda^{[1]} \to \mathbb{C}$ is uniquely defined by

$$\langle\Delta| \cdot |\Lambda\rangle = 1,$$
$$\langle u|L_n \cdot |v\rangle = \langle u| \cdot L_n|v\rangle \equiv \langle u|L_n|v\rangle,$$

where $\langle u| \in M_\Delta^*$, $|v\rangle \in M_\Lambda^{[1]}$.
Because, L_n for $n > 0$ acts on $|\Lambda\rangle$ diagonally and L_n for $n \leq 0$ acts on $\langle\Delta|$ diagonally.

Building block

The building block of $\tau_{\mathrm{VI}}(t)$ is the four point regular conformal block with $c = 1$:

$$\langle \theta_\infty^2 | \cdot \left(\Phi^{\theta_1^2}_{\theta_\infty^2, \sigma^2}(1) \Phi^{\theta_t^2}_{\sigma^2, \theta_0}(t) | \theta_0^2 \rangle \right).$$

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So, it is natural to expect that a building block of $\tau_{\rm V}(t)$ is the irregular conformal block having one irregular singular point and two regular singular points with $c = 1$:

$$\langle\theta_\infty^2| \cdot \left(\Phi^{\theta_t^2}_{(\Lambda_1-\beta,\Lambda_2),(\Lambda_1,\Lambda_2)}(t)|(\Lambda_1,\Lambda_2)\rangle\right),$$

$$\left(\langle\theta_\infty^2|\Phi^{*,\theta_t^2}_{\theta_\infty^2,\sigma^2}(t)\right)\cdot|(\Lambda_1,\Lambda_2)\rangle.$$

Theorem

A series expansion of the Painlevé V tau function at the irregular singular point ∞ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} s^n (-1)^{n(n+1)/2} G(1\pm\theta_0+\theta-\beta-n) G(1+\theta_t\pm(\beta+n))$$
$$\times \langle\theta_0^2| \cdot \left(\Phi^{\theta_t^2}_{(\theta-\beta-n,1/4),(\theta,1/4)}(t^{-1})|(\theta,1/4)\rangle\right).$$

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We prove this theorem by confluence limit.

A three-point irregular conformal block with $c=1$ is expanded as

$$\begin{aligned}
&\langle \theta_0^2 | \cdot \left(\Phi^{\theta_t^2}_{(\theta,1/4),(\theta-\beta,1/4)}(1/t) | (\theta,1/4)\rangle \right) \\
&= t^{2\theta_t^2+2\beta(\theta-\beta)} e^{\beta t} \left(1 + 2\left(2\beta^3 - 3\beta^2\theta + \beta\theta^2 - \beta\theta_0^2 - \beta\theta_t^2 + \theta\theta_t^2\right) t^{-1} \right. \\
&+ 2\left(4\beta^6 - 12\beta^5\theta + 13\beta^4\theta^2 - 4\beta^4\theta_0^2 - 4\beta^4\theta_t^2 + 5\beta^4 - 6\beta^3\theta^3 + 6\beta^3\theta\theta_0^2\right. \\
&+ 10\beta^3\theta\theta_t^2 - 10\beta^3\theta + \beta^2\theta^4 - 2\beta^2\theta^2\theta_0^2 - 8\beta^2\theta^2\theta_t^2 + 6\beta^2\theta^2 + \beta^2\theta_0^4 \\
&+ 2\beta^2\theta_0^2\theta_t^2 - 3\beta^2\theta_0^2 + \beta^2\theta_t^4 - 3\beta^2\theta_t^2 + 2\beta\theta^3\theta_t^2 - \beta\theta^3 \\
&\left.\left. -2\beta\theta\theta_0^2\theta_t^2 + \beta\theta\theta_0^2 - 2\beta\theta\theta_t^4 + 5\beta\theta\theta_t^2 + \theta^2\theta_t^4 - 2\theta^2\theta_t^2 + \theta_0^2\theta_t^2\right) t^{-2} + \cdots \right).
\end{aligned}$$

It is natural to expect that these irregular conformal blocks have combinatorial expressions.

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Fortunately, the coefficient of t^{-1} is expressed as

$$\begin{aligned}
&2\left(2\beta^3-3\beta^2\theta+\beta\theta^2-\beta\theta_0^2-\beta\theta_t^2+\theta\theta_t^2\right)\\
&=2(\beta-\theta)\left(\beta^2-\theta_t^2\right)+2\beta\left((\theta-\beta)^2-\theta_0^2\right).
\end{aligned}$$

We also have that the coefficient of t^{-2} is equal to

$$\begin{aligned}
&\frac{1}{2}(\theta-\beta)(2(\theta-\beta)+1)\left(\beta^2-\theta_t^2\right)\left((\beta-1)^2-\theta_t^2\right)\\
&+\frac{1}{2}(\theta-\beta)(2(\theta-\beta)-1)\left(\beta^2-\theta_t^2\right)\left((\beta+1)^2-\theta_t^2\right)\\
&+2(2(\theta-\beta)\beta-1)\left(\beta^2-\theta_t^2\right)\left((\theta-\beta)^2-\theta_0^2\right)\\
&+\beta(2\beta+1)\left((\theta-\beta)^2-\theta_0^2\right)\left((\theta-\beta-1)^2-\theta_0^2\right)\\
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&+2(2(\theta-\beta)\beta-1)\left(\beta^2-\theta_t^2\right)\left((\theta-\beta)^2-\theta_0^2\right)\\
&+\beta(2\beta+1)\left((\theta-\beta)^2-\theta_0^2\right)\left((\theta-\beta-1)^2-\theta_0^2\right)\\
&+\beta(2\beta-1)\left((\theta-\beta)^2-\theta_0^2\right)\left((\theta-\beta+1)^2-\theta_0^2\right).
\end{aligned}$$

We put

$$M_{\lambda,\mu}=\prod_{(i,j)\in\lambda}(2(\beta-\theta)+i-j)\prod_{(i,j)\in\mu}(-2\beta+i-j),$$

$$N_{\lambda,\mu}=(-1)^{|\mu|}\prod_{(i,j)\in\lambda}\frac{(\beta+i-j)^2-\theta_t^2}{h_\lambda(i,j)^2}\prod_{(i,j)\in\mu}\frac{(\theta-\beta+i-j)^2-\theta_0^2}{h_\mu(i,j)^2}.$$

Conjecture (N, arXiv:1611.08971)

A three-point irregular conformal block with two regular singular points t, ∞ and one irregular singular point 0 of rank one admits the following combinatorial formula

$$\langle\theta_0^2| \cdot \left(\Phi^{\theta_t^2}_{(\theta,1/4),(\theta-\beta,1/4)}(t)|(\theta,1/4)\rangle\right)$$

$$= t^{-2\theta_t^2-2\beta(\theta-\beta)} e^{\frac{\beta}{t}} \sum_{\lambda,\mu\in\mathbb{Y}} t^{|\lambda|+|\mu|} \sum_{\substack{\nu\subset\lambda,\eta\subset\mu,\\ |\nu|=|\eta|}} (-1)^{|\nu|} c^{\nu,\eta}_{\lambda,\mu} M_{\lambda/\nu,\mu/\eta} N_{\lambda,\mu},$$

where $c^{\nu,\eta}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$, as an expansion at the irregular singular point 0.

From Gauss to Kummer

The Gauss hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+(\gamma-(\alpha+\beta+1)x)\frac{dy}{dx}-\alpha\beta y=0$$

admit a confluence limit as

$$\beta\to\infty,\quad x\to\frac{x}{\beta}.$$

Taking the limit, we obtain the Kummer confluent hypergeometric equation

$$x\frac{d^2y}{dx}+(\gamma-x)\frac{dy}{dx}-\alpha y=0.$$

Limit of solutions

The Gauss hypergeometric equation has the following local solutions:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n}x^n,\quad x^{1-\gamma}\sum_{n=0}^{\infty}\frac{(\alpha-\gamma+1)_n(\beta-\gamma+1)_n}{(2-\gamma)_n(1)_n}x^n,\quad (x=0),$$

$$x^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha-\gamma+1)_n}{(\alpha-\beta+1)_n(1)_n}x^{-n},\quad x^{-\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n(\beta-\gamma+1)_n}{(\beta-\alpha+1)_n(1)_n}x^{-n},\quad (x=\infty),$$

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where $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$. It is easy to see that the three solutions except the last one admit limits by $\beta\to\infty$, $x\to x/\beta$. The last one is transformed to

$$x^{-\beta}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_n(1-\alpha)_n}{(\beta-\alpha+1)_n(1)_n}x^{-n}.$$

Then, we have the limit by $\beta\to\infty$, $x\to x/\beta$.

Consider

$$|R^{(2)}\rangle = \Phi^{\Delta_3}_{\Delta_4,\Delta}(w)\Phi^{\Delta_2}_{\Delta,\Delta_1}(z)|\Delta_1\rangle.$$

In what follows, we let w go to zero, while z is in a general position. Then $|R^{(2)}\rangle$ becomes an expansion of z at the irregular singular point zero. We already know how to take a limit of $\Phi^{\Delta_3}_{\Delta_4,\Delta}(w)|\Delta\rangle$ and the coefficients $R_k(w)$ of z^k $(k \geq 1)$ in

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diverge. Instead, Gaiotto and Teschner suggested a rearranged expansion of $|R^{(2)}\rangle$:

$$|R^{(2)}\rangle = z^{\Delta-\Delta_2-\Delta_1} w^{\Delta_4-\Delta_3-\Delta} \left(1 - \frac{z}{w}\right)^A \sum_{k=0}^{\infty} z^k |R^{(1)}_k\rangle$$

for some constant A in Appendix D of [Gaiotto, Teschner 2012].

The condition of the limit of $|R_0^{(1)}\rangle$ is

$$\Delta_3 - \Delta = \frac{\Lambda_1}{\epsilon} + O(1), \quad 2\Delta_3 - \Delta = \frac{\Lambda_2}{\epsilon^2} + O(\epsilon^{-1}) \quad (\epsilon \to 0).$$

The resulting vector $|I^{(1)}\rangle = \lim_{\epsilon\to 0}|R_0^{(1)}\rangle$ with $w = \epsilon$ satisfies

$$L_1|I^{(1)}\rangle = \Lambda_1|I^{(1)}\rangle, \quad L_2|I^{(1)}\rangle = \Lambda_2|I^{(1)}\rangle, \quad L_n|I^{(1)}\rangle = 0 \quad (n > 2).$$

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Also $|R_k^{(1)}\rangle$ satisfy for $n > 0$

$$\begin{aligned}&(L_n - w^n(w\partial_w + \Delta_4 + n\Delta_3 - \Delta))|R_k^{(1)}\rangle \\ &= A\sum_{s=1}^{n-1} w^{n-s}|R_{k-s}^{(1)}\rangle + (A + \Delta + n\Delta_2 - \Delta_1 + k - n)|R_{k-n}^{(1)}\rangle\end{aligned}$$

The coefficients in the left hand side admit a limit. Also the coefficients in the right hand side admit a limit by

$$A = O(\epsilon^{-1}), \quad A + \Delta - \Delta_1 = O(1), \quad \epsilon \to 0.$$

Furthermore, if we set

$$A\epsilon = -\beta + O(\epsilon), \quad A + \Delta - \Delta_0 = \alpha + \Delta_z, \tag{3.1}$$

where

$$\alpha = -\frac{\beta\Lambda_1}{2\Lambda_2} - 2\Delta_2, \tag{3.2}$$

then the limits of the recursion relations for $|R_k^{(1)}\rangle$ for L_n $(n \geq 1)$ take exactly the same forms for the vectors v_k of $\Phi^{\Delta_z}_{(\Lambda_1,\Lambda_2),(\Lambda_1+\beta,\Lambda_2)}(z)$: $M^{[1]}_{(\Lambda_1+\beta,\Lambda_2)} \to M^{[1]}_{(\Lambda_1,\Lambda_2)}$ such that

$$\Phi^{\Delta_z}_{(\Lambda_1,\Lambda_2),(\Lambda_1+\beta,\Lambda_2)}(z)|((\Lambda_1+\beta,\Lambda_2))\rangle = z^\alpha e^{\beta/z} \sum_{k=0}^{\infty} v_k z^k,$$

which are

$$\begin{aligned}
(L_1 - \Lambda_1)v_k =& (\alpha + 2\Delta_z + k - 1)v_{k-1}, \\
(L_2 - \Lambda_2)v_k =& -\beta v_{k-1} + (\alpha + 3\Delta_z + k - 2)v_{k-2}, \\
L_n v_k =& -\beta v_{k-n+1} + (\alpha + (n+1)\Delta_z + k - n)v_{k-n} \quad (n > 2).
\end{aligned}$$

The uniqueness of v_k proved in [N, 2015] implies that all $|R_k^{(1)}\rangle$ converge.

We are taking a limit of the series expansion of Painlevé VI tau function at $t = 0$:

$$\tau_{\mathrm{VI}}(t) = \sum_{n\in\mathbb{Z}} s^n C\begin{pmatrix} \theta_1, \theta_t \\ \theta_\infty, \sigma+n, \theta_0 \end{pmatrix} \mathcal{F}\begin{pmatrix} \theta_1, \theta_t & \\ \theta_\infty, \sigma+n, \theta_0 & ; \dfrac{t}{w} \end{pmatrix}$$

as $w(=\epsilon)$ goes to 0. We know how to take a limit of

$$\mathcal{F}\begin{pmatrix} \theta_1, \theta_t & \\ \theta_\infty, \sigma+n, \theta_0 & ; \dfrac{t}{w} \end{pmatrix} = w^{\theta_\infty^2-\theta_1^2-(\sigma+n)^2} t^{(\sigma+n)^2-\theta_t^2-\theta_0^2}\left(1-\frac{t}{w}\right)^{A(n)} \sum_{k=0}^{\infty} R_k(w) t^k.$$

Then, from $w^{-n^2-A(n)}$, we have ϵ^{-2n^2}.

Using

$$G(1+x+n) = G(1+x)\prod_{i=1}^{n}\Gamma(x)\prod_{j=1}^{n+1-i}(x+n+1-i-j),$$

$$G(1+x-n) = G(1+x)\prod_{i=0}^{n-1}\Gamma(x)^{-1}\prod_{j=0}^{n-2-i}(x-n+1+i+j)$$

for $n>0$, we obtain

$$C\begin{pmatrix} \theta_1,\theta_t \\ \theta_\infty,\sigma+n,\theta_0 \end{pmatrix} = PQ^n\epsilon^{2n^2}C\left(\theta_\infty,\beta+n,\theta_t,\theta\right)(1+O(\epsilon)),$$

where P and Q are independent to n. □