Painlevé functions,conformal blocks and combinatorics II

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Joint work with
O. Lisovyy [1608.00958]
A. Marshakov [1605.04554]

Reminder

For $n = 4$ points

$$\tau = \det(1 + K)$$

$$K = \begin{pmatrix} 0 & a^{[2]} \\ d^{[1]} & 0 \end{pmatrix}$$

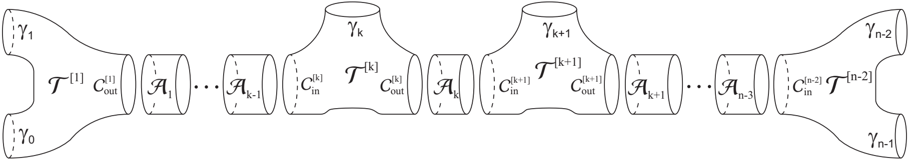
$$(\mathsf{a}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}(z, z')\, g(z')\, dz', \qquad \mathsf{a}(z, z') = \frac{\Psi^{[R]}(z)\, \Psi^{[R]}(z')^{-1} - \mathbb{1}}{z - z'},$$

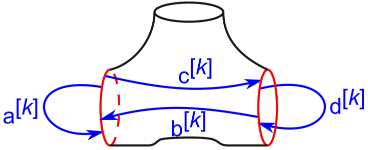
$$(\mathsf{d}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{d}(z, z')\, g(z')\, dz', \qquad \mathsf{d}(z, z') = \frac{\mathbb{1} - \Psi^{[L]}(z)\, \Psi^{[L]}(z')^{-1}}{z - z'}.$$

Plan of the talk

- Fredholm determinant representation for the arbitrary number of points
- Combinatorial description of diagonal minor expansion
- Structure of the matrix elements
- Computation of determinants
- Nekrasov functions
- Fermionic construction [P.G., A. Marshakov [1605.04554]].

Generalization to higher number of points





$$\left(\mathsf{a}^{[k]}g\right)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}^{[k]}_{\mathrm{in}}} \frac{\left[\Psi^{[k]}_{+}(z)\,\Psi^{[k]}_{+}(z')^{-1} - \mathbb{1}\right] g(z')\,dz'}{z-z'}$$

$$\left(\mathsf{b}^{[k]}g\right)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}^{[k]}_{\mathrm{out}}} \frac{\Psi^{[k]}_{+}(z)\,\Psi^{[k]}_{+}(z')^{-1} g(z')\,dz'}{z-z'}$$

$$\left(\mathsf{c}^{[k]}g\right)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}^{[k]}_{\mathrm{in}}} \frac{\Psi^{[k]}_{+}(z)\,\Psi^{[k]}_{+}(z')^{-1} g(z')\,dz'}{z-z'}$$

$$\left(\mathsf{d}^{[k]}g\right)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}^{[k]}_{\mathrm{out}}} \frac{\left[\Psi^{[k]}_{+}(z)\,\Psi^{[k]}_{+}(z')^{-1} - \mathbb{1}\right] g(z')\,dz'}{z-z'}$$

Diagonal minors

$$K_{\vec{I},\vec{J}} := \begin{array}{c|cccccccccc}
 & I_1 & J_1 & I_2 & J_2 & I_3 & J_3 & \cdot & \cdot & I_{n-3} & J_{n-3} \\
\hline
I_1 & 0 & (\mathrm{a}^{[2]})^{I_1}_{J_1} & (\mathrm{b}^{[2]})^{I_1}_{I_2} & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
J_1 & (\mathrm{d}^{[1]})^{J_1}_{I_1} & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
I_2 & 0 & 0 & 0 & (\mathrm{a}^{[3]})^{I_2}_{J_2} & (\mathrm{b}^{[3]})^{I_2}_{I_3} & 0 & \cdot & \cdot & 0 & 0 \\
J_2 & 0 & (\mathrm{c}^{[2]})^{J_2}_{J_1} & (\mathrm{d}^{[2]})^{J_2}_{I_2} & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
I_3 & 0 & 0 & 0 & 0 & 0 & (\mathrm{a}^{[4]})^{I_3}_{J_3} & \cdot & \cdot & \cdot & \cdot \\
J_3 & 0 & 0 & 0 & (\mathrm{c}^{[3]})^{J_3}_{J_2} & (\mathrm{d}^{[3]})^{J_3}_{I_3} & 0 & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (\mathrm{b}^{[n-3]})^{I_{n-2}}_{I_{n-3}} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
I_{n-3} & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & (\mathrm{a}^{[n-2]})^{I_{n-3}}_{J_{n-3}} \\
J_{n-3} & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & (\mathrm{c}^{[n-3]})^{J_{n-3}}_{J_{n-4}} & (\mathrm{d}^{[n-3]})^{J_{n-3}}_{I_{n-3}} & 0
\end{array}$$

$$|I_k| = |J_k|$$

Diagonal minors

$$
K_{\vec{I},\vec{J}} := \begin{pmatrix}
0 & (\mathsf{a}^{[2]})^{I_1}_{J_1} & (\mathsf{b}^{[2]})^{I_1}_{I_2} & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
(\mathsf{d}^{[1]})^{J_1}_{I_1} & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & (\mathsf{a}^{[3]})^{I_2}_{J_2} & (\mathsf{b}^{[3]})^{I_2}_{I_3} & 0 & \cdot & \cdot & 0 & 0 \\
0 & (\mathsf{c}^{[2]})^{J_2}_{J_1} & (\mathsf{d}^{[2]})^{J_2}_{I_2} & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (\mathsf{a}^{[4]})^{I_3}_{J_3} & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & (\mathsf{c}^{[3]})^{J_3}_{J_2} & (\mathsf{d}^{[3]})^{J_3}_{I_3} & 0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (\mathsf{b}^{[n-3]})^{I_{n-2}}_{I_{n-3}} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & (\mathsf{a}^{[n-2]})^{I_{n-3}}_{J_{n-3}} \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & (\mathsf{c}^{[n-3]})^{J_{n-3}}_{J_{n-4}} & (\mathsf{d}^{[n-3]})^{J_{n-3}}_{I_{n-3}} & 0
\end{pmatrix}
$$

Permutation of rows:

$$
\tilde{K}_{\vec{I},\vec{J}} = (\mathsf{d}^{[1]})^{J_1}_{I_1} \oplus \begin{pmatrix} (\mathsf{a}^{[2]})^{I_1}_{J_1} & (\mathsf{b}^{[2]})^{I_1}_{I_2} \\ (\mathsf{c}^{[2]})^{J_2}_{J_1} & (\mathsf{d}^{[2]})^{J_2}_{I_2} \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} (\mathsf{a}^{[n-3]})^{I_{n-2}}_{J_{n-4}} & (\mathsf{b}^{[n-3]})^{I_{n-2}}_{I_{n-3}} \\ (\mathsf{c}^{[n-3]})^{J_{n-3}}_{J_{n-4}} & (\mathsf{d}^{[n-3]})^{J_{n-3}}_{I_{n-3}} \end{pmatrix} \oplus (\mathsf{d}^{[n-2]})^{I_{n-3}}_{J_{n-3}}
$$

Combinatorial expansion

$$Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right) := (-1)^{|I_k|}\det\begin{pmatrix} \left(\mathsf{a}^{[k]}\right)^{I_{k-1}}_{J_{k-1}} & \left(\mathsf{b}^{[k]}\right)^{I_{k-1}}_{I_k} \\ \left(\mathsf{c}^{[k]}\right)^{J_k}_{J_{k-1}} & \left(\mathsf{d}^{[k]}\right)^{J_k}_{I_k} \end{pmatrix}$$

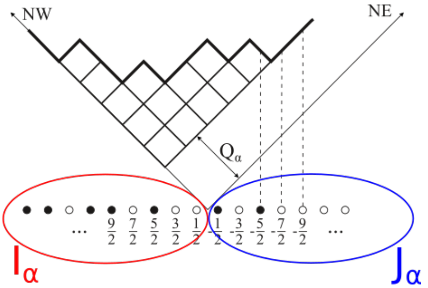
$$\tau = \sum_{(\vec{I},\vec{J})\in \mathrm{Conf}_+} \prod_{k=1}^{n-2} Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right).$$

Maya diagrams and charged Young diagrams

$$I = \{(\alpha_1, p_1), (\alpha_2, p_2), \dots\}, \qquad I = \bigcup_{\alpha=1}^{N} I_\alpha$$

$$J = \{(\beta_1, -q_1), (\beta_2, -q_2), \dots\}, \qquad J = \bigcup_{\alpha=1}^{N} J_\alpha$$

$$|I_\alpha| - |J_\alpha| = Q_\alpha, \qquad \sum_{\alpha=1}^{N} Q_\alpha = 0$$



$$I_\alpha = \{\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{13}{2}\}, \qquad J_\alpha = \{-\frac{1}{2}, -\frac{5}{2}\}, \qquad Q_\alpha = 2$$

Combinatorial expansion-2

$$\{(I,J): |I|=|J|\} = \{(Y_1,\ldots,Y_N); (Q_1,\ldots Q_N): \sum Q_\alpha = 0\}$$

$$\tau = \sum_{\vec{Q}_1,\ldots\vec{Q}_{n-3}\in\mathfrak{Q}_N} \sum_{\vec{Y}_1,\ldots\vec{Y}_{n-3}\in\mathbb{Y}^N} \prod_{k=1}^{n-2} Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}\left(\mathcal{T}^{[k]}\right),$$

Computation of matrix elements

$$\mathsf{a}^{[k]}\left(z,z'\right) := \frac{\Psi_{+}^{[k]}(z)\,\Psi_{+}^{[k]}(z')^{-1} - \mathbb{1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_{+}} \mathsf{a}^{[k]}{}_{-q}^{\;p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}$$

$$\mathcal{L}_0 = z\partial_z + z'\partial_{z'} + 1.$$

$$\mathcal{L}_0 \mathsf{a}^{[k]}\left(z,z'\right) = \frac{\left(z\partial_z + z'\partial_{z'}\right)\Psi_{+}^{[k]}(z)\,\Psi_{+}^{[k]}(z')^{-1}}{z-z'} =$$

$$= \left[\mathsf{a}^{[k]}\left(z,z'\right), \mathfrak{S}_{k-1}\right] - \frac{\Psi_{+}^{[k]}(z)}{z-a_k} a_k A_1^{[k]} \frac{\Psi_{+}^{[k]}(z')^{-1}}{z'-a_k} \quad - \text{finite rank}$$

$$\boxed{\mathsf{a}^{[k]}{}_{-q;\beta}^{\;p;\alpha} = \sum_{r=1}^{\mathfrak{r}^{[k]}} \frac{\left(\psi_r^{[k]}\right)^{p;\alpha}\left(\bar{\psi}_r^{[k]}\right)_{q;\beta}}{p+q+\sigma_{k-1,\alpha}-\sigma_{k-1,\beta}}}$$

Nekrasov functions

When $\mathfrak{r}^{[k]} = 1$ everything (up to diagonal factors) reduces to Cauchy determinant:

$$\det\left(\frac{1}{x_\imath^{[k]} - y_\jmath^{[k]}}\right)_{\jmath\in J_{k-1}\sqcup I_k}^{\imath\in I_{k-1}\sqcup J_k} = \frac{\prod\limits_{i<j}(x_i - x_j)\prod\limits_{i>j}(y_i - y_j)}{\prod\limits_{ij}(x_i - y_j)}$$

It can be computed and proved to coincide with Nekrasov functions:

$$\boxed{Z_{\vec{Y},\vec{Q}}^{\vec{Y}',\vec{Q}'}(\mathcal{T}) = \pm e^{\vec{\beta}'\cdot\vec{Q}'+\vec{\beta}\vec{Q}}\frac{C(\vec{\sigma}'+\vec{Q}',\vec{\sigma}+\vec{Q})}{C(\vec{\sigma}',\vec{\sigma})}\sqrt{Z_{vec}(\vec{\sigma}'+\vec{Q}'|\vec{Y}')}Z_{bif}(\vec{\sigma}'+\vec{Q}',\vec{\sigma}+\vec{Q}|\vec{Y}',\vec{Y})\sqrt{Z_{vec}(\vec{\sigma}+\vec{Q}|\vec{Y})}} =$$

$$= \frac{\pm\prod\limits_{\alpha,\beta=1}^{N} Z_b(\sigma'_\alpha - \sigma_\beta + Q'_\alpha - Q_\beta|Y'_\alpha, Y_\beta)}{\prod\limits_{\alpha<\beta}^{N}\left[Z_b(\sigma'_\alpha - \sigma'_\beta + Q'_\alpha - Q'_\beta|Y'_\alpha, Y'_\beta)Z_b(\sigma_\beta - \sigma_\alpha + Q_\beta - Q_\alpha|Y_\beta, Y_\alpha)\right]\prod\limits_{\alpha=1}^{N}\sqrt{Z_b(0|Y_\alpha, Y_\alpha)Z_b(0|Y'_\alpha, Y'_\alpha)}}\times$$

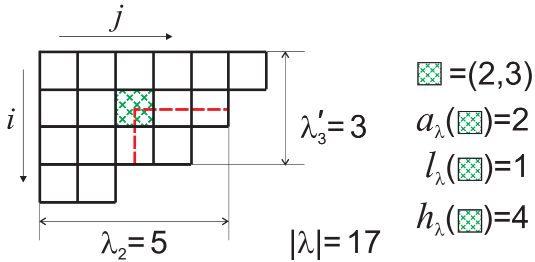
$$\times\frac{\prod\limits_{\alpha\beta} C(\sigma'_\alpha - \sigma_\beta|Q'_\alpha, Q_\beta)}{\prod\limits_{\alpha<\beta}^{N}\left[C(\sigma'_\alpha - \sigma'_\beta|Q'_\alpha, Q'_\beta)C(\sigma_\beta - \sigma_\alpha|Q_\beta, Q_\alpha)\right]}$$

Where

$$C(\nu|Q', Q) = \frac{G(1+\nu+Q'-Q)}{G(1+\nu)\Gamma(1+\nu)^{Q'-Q}}$$

and

$$Z_b(\nu|Y',Y) = \prod_{s\in Y'}(\nu+1+a_{Y'}(s)+l_Y(s))\prod_{t\in Y}(\nu-1-a_Y(t)-l_{Y'}(t))$$



Fermionic algebra

$$\psi^{\boldsymbol{\sigma}}_{\alpha}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\psi^{\boldsymbol{\sigma}}_{\alpha,n}}{z^{n+\frac{1}{2}+\sigma_\alpha}} \qquad \psi^{*\boldsymbol{\sigma}}_{\alpha}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\psi^{*\boldsymbol{\sigma}}_{\alpha,n}}{z^{n+\frac{1}{2}-\sigma_\alpha}}$$

$$\{\psi^*_{\alpha,k}, \psi_{\beta,m}\} = \delta_{\alpha\beta}\delta_{k+m,0} \qquad \psi_{\alpha,p>0}|\boldsymbol{\sigma}\rangle = 0 \qquad \psi^*_{\alpha,p>0}|\boldsymbol{\sigma}\rangle = 0$$

We define such $V_{\boldsymbol{\nu}}(t)$ that

$$\langle \vec{Y}_{k-1}, \vec{Q}_{k-1}|V_{\boldsymbol{\nu}}(t)|\vec{Y}_k, \vec{Q}_k\rangle = Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}\left(\mathcal{T}^{[k]}\right)$$

Then

$$\tau(a_1, \ldots, a_n) = \langle 0|V_1(a_1)\ldots V_n(a_n)|0\rangle$$

Axiomatic definition of the vertex operator

Operator V_ν now will act from the fermionic Fock module $\mathcal{F}_{\boldsymbol{\sigma}}$ to $\mathcal{F}_{\boldsymbol{\theta}}$. It is defined by two axioms

1. $V_\nu(t)$ is a group-like element: $V_\nu(t)^{-1}\psi^{\boldsymbol{\theta}}_{\alpha p}V_\nu(t) \in \mathrm{Span}(\psi^{\boldsymbol{\sigma}}_{\beta q}\Big| \forall\beta, q)$

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2. All 2-fermionic correlators give the solution for the 3-point Riemann-Hilbert problem in the different regions

$$\langle\boldsymbol{\theta}|\mathcal{R}V^{\boldsymbol{\nu}}(t)\psi^{*\boldsymbol{\sigma}}_{\alpha}(z)\psi^{\boldsymbol{\sigma}}_{\beta}(w)|\boldsymbol{\sigma}\rangle = \frac{[\phi(z)\phi(w)^{-1}]_{\alpha\beta}}{z-w}, \qquad |z|\leq t,\ |w|\leq t$$

$$\langle\boldsymbol{\theta}|\mathcal{R}\psi^{*\boldsymbol{\theta}}_{\dot\alpha}(z)\psi^{\boldsymbol{\theta}}_{\dot\beta}(w)V^{\boldsymbol{\nu}}(t)|\boldsymbol{\sigma}\rangle = \frac{[\tilde\phi(z)\tilde\phi(w)^{-1}]_{\dot\alpha\dot\beta}}{z-w}, \qquad |z|\geq t,\ |w|\geq t$$

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$$-\langle\boldsymbol{\theta}|\mathcal{R}\psi^{\boldsymbol{\theta}}_{\dot\beta}(w)V^{\boldsymbol{\nu}}(t)\psi^{*\boldsymbol{\sigma}}_{\alpha}(z)|\boldsymbol{\sigma}\rangle = \frac{[\phi(z)\tilde\phi(w)^{-1}]_{\alpha\dot\beta}}{z-w}, \qquad |z|\leq t,\ |w|\geq t$$

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Where $\phi(z)$ (which is $N\times N$ matrix) solves the system

$$\frac{d}{dz}\phi(z) = \phi(z)\left(\frac{A_0}{z} + \frac{A_t}{z-t}\right)$$

with $A_0 \sim \boldsymbol{\sigma}$, $A_t \sim \boldsymbol{\nu}$, $A_\infty \sim \boldsymbol{\theta}$, and normalized such that $\phi(z)\sim z^{A_0}, z\to 0$. $\tilde\phi(z)$ solves the same system, but $\tilde\phi(z)\sim z^{A_\infty}, z\to\infty$.

They are related by $\tilde\phi(z) = C\phi(z)$ when $|z|=t$

Proof of the primarity

Fermionic realization of $W_N \oplus H$

One may take, for example,

$$U_k(z) = \sum_{\alpha=1}^{N} (\psi^*_\alpha(z)\partial^{k-1}\psi_\alpha(z))$$

Only $U_1(z), \ldots, U_N(z)$ are independent, others are expressed in some non-linear way. For example: $((\psi^*(z)\psi(z))(\psi^*(z)\psi(z))) = (\partial\psi^*(z)\psi(z)) - (\psi^*(z)\partial\psi(z))$

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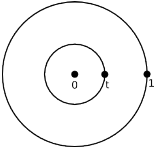

- $U_k^{\theta}(z)V^{\nu}(t) = V^{\nu}(t)U_k^{\sigma}(z)$ when $|z| = t$ because $U_k(z)$ are invariant with respect to the replacement $\psi_\alpha(z) \mapsto \sum_\beta C_{\alpha\beta}\psi_\beta(z)$, $\psi^*_\alpha(z) \mapsto \sum_\beta C^{-1}_{\beta\alpha}\psi_\beta(z)$

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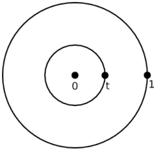

- $U_k^{\boldsymbol{\theta}}(z)V^{\boldsymbol{\nu}}(t) = V^{\boldsymbol{\nu}}(t)U_k^{\boldsymbol{\sigma}}(z)$ when $|z| = t$ because $U_k(z)$ are invariant with respect to the replacement $\psi_\alpha(z) \mapsto \sum_\beta C_{\alpha\beta}\psi_\beta(z)$, $\psi^*_\alpha(z) \mapsto \sum_\beta C^{-1}_{\beta\alpha}\psi_\beta(z)$
- $\langle\boldsymbol{\theta}|\ldots U_k(z)V^{\boldsymbol{\nu}}(t)\ldots|\boldsymbol{\sigma}\rangle = \left(\frac{u_k(\boldsymbol{\nu})}{(z-t)^k} + \text{less singular}\right)\langle\boldsymbol{\theta}|\ldots V^{\boldsymbol{\nu}}(t)\ldots|\boldsymbol{\sigma}\rangle$
  Last formula completes the proof, so V is primary.

Adjoint action

$$V^{\nu}(1)\psi^{\sigma}_{\alpha}(z)V^{\nu}(1)^{-1}=\sum_{\beta}C^{\beta}_{\alpha}\psi^{\sigma}_{\beta}(z)\,, \qquad |z|=1$$

$$V^{\nu}(1)\psi^{\sigma}_{\alpha,n}V^{\nu}(1)^{-1}=\sum_{\beta,m}C^{\beta}_{\alpha}\oint\limits_{|z|=1}\frac{dz}{2\pi i}z^{n-m-1+\sigma_\alpha-\theta_\beta}\psi^{\theta}_{m,\beta}=\sum_{\beta,m}C^{\beta}_{\alpha}\frac{-i(e^{2\pi i(\sigma_\alpha-\theta_\beta)}-1)}{2\pi(n-m+\sigma_\alpha-\theta_\beta)}\psi^{\theta}_{\beta,m}$$

Formula $V_{\nu}(1)\approx {}^{\times}_{\times}\exp(\sum_{\alpha\beta pq}(\frac{\tilde{C}_{\alpha\beta}}{p-q+\sigma_\alpha-\theta_\beta}-\delta_{\alpha\beta}\delta_{pq})\psi^{*}_{\beta,-q}\psi_{\alpha,p})^{\times}_{\times}$ is an immediate consequence of this identity.

Adjoint action

$$V^{\nu}(1)\psi_{\alpha}^{\boldsymbol{\sigma}}(z)V^{\nu}(1)^{-1} = \sum_{\beta} C_{\alpha}^{\beta}\psi_{\beta}^{\boldsymbol{\sigma}}(z)\,, \qquad |z| = 1$$

$$V^{\nu}(1)\psi_{\alpha,n}^{\boldsymbol{\sigma}}V^{\nu}(1)^{-1} = \sum_{\beta,m} C_{\alpha}^{\beta}\oint\limits_{|z|=1}\frac{dz}{2\pi i}z^{n-m-1+\sigma_\alpha-\theta_\beta}\psi_{m,\beta}^{\boldsymbol{\theta}} = \sum_{\beta,m} C_{\alpha}^{\beta}\frac{-i(e^{2\pi i(\sigma_\alpha-\theta_\beta)}-1)}{2\pi(n-m+\sigma_\alpha-\theta_\beta)}\psi_{\beta,m}^{\boldsymbol{\theta}}$$

Formula $V_{\nu}(1) \approx \substack{\times\\\times}\exp(\sum_{\alpha\beta pq}(\frac{\tilde{C}_{\alpha\beta}}{p-q+\sigma_\alpha-\theta_\beta} - \delta_{\alpha\beta}\delta_{pq})\psi_{\beta,-q}^{*}\psi_{\alpha,p})\substack{\times\\\times}$ is an immediate consequence of this identity.

Another representation

As a consequence of its definition, V_{ν} can be rewritten as

$$V_{\nu}(t) = \mathbf{:}\exp(\sum_{\alpha\beta pq}\mathcal{K}_{\alpha q,\beta p}\psi_{\beta,-q}^{*}\psi_{\alpha,p})\mathbf{:}\,, \qquad \text{where } \mathbf{:}\psi_{\alpha,p}^{*}\psi_{\beta,-q}\mathbf{:} = -\psi_{\beta,-q}\psi_{\alpha,p}^{*}$$

Where, for example, $\sum_{\substack{p,q>0\\ \alpha\beta}}\mathcal{K}_{\alpha,-q,\beta,p}z^{q-\sigma_\alpha-\frac{1}{2}}w^{p+\sigma_\beta-\frac{1}{2}} = \frac{[\phi(z)\phi(w)^{-1}]_{\alpha\beta}}{z-w}$

Therefore, to get matrix K one should be able either to compute minors of quasi-Cauchy matrices, or to solve the 3-pt Riemann-Hilbert problem.

Thank you for your attention!