Nahm's equations and transverse Hilbert schemes

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Hilbert schemes of points

Let X be a complex manifold.

- The Hibert scheme $X^{[n]}$ of n points on X parameterises ideals $I \subset \mathcal{O}_X$ such that $\mathcal{O}_X/I$ has 0-dimensional support and $\dim H^0(\mathcal{O}_X/I) = n$.
- It is a partial resolution of singularities of the symmetric power $S^nX = X^n/\Sigma_n$.
- $X^{[2]}$ is the blow-up of the diagonal in $S^2X \implies X^{[2]}$ is always smooth.
- If $\dim_{\mathbb{C}} X = 2$, then $X^{[n]}$ is smooth for each n (Fogarty).
- If $\dim_{\mathbb{C}} X = 2$ and X has a complex symplectic form ω, then $X^{[n]}$ has a canonical complex symplectic form $\omega^{[n]}$ (Beauville).
- $\implies$ (together with the Calabi-Yau them) if X is a compact Ricci-flat Kähler complex surface, then $X^{[n]}$ is hyperkähler for each n.
- What about noncompact X? True for ALE spaces (i.e. $X^{[n]}$ is hyperkähler) (Nakajima, quiver varieties)

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A toy example

$X = \mathbb{C}^2$. Nakajima describes $(\mathbb{C}^2)^{[n]}$ as follows: it is the manifold of triples $(B_1, B_2, i) \in \mathrm{Mat}_{n,n}(\mathbb{C}) \times \mathrm{Mat}_{n,n}(\mathbb{C}) \times \mathbb{C}^n$ such that $[B_1, B_2] = 0$ and there is no proper subspace S of $\mathbb{C}^n$ such that $i \in S$, $B_1 S \subset S$, $B_2 S \subset S$, **modulo** the $GL(n, \mathbb{C})$-action (conjugation on $\mathrm{Mat}_{n,n}(\mathbb{C})$, standard on $\mathbb{C}^n$).

The correspondence between such an orbit and an ideal of colength n in $\mathbb{C}[z_1, z_2]$ is given by

$$(B_1, B_2, i) \mapsto I = \{f \in \mathbb{C}[z_1, z_2]; f(B_1, B_2)i = 0\}.$$

For example, assume that B_1 and B_2 are simultaneously diagonalisable: $B_1 = \mathrm{diag}(x_1, \ldots, x_n)$, $B_2 = \mathrm{diag}(y_1, \ldots, y_n)$. The stability condition and the residual action of invertible diagonal matrices imply that i can be taken to be $(1, \ldots, 1)^T$, and then, again by the stability condition, we must have $(x_i, y_i) \neq (x_j, y_j)$ if $i \neq j$, i.e. simultaneously diagonalisable B_1, B_2 correspond to n distinct points in $\mathbb{C}^2$.

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Generally, two commuting matrices can be simultaneously made upper-triangular:

$$B_1 = \begin{pmatrix} x_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_n \end{pmatrix}, \quad B_2 = \begin{pmatrix} y_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & y_n \end{pmatrix}.$$

The Hilbert-Chow morphism $X^{[n]} \to S^n X$ is given by

$$(B_1, B_2, i) \to \{(x_1, y_1), \dots, (x_n, y_n)\}.$$

The off-diagonal entries (with coords i, j such that $(x_i, y_i) = (x_j, y_j)$) correspond to (multi)-tangent directions. For example, the ideal I corresponding to

$$B_1 = \begin{pmatrix} x & \alpha \\ 0 & x \end{pmatrix}, \quad B_2 = \begin{pmatrix} y & \beta \\ 0 & y \end{pmatrix}, \quad i = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $(\alpha, \beta) \neq (0, 0)$ is generated by $(z_1 - x)^2$, $(z_2 - y)^2$, and $\beta(z_1 - x) - \alpha(z_2 - y)$, i.e. I consists of $f \in \mathbb{C}[z_1, z_2]$ such that

$$f(x, y) = 0, \quad \left(\alpha \frac{\partial f}{\partial z_1} + \beta \frac{\partial f}{\partial z_2}\right)(x, y) = 0.$$

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Transverse Hilbert schemes

- A construction due to Atiyah and Hitchin which often produces complete hyperkähler metrics on open subsets of $X^{[n]}$.
- Setting: X - complex manifold, C - 1-dimensional complex manifold, $\pi : X \to C$ a surjective holomorphic map.
- $X_\pi^{[n]}$ is an open subset of $X^{[n]}$ consisting of those $Z \in X^{[n]}$ such that $\pi_* O_Z$ is a cyclic O_C sheaf.
- $\iff$ $\pi : Z \to \pi(Z)$ is an isomorphism onto its scheme-theoretic image, i.e. $\dim O_{\pi(Z)} = n$.

Locally, a nbhd of an $t_0 \in \pi(Z)$ is of the form $\mathbb{C}[t]/(t^m)$ for some $m \leq n$. Since $\pi : Z \to \pi(Z)$ is an isomorphism, there exists a $\phi : \mathbb{C}[t]/(t^m) \to X$, the image of which is an open subset of Z. Such a ϕ is an equivalence class of local sections of π, truncated up to order m.

Let us call $X_\pi^{[n]}$ the Hilbert scheme of n points transverse to π.

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Affine case

Suppose that $X \subset \mathbb{C}^k$ is an affine variety and $\pi : X \to \mathbb{C}$ is a restriction of a polynomial. W.l.o.g. we can assume that $\pi(w_1, \ldots, w_k) = w_k$.

Then $X_\pi^{[n]}$ is an affine variety in $\mathbb{C}^{kn}$: we identify $\mathbb{C}^{kn}$ with k-tuples of polynomials $(p_1(z), \ldots, p_{k-1}(z), q(z))$ with $\deg p_i \le n-1$ and $q(z)$ a monic polynomial of degree n. Let I be the defining ideal of X. Then $X_\pi^{[n]}$ is defined by the equations:

$$\forall_{f \in I} \quad f(p_1(z), \ldots, p_{k-1}(z), z) = 0 \mod q(z).$$

Examples. 1) $X = \mathbb{C}^* \times \mathbb{C}$ and π - projection onto the second factor, i.e. $X = \{(x, y, z) \in \mathbb{C}^3;\ xy = 1\}$ and $\pi(x, y, z) = z$. According to the above, $X_\pi^{[n]}$ consists of triples $(x(z), y(z), q(z))$ of polynomials with $\deg x, \deg y \le n-1$ and q a monic polynomial of degree n such that $x(z)y(z) = 1 \mod q(z)$. In other words $x(z)$ (or $y(z)$) does not vanish at any of the roots of $q(z)$ and, consequently, $X_\pi^{[n]}$ is isomorphic to the space of based rational maps of degree n.

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2) $X = \mathbb{C}^2$ and $\pi(x,y) = xy$. Recall the description of $(\mathbb{C}^2)^{[n]}$ as $\{(B_1, B_2, i); \dots\}/GL_n(\mathbb{C})$. The ideal of $\pi(Z)$ consists of $g \in \mathbb{C}[z]$ such that $\pi^* g = 0$, i.e. $g(B_1 B_2) = 0$. We require $\dim \mathcal{O}_{\pi(Z)} = n$, which means that $B_1 B_2$ is a regular matrix.

Thus $(\mathbb{C}^2)^{[n]}_\pi$ consists of $GL(n,\mathbb{C})$-orbits of triples (B_1, B_2, i) as above and such that $B_1 B_2$ is a regular matrix. Every conjugacy class of regular matrices contains a unique companion matrix S, i.e. a matrix of the form

Back

$$\begin{pmatrix} 0 & \dots & 0 & q_0 \\ 1 & \ddots & 0 & q_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & q_{n-1} \end{pmatrix}. \qquad (1)$$

If $B_1 B_2 = S$, then we can conjugate B_1 and B_2 by an element of the centraliser of S in order to make the vector i equal to $e_1 = (1, 0, \dots, 0)^T$.

Thus $(\mathbb{C}^2)^{[n]}_\pi$ is isomorphic to the variety of triples (B_1, B_2, S) of $n \times n$ matrices, such that S is of the form (1), B_1, B_2 commute, and $B_1 B_2 = S$.

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2) $X = \mathbb{C}^2$ and $\pi(x,y) = xy$. Recall the description of $(\mathbb{C}^2)^{[n]}$ as $\{(B_1, B_2, i); \dots\}/GL_n(\mathbb{C})$. The ideal of $\pi(Z)$ consists of $g \in \mathbb{C}[z]$ such that $\pi^* g = 0$, i.e. $g(B_1 B_2) = 0$. We require $\dim \mathcal{O}_{\pi(Z)} = n$, which means that $B_1 B_2$ is a regular matrix.

Thus $(\mathbb{C}^2)^{[n]}_{\pi}$ consists of $GL(n,\mathbb{C})$-orbits of triples (B_1, B_2, i) as above and such that $B_1 B_2$ is a regular matrix. Every conjugacy class of regular matrices contains a unique companion matrix S, i.e. a matrix of the form

Back

$$\begin{pmatrix} 0 & \dots & 0 & q_0 \\ 1 & \ddots & 0 & q_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & q_{n-1} \end{pmatrix}. \tag{1}$$

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3) Let X be the double cover of the Atiyah-Hitchin manifold, i.e. an affine surface in $\mathbb{C}^3$ defined by the equation $x^2 - zy^2 = 1$. Again, $X_\pi^{[n]}$ is the variety of triples of polynomials $x(z), y(z), q(z)$ of degrees $\leq n-1, \leq n-1$ and n and q monic, such that $x(z), y(z), z$ satisfy the defining equation modulo $q(z)$. Alternatively, consider the quadratic extension $z = u^2$, so that the defining equation becomes $(x+uy)(x-uy) = 1$. If $x(z)$ and $y(z)$ are polynomials of degree $\leq n-1$, then $x(z) \pm uy(z) = x(u^2) \pm uy(u^2)$ and $q(z) = q(u^2)$. In other words, $q(u^2)$ is a polynomial of degree $2n$ with all coefficients of odd powers equal to 0 and $p(u) = x(u^2) + uy(u^2)$ is a polynomial of degree $\leq 2n-1$ satisfying $p(u)p(-u) = 1$ at every root u of $q(u^2)$. Thus $X_\pi^{[n]}$ is the space of degree $2n$ based rational maps of the form $p(u)/q(u^2)$ with p satisfying the above condition.

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Atiyah-Hitchin: do the construction $X \mapsto X_\pi^{[n]}$ on fibres of the twistor space of X.

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Twistor spaces

Let X be a 4-dimensional hyperkähler manifold. Its twistor space Z (diffeomorphic to $X \times S^2$) is a complex 3-fold with a holomorphic projection $\pi : Z \to \mathbb{P}^1$ and an antiholomorphic involution $\tau : Z \to Z$ covering the antipodal map on $\mathbb{P}^1$. There is also an $O(2)$-valued complex symplectic form ω along the fibres of π.

The manifold M and its hyperkähler structure are encoded in Z. In particular, M corresponds to a connected component of the Kodaira manifold of all τ-invariant sections of π with normal bundle $\simeq O(1) \oplus O(1)$. Each fibre of Z is biholomorphic to M with one of the complex structures.

Examples: 1) X - flat $\mathbb{R}^4 \implies Z = |O(1) \oplus O(1)|$ (total space of a vector bundle).
2) $X = S^1 \times \mathbb{R}^3 \implies Z = |L^c| \backslash \{0\}$, where L^c is a line bundle over $T\mathbb{P}^1$ with transition function $\exp\{-c\eta/\zeta\}$ (here ζ is the affine coordinate on $\mathbb{P}^1 \backslash \{0\}$ and η the induced fibre coordinate on $T(\mathbb{P}^1 \backslash \{0\})$).
3) X - $\mathbb{R}^4$ with the Taub-NUT metric $\implies$

$$Z = \{(x, y, z) \in (L^c(1) \oplus L^{-c}(1)) \oplus O(2);\ xy = z\}.$$

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Hyperkähler metrics on $X_\pi^{[n]}$

X - 4-dimensional hyperkähler manifold; for each complex structure a holomorphic map $X \to \mathbb{C} \implies$ a holomorphic map $p : Z \to |\mathcal{O}(2r)|$, $r \geq 1$.

Apply the transverse Hilbert scheme construction fibrewise to get a new twistor space $Z_p^{[n]}$ with all the properties of the twistor space of a hyperkähler manifold except, perhaps, existence of sections of $Z_p^{[n]} \to \mathbb{P}^1$ with correct normal bundle.

A section of $Z_p^{[n]} \to \mathbb{P}^1$ is the same as a curve C of degree n in $\mathcal{O}(2r)$ together with a lift $\hat{C}$ to Z. The condition that the normal bundle of the section is the sum of $\mathcal{O}(1)$'s is equivalent to $H^*\left(C, N_{\hat{C}/Z}(-2)\right) = 0$

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2) X - ALE space of type A or D (i.e. surfaces $xy = z^{k+1} + \dots$, $x^2 - zy^2 = z^{n-1} + \dots$), $\pi(x,y,z) = z$. Then $X_\pi^{[n]}$ (with a bound on n) carries a complete hK metric (Manolescu for A_{2d}, Jackson - others).
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$\mathfrak{g}$ - Lie algebra of a compact group G.
Nahm's equations are ODE for quadruples of $\mathfrak{g}$-valued smooth functions $T_i(t)$, $i = 0, 1, 2, 3$:

$$\dot{T}_1 + [T_0, T_1] = [T_2, T_3]$$

and two further equations given by cyclic permutations of indices $1, 2, 3$.
We are interested in solutions on $(0, 1]$ such that T_0 is smooth at $t = 0$ and $T_i(t) = \alpha_i / t + \text{smooth}$, $i = 1, 2, 3$, where $\alpha_1, \alpha_2, \alpha_3$ is an $\mathfrak{su}(2)$-triple in $\mathfrak{g}$.
The group of smooth gauge transformations $g(t)$ on $[0, 1]$ with $g(0) = g(1) = 1$ acts on the set of solutions and the resulting moduli space has a natural complete hyperkähler metric.
With respect to any complex structure, say I, this manifold is biholomorphic to $S(f) \times G^{\mathbb{C}}$ where $S(f)$ is the *Slodowy slice* to the nilpotent adjoint orbit of $f = \alpha_2 + i\alpha_3$, i.e. $S(f) = f + Z(f^*)$. For example, if $G = U(n)$ and f is the standard nilpotent element of rank $n-1$, then $S(f)$ is the set of matrices of the form:

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The nice thing is that the complex structures of these quotients are easily identified (under mild assumptions):
if $\mu : M \to \mathfrak{g}^{\mathbb{C}}$ is the complex moment map for the $G^{\mathbb{C}}$-action on M, then this hyperkähler quotient is biholomorphic to $\mu^{-1}\big(S(f)\big)$.

Examples: 1) $G = U(n)$, f a regular nilpotent element, M - second copy of $S(f) \times G^{\mathbb{C}}$: get the moduli space of $SU(2)$-monopoles of charge n.

2) $\mathfrak{g}$ of type A, D or E, f a subregular nilpotent orbit, M a regular semisimple adjoint orbit of $G^{\mathbb{C}}$ with its Kronheimer's metric: get the ALE-spaces of type A, D or E.

3) $\mathfrak{g}$ and M as above of type A or D, f - more general: get the transverse Hilbert schemes of points on ALE-spaces (Seidel-Smith, Manolescu, Jackson).

4) $G = SU(2) \times SU(k-2)$, f - regular nilpotent, M - regular semisimple orbit of $SL(k, \mathbb{C})$: get the ALF-spaces of type D_k.

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4) $G = SU(2) \times SU(k-2)$, f - regular nilpotent, M - regular semisimple orbit of $SL(k, \mathbb{C})$: get the ALF-spaces of type D_k.

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The nice thing is that the complex structures of these quotients are easily identified (under mild assumptions):

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We can identify M with pairs B_1, B_2 of complex matrices; then the two $\mathfrak{gl}(n, \mathbb{C})$-valued moment maps are given by $B_1 B_2$ and $B_2 B_1$. Thus the transverse Hilbert scheme is the set of (B_1, B_2) such that $B_1 B_2$ and $B_2 B_1$ is a companion matrix. It follows that this is the same companion matrix and we obtain $(\mathbb{C}^2)^{[n]}_{\pi}$.

The transverse Hilbert schemes of points on the D_0, D_1, D_2-surfaces arise this way for $G = U(n)$, f - a regular nilpotent element, and M product of two minimal semisimple adjoint orbits.

For the D_0- and D_1-surface there is an alternative description, as a hyperkähler submanifold of the moduli space of magnetic $SU(2)$-monopoles of charge $2n$ for D_0 and charge $2n+1$ for D_1.

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