Conservation laws and stability of higher-derivative theories

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5th String Theory Meeting in the Greater Tokyo Area,
December 02, 2016

based on arXiv: 1407.8481, 1510.02007, 1510.08365

Background: Ostrogradski instability

Given the set of fields $\varphi^J(t)$, consider the Lagrangian theory with the action

$$S = \int L(\varphi, \dot{\varphi}, \ddot{\varphi})\, dt\,.$$

Then, the canonical (or Noether) energy of the theory reads

$$E = \ddot{\varphi}^J \frac{\partial L}{\partial \ddot{\varphi}^J} + \dot{\varphi}^J\Big(\frac{\partial L}{\partial \dot{\varphi}^J} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{\varphi}^J}\Big) - L = \frac{\partial^2 L}{\partial \ddot{\varphi}^I \partial \ddot{\varphi}^J}\dot{\varphi}^I \dddot{\varphi}^J + \dots\,,$$

dots denote the terms, which depend on $\ddot{\varphi}, \dot{\varphi}, \varphi$.

The energy contains $\dddot{\varphi}^J$ in a linear way. If the Hesse matrix is non-degenerate,

$$\det \frac{\partial^2 L}{\partial \ddot{\varphi}^I \partial \ddot{\varphi}^J} \neq 0\,,$$

the third derivates $\dddot{\varphi}^J$'s are independent Cauchy data and, thus, the energy cannot be bounded from below.

The corresponding quantum theory does not have a well-defined vacuum state with the lowest energy. Such theory is unstable.

Idea

- Use of alternative Hamiltonian formulation and quantization scheme

Key steps

- Construct the most general conserved quantity. If the theory has several integrals of motion, take their combination with indefinite coefficients.

- Take the general conserved quantity as the anastz of the Hamiltonian. Find the Poisson bracket. At this step, the theory may appear to be poly-Hamiltonian.

- Consider the interactions which are compatible with alternative Hamiltonian formulations.

If one of alternative Hamiltonian formulations has the bounded from below Hamiltonian, the corresponding theory is stable from the classical and quantum viewpoints.

K. Bolonek and P. Kosinski, Acta Phys. Polon. B 36 (2005) 2115

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Critical problems

• How to construct the most general conserved quantity in a higher-derivative theory? What conserved quantities are relevant?

• How to find the Poisson bracket? When does the solution to the problem exist?

• Is there algorithmic procedure for construction of stable interactions?

And, finally,

• Can one see that the higher-derivative theory is stable already at the first step of alternative Hamiltonian formulation construction?

In this talk, we answer these questions and consider the alternative Hamiltonian formulation construction for the certain class of higher derivative models.

Plan

The plan of the talk is follows:

- Warm-up example: the Pais-Uhlenbeck oscillator
- Derived theories
    - Symmetries and conservation laws
    - Hamiltonian formulation (condition of existence)
    - Stable interactions
- Field-theoretical example: 3rd order extensions of the Chern-Simons model
- Summary

4th order Pais-Uhlenbeck oscillator

- Lagrangian

$$L = \frac{1}{2}\frac{1}{\omega_2^2 - \omega_1^2}\Big(\ddot{\varphi}^2 - (\omega_1^2 + \omega_2^2)\dot{\varphi}^2 + \omega_1^2\omega_2^2\varphi^2\Big)\,, \qquad \overset{(n)}{\varphi} = \frac{d^n\varphi}{dt^n}\,, \qquad 0 < \omega_1 < \omega_2\,.$$

with $\varphi(t)$ being the dynamical variable; ω's are the frequencies of oscillations.

The equation of motion reads

$$\frac{1}{\omega_2^2 - \omega_1^2}\Big(\frac{d^2}{dt^2} + \omega_1^2\Big)\Big(\frac{d^2}{dt^2} + \omega_2^2\Big) = \frac{1}{\omega_2^2 - \omega_1^2}(\overset{(4)}{\varphi} + (\omega_1^2 + \omega_2^2)\ddot{\varphi} + \omega_1^2\omega_2^2\varphi) = 0\,.$$

- The Hesse matrix

$$\frac{\partial^2 L}{\partial\ddot{\varphi}\partial\ddot{\varphi}} = \frac{1}{\omega_2^2 - \omega_1^2} \neq 0$$

is non-degenerate, so the theory is non-singular.

- The canonical energy

$$E = \ddot{\varphi}\frac{\partial L}{\partial\ddot{\varphi}} + \dot{\varphi}\Big(\frac{\partial L}{\partial\dot{\varphi}} - \frac{d}{dt}\frac{\partial L}{\partial\ddot{\varphi}}\Big) - L = \frac{\ddot{\varphi}^2 - (\omega_1^2 + \omega_2^2)\dot{\varphi}^2 - 2\dot{\varphi}\dddot{\varphi} - \omega_1^2\omega_2^2\varphi^2}{2(\omega_2^2 - \omega_1^2)}\,.$$

It is unbounded from below, because it is linear in $\dddot{\varphi}$.

Integrals of motion

In a general case, the Pais-Uhlenbeck oscillator has two integrals of motion

$$J_1 = \frac{1}{2}\left(\frac{\dddot{\varphi}+\omega_2^2\dot{\varphi}}{\omega_2^2-\omega_1^2}\right)^2 + \frac{\omega_1^2}{2}\left(\frac{\ddot{\varphi}+\omega_2^2\varphi}{\omega_2^2-\omega_1^2}\right)^2\,, \qquad J_2 = J_1\Big|_{\omega_1\leftrightarrow\omega_2}\,.$$

Both J_1 and J_2 are non-negative, $J_1, J_2 \geq 0$, and independent.

The general conserved quantity is the linear combination of these integrals of motion

$$J_{\alpha_1,\alpha_2} = \alpha_1 J_1 + \alpha_2 J_2\,.$$

Here, α_1 and α_2 are some real constants.

$\alpha_1 = -\alpha_2 = 1$ corresponds to the canonical energy. It is not bounded from below because J_1 and J_2 contributions have different signs.

For $\alpha_1, \alpha_2 \geq 0$, the integral of motion J_{α_1,α_2} is a bounded quantity.

The Noether theorem relates J_1 and J_2 to the following symmetries

$$\delta_1\varphi = -\frac{\dddot{\varphi}+\omega_2^2\dot{\varphi}}{\omega_2^2-\omega_1^2}\,, \qquad \delta_2\varphi = -\frac{\dddot{\varphi}+\omega_1^2\dot{\varphi}}{\omega_2^2-\omega_1^2}\,.$$

The Pais-Uhlenbeck model is equivalent to the system of two harmonic oscillators

$$\ddot{x} + \omega_1^2 x = 0\,, \qquad \ddot{y} + \omega_2^2 y = 0\,.$$

The following formulas establish correspondence between the solutions to the theories

$$\varphi(t) = x(t) + y(t), \qquad x(t) = \frac{\ddot{\varphi} + \omega_2^2\varphi}{\omega_2^2 - \omega_1^2}\,, \qquad y(t) = \frac{\ddot{\varphi} + \omega_1^2\varphi}{\omega_1^2 - \omega_2^2}\,.$$

(Note: this correspondence is one-to-one only on the mass shell)

In the harmonic coordinates, the integrals of motion J_1 and J_2 take the form

$$J_1 = \frac{1}{2}\dot{x}^2 + \frac{\omega_1^2}{2}x^2\,, \qquad J_2 = \frac{1}{2}\dot{y}^2 + \frac{\omega_2^2}{2}y^2\,.$$

The following symmetries related to the integrals of motion

$$\delta_1\varphi = -\dot{x}\,, \qquad \delta_2\varphi = -\dot{y}$$

The integrals J_1 and J_2 are the "energies" of oscillatory modes x and y.

Alternative Hamiltonian formulations

Introduce the collective notation $z_a = \{\varphi, \dot{\varphi}, \ddot{\varphi}, \dddot{\varphi}\}$ for the phase-space variables.

Then the bi-Hamiltonian formulation (Bolonek and Kosinski, 2005) reads

$$\dot{z}^a = \{z^a, J_{\alpha_1,\alpha_2}\}_{\alpha_1,\alpha_2}\,,$$

with α_1, α_2 being the parameters. We assume that $\alpha_1\alpha_2 \neq 0$.

The Poisson bracket $\{\ ,\ \}_{\alpha_1,\alpha_2}$ is defined by the relations

$$\{\varphi, \dot{\varphi}\}_{\alpha_1,\alpha_2} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\,, \quad \{\varphi\,, \dddot{\varphi}\}_{\alpha_1,\alpha_2} = -\{\dot{\varphi}, \ddot{\varphi}\}_{\alpha_1,\alpha_2} = \frac{\omega_1^2}{\alpha_1} + \frac{\omega_2^2}{\alpha_2}$$
$$\{\ddot{\varphi}\,, \dddot{\varphi}\}_{\alpha_1,\alpha_2} = \frac{\omega_1^4}{\alpha_1} + \frac{\omega_2^4}{\alpha_2}\,, \quad \{\varphi, \ddot{\varphi}\}_{\alpha_1,\alpha_2} = \{\dot{\varphi}, \dddot{\varphi}\}_{\alpha_1,\alpha_2} = 0\,.$$

$\alpha_1 = -\alpha_2 = 1$ corresponds to the Ostrogradski's Hamiltonian formulation.

$\alpha_1 = \alpha_2 = 1$ corresponds to the "alternative" and stable Hamiltonian formulation.

Ostrogradski and "alternative" Hamiltonian formulations are not equivalent.

Stable interactions

(D.S.K., S.L. Lyakhovich, A.A. Sharapov, 2014)

$$\overset{(4)}{\varphi} + (\omega_1^2+\omega_2^2)\ddot{\varphi} + \omega_1^2\omega_2^2\varphi + U'(x-y) = 0\,, \qquad J = J_{1,1} + E_U(x-y)\,,$$

$$U' = \frac{\partial U}{\partial \varphi} - \frac{d}{dt}\frac{\partial U}{\partial \dot{\varphi}}\,, \qquad E_U = \dot{\varphi}\frac{\partial U}{\partial \dot{\varphi}} - U\,, \qquad U = U(\dot{\varphi},\varphi)\,.$$

(D.S.K., S.L. Lyakhovich, A.A. Sharapov, 2016)

$$\overset{(4)}{\varphi} + (\omega_1^2+\omega_2^2)\ddot{\varphi} + \omega_1^2\omega_2^2\varphi + 2\ddot{U}' + (\omega_1^2+\omega_2^2)U' = 0\,,$$

$$J = J_{1,1}(x+\frac{1}{2}U, y+\frac{1}{2}U) + E_U\,,$$

$$U' = \frac{\partial U}{\partial \varphi} - \frac{d}{dt}\frac{\partial U}{\partial \dot{\varphi}}\,, \qquad E_U = \dot{\varphi}\frac{\partial U}{\partial \dot{\varphi}} - U\,, \qquad U = U(\dot{\varphi},\varphi)\,.$$

(D.S.K., S.L. Lyakhovich, to appear in Russ. Phys. J., 2017)

These two interactions are equivalent on the mass shell.

Part 2.

Conservation laws and stability of derived field theories

Linear field theories

We use the following notation.

- Spacetime is n-dim. Minkowski space with local coordinates $x^\mu\,, \mu=\overline{0, n-1}$.
- Signature of metric is $\eta_{\mu\nu}=\mathtt{diag}\{+\,,-\,,\ldots\,,-\}$.
- Set of fields is $\varphi=\{\varphi^J(x)\}$. The multi-index J includes all discrete indices.
- The theory admits appropriate constant metrics to rise and lower multi-indices.

In this setting, any local linear system of field equations can be represented in the following form:

$$M_{IJ}(\partial)\varphi^J=0.$$

Here, $M=\{M_{IJ}\}$ is the matrix whose entries are polynomials in ∂_μ. If ∂_μ is the partial derivative w.r.t. to the coordinate x^μ, we have the system of PDEs.

- M is known as the wave operator.

Derived field theories

Linear field theory is called a derived theory if its wave operator is the polynomial of a formally self-adjoint operator W of lower order.

- Equation of motion of a derived can be represented in two alternative forms

$$M_{IJ}\varphi^J = 0\,,$$

$$M(W) = \sum_{k=0}^{N} l_k W^k = l_N \prod_{a=1}^{r}(W-\omega_a)^{p_a} \prod_{\alpha=r+1}^{r+s}(W^2-(\omega_\alpha+\overline{\omega}_\alpha)W+|\omega_\alpha|^2)^{p_\alpha}\,.$$

Here l_k are some constants and $\omega_a, \omega_\alpha, \overline{\omega}_\alpha$ be the roots of the polynomial $M(W)$ with the multiplicities p_a and p_α. We use the collective notation $A=\{a,\alpha\}$ for the indices labelling roots.

- The operator W is self-adjoint if $W^\dagger = W$. The conjugation rule is defined by

$$W^\dagger{}_{IJ}(\partial) = W(-\partial)_{JI}\,.$$

- Any derived theory is Lagrangian with the Lagrangian

$$L = \frac{1}{2}\varphi^I M_{IJ}\varphi^J\,.$$

Examples of derived theories

- Scalar field with higher derivatives

$$\varphi^J(x) = \varphi(x)\,, \qquad W = \Box\,, \qquad M = (\Box + m_1^2)(\Box + m_2^2)\,.$$

$$\text{EoM:} \qquad (\Box + m_1^2)(\Box + m_2^2)\varphi = 0\,.$$

- Bopp-Podolski generalized electrodynamics

$$\varphi^J(x) = \varphi^\mu(x)\,, \quad W = \Box - \partial\partial\cdot\,, \quad M = W(W + m^2)\,.$$

$$\text{EoM:} \qquad (\Box + m^2)\partial^\mu F_{\mu\nu} = 0\,, \qquad F_{\mu\nu} = \partial_\mu\varphi_\nu - \partial_\nu\varphi_\mu\,.$$

- Odd-order Pais-Uhlenbeck oscillator

$$\varphi^J(x) = \{\varphi^i(t)\}, \quad W_{ij} = \varepsilon_{ij}\frac{d}{dt}\,, \quad \varepsilon_{ij} = -\varepsilon_{ji}\,, \quad \varepsilon_{12} = 1\,.$$

$$\text{EoM :} \qquad M_{ij}\varphi^j = \prod_{a=1}^{r}\Big(\frac{d^2}{dt^2} + \omega_a^2\Big)\varepsilon_{ij}\dot{\varphi}^j = 0\,, \qquad i,j = 1,2.$$

Reduction of order

Introduce the cofactors Ω_A by the formulas

$$\Omega_A = \prod_{a\neq A}(W-\omega_a)^{p_a}\prod_{\alpha\neq A}(W^2-(\omega_\alpha+\overline{\omega}_\alpha)+|\omega|^2_\alpha)^{q_\alpha}\,.$$

Then, the derived theory is equivalent to the system of equations of lower order

$$(W-\omega_a)^{p_a}\xi_a=0\,,\qquad (W^2-(\omega_\alpha+\overline{\omega}_\alpha)+|\omega|^2_\alpha)^{p_\alpha}\xi_\alpha=0\,.$$

for the unknown fields $\xi_A=\{\xi_A^J(x)\}$.

The following relations establish a correspondence between solutions of the models

$$\xi_A=\Omega_A\varphi\,,\qquad \varphi=\sum_A\Lambda_A\xi_A\,,$$

where the matrix differential operators Λ_A are defined from the equation:

$$\sum_A\Lambda_A\Omega_A=1\,,\qquad \Lambda_A=\sum_{k=0}^{N_A}\lambda_A^kW^k\,,\qquad N_A=p_a-1\text{ or }2p_\alpha-1$$

Symmetries and conservation laws

Provided by W is space-time translation invariant, $[W, \partial] = 0$, the following transformations keep the action of derived theory invariant:

$$\delta_A \varphi = \partial_\mu \xi_A \,, \qquad \delta_A S = 0 \,.$$

If $N_A > 0$ (this is the case of multiple real and complex rings), then the following transformations are also symmetries of the action:

$$\delta_A^k \varphi = \partial_\mu W^k \xi_A \,, \qquad k = 1, \ldots, N_A \,.$$

- These two groups include exactly $N \times n$ symmetries.

Corresponding Noether's conserved currents combine into N second-rank conserved tensors

$$(\Theta_A^k)^\mu{}_\nu \,, \qquad k = 0, \ldots, N_A \,.$$

Here, $k = 0$ corresponds to the first group of symmetries. These conserved tensors Θ_A^0 are the "energy-momentum tensors" for the fields ξ_A.

General conservation law

Since the stability is concerned, the conserved quantities associated to the time translations are relevant.

Introduce conserved quantities are related to the time-translation symmetry

$$J_A^k = \int d^{n-1}x (\Theta_A^k)^0{}_0 \,.$$

Then, the general ansatz for conserved quantity reads

$$J_\alpha = \sum_A \sum_{k=1}^{N_a} \alpha_A^k J_A^k \,,$$

with α_A^k being some constants. The number of independent parameters α_A^k is N.

- The canonical energy $E = J$ corresponds to the choice $\alpha_A^k = -\lambda_A^k$.
- Depending on the values of α's the conserved quantity J may be bounded.
- Some conserved tensors J_A^k may be trivial.

Hamiltonian formalism

Conjecture

Let $J_{\bar{\alpha}}$ be the conserved quantity for some fixed value of the parameters $\overline{\alpha}_A^k$, i.e.

$$J_{\bar{\alpha}} = \sum_A \sum_{k=0}^{N_a} \overline{\alpha}_A^k J_A^k \,.$$

Then $J_{\bar{\alpha}}$ is a Hamiltonian of the derived theory w.r.t. some Poisson bracket if and only if

$$\overline{\alpha}_A^0 \neq 0$$

for all A's.

Corollary 1

Any derived theory is poly-Hamiltonian. There are N free parameters $\overline{\alpha}_A^k$ in the Hamiltonian.

Corollary 2

The Hamiltonian $J_{\overline{\alpha}}$ may be bounded or unbounded depending on the values of parameters $\overline{\alpha}_A^k$. If $J_{\bar{\alpha}} \geq 0$, the theory is stable classically and quantum mechanically.

Stable interactions

The derived theory admits two types of (stable) nonlinear extensions

1) $$M\varphi + U'(V\varphi) = 0\,, \qquad J' = J_{\bar\alpha}(\varphi) + E_U(V\varphi)\,.$$

2) $$M\varphi + VU'(\varphi) = 0\,, \qquad J' = J_{\bar\alpha}(\varphi - VU) + E_U(\varphi)\,.$$

Here,

- U' is the Euler-Lagrange derivative of interaction function $U = U(\varphi, \partial\varphi, ..)$
- E_U is the canonical energy of the interaction;
- $V(W)$ is the matrix differential operator such that

$$V\Omega = -1 \qquad (\mathrm{Mod}\,M)\,, \qquad V = \sum_{k=0}^{N-1} v^k W^k\,, \qquad \Omega = \sum_A \sum_{k=1}^{N_A} \alpha_A^k W^k \Omega_A\,;$$

- These two types of nonlinear extensions are equivalent on the mass shell.

Stability analysis procedure for a derived theory

The main four steps are follows:

• Write out the Lagrangian and equations of motion. Find the wave operator M.

• Identify the operator W and structure of roots of the polynomial $M(W)$.

• Construct the symmetries associated with the factors of M and conservation laws.

• Write out the general conserved quantity. Identify the range of parameters where it is bounded.

If conditions for the parameters ensure existence of Hamiltonian formulation with bounded from below Hamiltonian, the theory is stable.

(Be attentive in case of multiple roots)

Part 3.

Field-theoretical example.

3rd-order extension of 3d Chern-Simons model

Action

$$S = \frac{1}{2}\int d^3x\Big(\frac{1}{m}\varepsilon^{\mu\nu\rho}F_\mu\partial_\nu F_\rho + l_2 F^\mu F_\mu + l_1 m F^\mu\varphi_\mu + l_0 m^2\varphi^\mu\varphi_\mu\Big)\,, \quad F^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu\varphi_\rho.$$

where φ^μ is the vector field on 3d Minkowski space, m is a constant with the dimension of mass, l_0, l_1, l_2 are some constant parameters.

Equations of motion

$$-\frac{1}{m}\Box\varepsilon^{\mu\nu\rho}\partial_\nu\varphi_\rho - l_2(\Box\delta^\mu{}_\nu - \partial^\mu\partial_\nu)\varphi^\nu + l_1 m\varepsilon^{\mu\nu\rho}\partial_\nu\varphi_\rho + l_0 m^2\varphi^\mu = 0\,.$$

We can rewrite the equations of motion in the form

$$\frac{1}{m^2}(W^3 + l_2 W^2 + l_1 W + l_0)\varphi = 0\,, \qquad W^{\mu\nu} = \varepsilon^{\mu\rho\nu}\partial_\rho\,,$$

so we are dealing with the derived theory.

The Chern-Simons operator W is obviously self-adjoint, $W^\dagger = W$.

Depending on the values of parameters m, l_2, l_1, l_0 four cases seem to be different.

- Polynomial $M(W)$ has three different real roots ω_a, $a = 1, 2, 3$.

$$l_2 = -\omega_1 - \omega_2 - \omega_3\,, \qquad l_1 = \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3\,, \qquad l_0 = -\omega_1\omega_2\omega_3\,.$$

- Simple and multiplicity 2 real roots ω_a, $a = 1, 2$: $p_1 = 1, p_2 = 2$.

$$l_2 = -\omega_1 - 2\omega_2\,, \qquad l_1 = 2\omega_1\omega_2 + \omega_2^2\,, \qquad l_0 = -\omega_1\omega_2^2\,.$$

- Real and complex roots ω_1 and ω_2.

$$l_2 = -\omega_1 - \omega_2 - \overline{\omega}_2\,, \qquad l_1 = \omega_1\omega_2 + \omega_1\overline{\omega}_2 + |\omega_2|^2\,, \qquad l_0 = -\omega_1|\omega_2|^2\,.$$

- Multiplicity 3 real root ω_1.

$$l_2 = -3\omega_1\,, \qquad l_1 = 3\omega_1^2\,, \qquad l_0 = -\omega_1^3\,.$$

The theory is stable in case of different real roots and multiple zero real root, and unstable otherwise.

The canonical energy is unbounded in all the cases.

Symmetries and conservation laws (three real roots)

- Symmetries

$$\delta_a\varphi = \Omega_a\varphi\,, \qquad \Omega_a = \prod_{b\neq a}(W-\omega_b)\,.$$

- Conservation laws

$$J_a^0 = J_a \approx \frac{\omega_a}{2}\int d^2x(\Omega_a\varphi)^2\,, \qquad (\Omega_a\varphi)^2 = (\Omega_a\varphi_0)^2 + (\Omega_a\varphi_1)^2 + (\Omega_a\varphi_2)^2.$$

(all conservation laws are sign-definite depending on the sign of the root ω_a)

General conserved quantity

$$J_\alpha = \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3\,, \qquad \alpha_a \equiv \alpha_a^0\,,$$

is bounded from below of $\alpha_a\omega_a \geq 0\,, \qquad a = 1, 2, 3.$

It is bounded for α_a's such that $\alpha_a\omega_a > 0$ (take any $\alpha_b \neq 0$ for $\omega_b = 0$).

In this case, the extension of Chern-Simons theory admits Hamiltonian formulation with bounded Hamiltonian. It is stable.

Symmetries and conservation laws (two real roots)

- Symmetries

$$\delta_1\varphi = (W-\omega_2)^2\varphi\,, \qquad \delta_2^p\varphi = (W-\omega_2)^p(W-\omega_1)\varphi\,, \qquad p = 0, 1.$$

- Conservation laws

$$J_1 \equiv J_1^0 \approx \frac{\omega_a}{2}\int d^2x(\Omega_1\varphi)^2\,, \qquad J_2^0 \approx \frac{1}{2}\int d^2x((W\Omega_2\varphi)^2 - \omega_2^2(W\Omega_2\varphi)^2)\,,$$

$$J_2^1 \approx \omega_2^2\int d^2x(W\Omega_2\varphi, \Omega_2\varphi)\,, \qquad \Omega_1 = (W-\omega_2)^2\,, \qquad \Omega_2 = W-\omega_1\,.$$

(J_2^0 is not bounded unless $\omega_2 = 0$, i.e. unless multiple root equals zero)

General conserved quantity

$$J_\alpha = \alpha_1 J_1 + \alpha_2^0 J_2^0 + \alpha_2^1 J_2^1\,, \qquad \alpha_1 \equiv \alpha_1^0\,,$$

is bounded from below if simultaneosly three conditions are satisfied

$$\alpha_1\omega_1 \geq 0\,, \qquad \alpha_2^0 \geq 0\,, \qquad \text{and} \qquad \omega_2 = 0\,.$$

The extension of Chern-Simons theory is stable if multiple root equals zero.

Some references on non-Lagrangian field theory

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Thank you for your attention!