Painlevé functions, conformal blocks and combinatorics

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Tokyo, 29/11/2016

based on:

Gamayun, Iorgov, OL, 1207.0787
Iorgov, OL, Teschner, 1401.6104
Gavrylenko, OL, 1608.00958

PAUL PAINLEVÉ

(1863–1933)

Richard Fuchs
(1873-1944)

BERTRAND GAMBIER

(1879–1954)

Painlevé equations are nonlinear 2nd order ODEs of the form

$$w'' = F(w, w', t)$$

where $F(w, w', t)$ is a rational function of w, w', t.

Their solutions $w\,(t; C_1, C_2)$ satisfy Painlevé property

- $w\,(t; C_1, C_2)$ do not have critical points depending on $C_1,\ C_2$

Example:

- $w' = w \quad \Longrightarrow \quad w = e^{t-C}$ ✓ (essential singularity $t = \infty$)
- $w' = w^2 \quad \Longrightarrow \quad w = \frac{1}{C-t}$ ✓ (movable pole)
- $w' = w^3 \quad \Longrightarrow \quad w \sim \frac{1}{\sqrt{t-C}}$ X (movable branchpoint)

Classification at order 1 [L. Fuchs, 1884]

The only ODE $w' = F(w, t)$ without movable critical points is the generalized Riccati equation

$$w' = p_2(t)\, w^2 + p_1(t)\, w + p_0(t)\,.$$

Lazarus Fuchs
(1833–1902)

Painlevé equations [P. Painlevé & B. Gambier, 1900–1910]:

$$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)(w')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right) w' + \\ + \frac{2w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{w^2} + \frac{\gamma(t-1)}{(w-1)^2} + \frac{\delta t(t-1)}{(w-t)^2}\right) \qquad (\mathrm{P}_{\mathrm{VI}})$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{t} + \frac{(w-1)^2}{t^2}\left(\alpha w + \frac{\beta}{w}\right) + \\ + \frac{\gamma w}{t} + \frac{\delta w(w+1)}{w-1}, \qquad (\mathrm{P}_{\mathrm{V}})$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4tw^2 + 2\left(t^2 - \alpha\right) w + \frac{\beta}{w}, \qquad (\mathrm{P}_{\mathrm{IV}})$$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{t} + \frac{\alpha w^2 + \beta}{t} + \gamma w^3 + \frac{\delta}{w}, \qquad (\mathrm{P}_{\mathrm{III}})$$

$$w'' = 2w^3 + tw + \alpha, \qquad (\mathrm{P}_{\mathrm{II}})$$

$$w'' = 6w^2 + t. \qquad (\mathrm{P}_{\mathrm{I}})$$

Confluence diagram:

$$\begin{tikzcd}
\mathrm{VI} \arrow[r] & \mathrm{V} \arrow[r] \arrow[d] & \mathrm{III}_1 \arrow[r] \arrow[d] & \mathrm{III}_2 \arrow[r] \arrow[d] & \mathrm{III}_3 \\
 & \mathrm{IV} \arrow[r] & \mathrm{II} \arrow[r] & \mathrm{I} &
\end{tikzcd}$$

- non-autonomous hamiltonian systems
- Bäcklund transformations
- connection formulæ

Painlevé VI:

$$\left(t(t-1)\zeta''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

- $\zeta(t) = t(t-1)\frac{d}{dt}\ln\tau$, where $\tau(t)$ is Painlevé VI tau function

(Special) solutions of Painlevé VI:

1. Hypergeometric Riccati family

$$\tau_{\mathrm{Riccati}}(t) = (1-t)^{-\frac{N\left(N+\nu+\nu'\right)}{2}} \det\left[A_{j-k}(t)\right]_{j,k=0}^{N-1},$$

$$A_m(t) = \frac{\Gamma(1+\nu')\, t^{\frac{\eta-m}{2}}(1-t)^{\nu}}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+\nu')}\,{}_2F_1\left[\begin{array}{c} -\nu, 1+\nu' \\ 1+\eta-m \end{array}\middle|\frac{t}{t-1}\right] + \frac{\xi\,\Gamma(1+\nu)\, t^{\frac{m-\eta}{2}}(1-t)^{\nu'}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+\nu)}\,{}_2F_1\left[\begin{array}{c} 1+\nu, -\nu' \\ 1-\eta+m \end{array}\middle|\frac{t}{t-1}\right]$$

- PVI parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \frac{1}{2}(\eta, N, -N-\nu-\nu', \nu-\nu'+\eta)$ depend on $\nu, \nu', \eta \in \mathbb{C}$ and $N \in \mathbb{Z}_{\geq 0}$
- 1-parameter family of initial conditions depending on $\xi \in \mathbb{C}$
- [Forrester, Witte, '02]

2. Elliptic Picard family

$$\tau_{\mathrm{Picard}}(t) = \frac{e^{i\pi\sigma^2\bar{\tau}}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\frac{\vartheta_3(\sigma\pi\bar{\tau}+\sigma'\pi|\bar{\tau})}{\vartheta_3(0|\bar{\tau})}, \qquad \bar{\tau} = \frac{iK'(t)}{K(t)}$$

- PVI parameters $(\theta_0,\theta_t,\theta_1,\theta_\infty) = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$
- 2-parameter family of initial conditions depending on $\sigma,\sigma' \in \mathbb{C}$
- [Kitaev, Korotkin, '98]

3. Algebraic solutions

$$\tau_{H_3'}(t) = \frac{(1-s)^{\frac{1}{20}}\,s^{\frac{1}{20}}\,(1+3s)^{\frac{1}{12}}}{(1+s)^{\frac{3}{20}}\,(1-3s)^{\frac{11}{300}}\,(1+4s-s^2)^{\frac{1}{25}}},$$

$$t = \frac{(s-1)^5(3s+1)^3(s^2+4s-1)}{(s+1)^5(3s-1)^3(s^2-4s-1)}.$$

- $(\theta_0,\theta_t,\theta_1,\theta_\infty) = \left(0,0,0,-\frac{1}{5}\right)$, 10 branches
- no parameters in the initial conditions
- great icosahedron solution from [Dubrovin, Mazzocco, '98]

4. Fredholm determinant solutions

$$\tau_{\mathrm{BD}}(t) = \det\left(\mathbf{1} - \lambda K\big|_{(0,t)}\right),$$

where continuous ${}_2F_1$ kernel $K(x,y) = \dfrac{\psi(x)\varphi(y) - \varphi(x)\psi(y)}{x-y}$ is defined by

$$\begin{aligned}\varphi(x) &= \quad x^{\theta_0}(1-x)^{\theta_1}\,{}_2F_1\left[\begin{array}{c}\theta_0+\theta_1+\theta_\infty, \theta_0+\theta_1-\theta_\infty \\ 2\theta_0\end{array};x\right],\\ \psi(x) &= x^{1+\theta_0}(1-x)^{\theta_1}\,{}_2F_1\left[\begin{array}{c}1+\theta_0+\theta_1+\theta_\infty, 1+\theta_0+\theta_1-\theta_\infty \\ 2+2\theta_0\end{array};x\right].\end{aligned}$$

- PVI parameters $(\theta_0, \theta_t = 0, \theta_1, \theta_\infty)$
- 1-parameter family of initial conditions depending on $\lambda \in \mathbb{C}$
- [Borodin, Deift, '01]

Solutions:

- Riccati: classical special functions

$$\begin{tikzcd}
{}_2F_1 \arrow[r] & {}_1F_1 \arrow[r] \arrow[d] & \text{Bessel} \arrow[r, dash, dashed] \arrow[d] & \varnothing \arrow[r, dash, dashed] \arrow[d, dash, dashed] & \varnothing \\
 & \text{Hermite} \arrow[r] & \text{Airy} \arrow[r, dash, dashed] & \varnothing &
\end{tikzcd}$$

- elliptic (PVI)
- algebraic
- transcendental (almost all solutions!)

Question 1:

Can the general solution of Painlevé VI be expressed in terms of a Fredholm determinant?

Digression: Monodromy preserving deformation

Consider rank N Fuchsian system on $\mathbb{P}^1$:

$$\partial_z \Phi = \Phi A(z),$$
$$A(z) = \sum_{\nu=1}^{n-1} \frac{A_\nu}{z - a_\nu}, \qquad A_\nu \in \mathfrak{sl}_N$$

- n regular singular points $a_1, \ldots, a_{n-1}, \infty$

Monodromy representation:

$$\rho : \pi_1(\mathbb{P}^1 \backslash \{a\}) \to SL_N(\mathbb{C})$$

- different choices of the basis of solutions $\Longrightarrow$ equivalent representations

Riemann-Hilbert correspondence:

$$\mathcal{RH}: \begin{array}{c}\text{parameter set } \mathcal{P} \\ \text{of the linear system}\end{array} \longrightarrow \begin{array}{c}\text{space } \mathcal{M} \\ \text{of monodromy data}\end{array}$$

Schlesinger equations:

$$\partial_{a_\mu} A_\nu = \frac{[A_\mu, A_\nu]}{a_\nu - a_\mu}, \qquad \mu \neq \nu$$

- non-autonomous hamiltonian system
- a_ν's play the role of times

Ludwig Schlesinger
(1864–1933)

- tau function generates hamiltonians of isomonodromic flows:

$$H_\mu := \partial_{a_\nu} \ln \tau(a) = \frac{1}{2} \operatorname{res}_{z=a_\nu} \operatorname{tr} A^2(z)$$

- $\operatorname{sp}(A_\nu)$ conserved due to Lax form

Consider moduli space of representations with fixed local monodromies.

$$\mathcal{M}_{\boldsymbol{\theta}} := \mathrm{Hom}(\pi_1(\mathbb{P}^1 \backslash \{a\}, SL(N, \mathbb{C}))/\sim$$

Example: $N = 2$

- Schlesinger $\Longrightarrow$ Garnier system $\mathcal{G}_{n-3}$
- $\dim \mathcal{M}_{\boldsymbol{\theta}} = 3(n-1) - 3 - n = 2(n-3)$
  (complete set of conserved quantities for $\mathcal{G}_{n-3}$!)
- $n = 4 \Longrightarrow$ Painlevé VI; $a = \{0, t, 1, \infty\}$

Monodromy provides a convenient labeling of Painlevé functions.

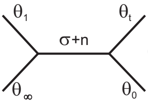
solution of Painlevé equations	=	construction of inverse map $\mathcal{RH}^{-1}$

General solution of PVI:

[Gamayun, Iorgov, OL, 1207.0787]

PVI tau function is a Fourier transform of $c = 1$ Virasoro conformal block:

$$\boxed{\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\, \mathcal{B}(\vec{\theta}, \sigma + n, t) = \sum_{n\in\mathbb{Z}} e^{in\eta}}$$



$$(t)$$

- $\mathcal{B}(\vec{\theta}, \sigma, t) = t^{\alpha} \sum_{k=0}^{\infty} B_k(\vec{\theta}, \sigma)\, t^k$, with B_k rational in $\vec{\theta}, \sigma$ and determined by commutation relations of Vir
- as $c \to \infty$, conformal block $\mathcal{B}(t) \sim {}_2F_1(t)$
- all 4 parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) \Longleftrightarrow$ external momenta
- 2 integration constants $(\sigma, \eta) \Longleftrightarrow$ internal momentum + Fourier conjugate variable
- explicit inverse of the Riemann-Hilbert map

CFT derivations:

[Iorgov, OL, Teschner, 1401.6104]

- understood in the framework of Liouville CFT and generalized to an arbitrary number of punctures (Garnier system)
- uses quantum monodromy of conformal blocks with additional level 2 degenerate insertions

[Bershtein, Shchechkin, 1406.3008]

- bilinear differential-difference equations for conformal blocks coming from an embedding $\mathrm{Vir}\oplus\mathrm{Vir}\subset\mathrm{NSR}\oplus\mathrm{F}$
- extends to arbitrary values of central charge c

AGT correspondence [Alday, Gaiotto, Tachikawa, '09]

$$\mathcal{B}(t) = \mathcal{Z}_{\text{inst}}(t) = \begin{array}{c}\text{combinatorial sum}\\ \text{over tuples of partitions}\end{array} \qquad \text{[Nekrasov, '04]}$$

- proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]
- provides explicit series representation for general Painlevé VI function!

Conjecture [Gamayun, Iorgov, OL, 1207.0787]

Complete expansion of Painlevé VI tau function at $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\mathcal{B}(\vec{\theta}, \sigma + n; t),$$

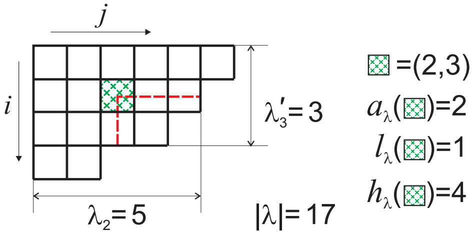
where the function $\mathcal{B}(\vec{\theta}, \sigma; t)$ is explicitly given by

$$\mathcal{B}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\vec{\theta},\sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta},\sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$

$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$



Young diagram associated to partition
$\lambda = \{6, 5, 4, 2\}$.

Question 2:

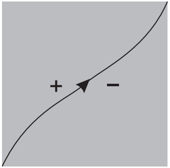
How to understand this combinatorial structure within the theory of monodromy preserving deformations? (without reference to CFT/gauge theory)

Scheme of the proof

Step 1: Represent the tau function of the Schlesinger system in the form of Fredholm determinant

- arbitrary rank N, arbitrary number n of regular singularities

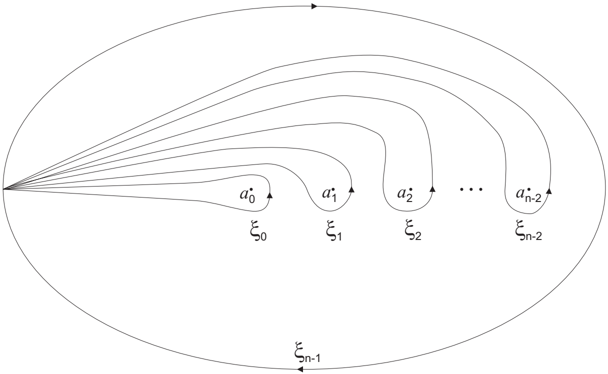
Riemann-Hilbert setup



- contour Γ on a Riemann surface Σ
- jump matrix $J : \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$

RHP defined by (Γ, J) is to find analytic invertible matrix function $\Psi : \Sigma \backslash \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$ whose boundary values satisfy

$$\Psi_+ = J \Psi_-$$

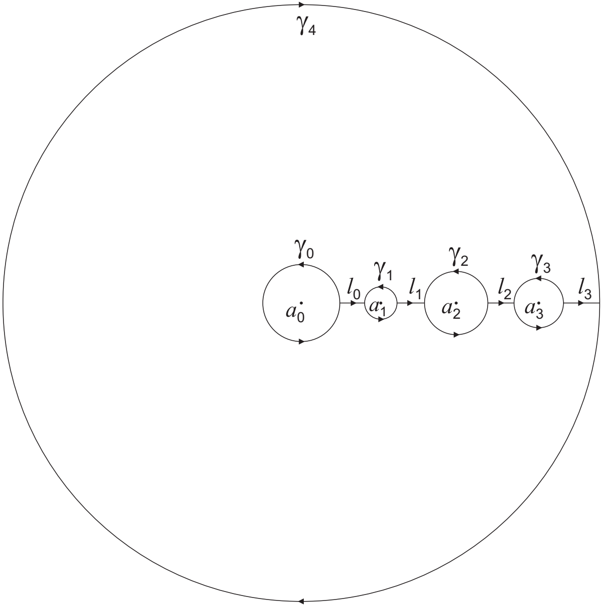


Monodromy representation $\rho : \pi_1\left(\mathbb{P}^1 \backslash a\right) \to \mathrm{GL}\,(N, \mathbb{C})$ generated by

$$M_k = \rho\left(\xi_k\right) = {M_{1\to k-1}}^{-1} M_{1\to k}$$

Assume that all $M_{1\to k} = M_1 \ldots M_k$ are diagonalizable,

$$M_{1\to k} = S_k e^{2\pi i \mathfrak{S}_k} S_k^{-1}, \qquad \mathfrak{S}_k = \mathrm{diag}\left\{\sigma_{k,1}, \ldots, \sigma_{k,N}\right\}.$$



Contour Γ for $n = 5$

Fundamental matrix solution

$$\Phi(z) = \begin{cases} \Psi(z), & z \text{ outside } \gamma_{1\ldots n}, \\ C_k (a_k - z)^{\Theta_k} \Psi(z), & z \text{ inside } \gamma_k, \quad k = 1, \ldots, n-1, \\ C_n (-z)^{-\Theta_n} \Psi(z), & z \text{ inside } \gamma_n. \end{cases}$$

- only piecewise constant jumps on $\mathbb{R}_{>0}$
- matrix $\Phi^{-1}\partial_z\Phi$ meromorphic on $\mathbb{P}^1$ with poles only possible at $a_1, \ldots, a_n$
- local analysis shows that

$$\partial_z\Phi = \Phi A(z), \qquad A(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}$$

with $A_k = \Psi(a_k)^{-1} \Theta_k \Psi(a_k)$

Jump data

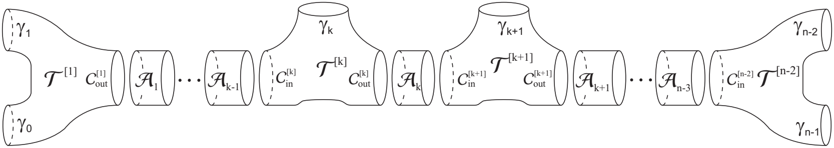
- local exponents: n diagonal non-resonant $N \times N$ matrices $\Theta_k = \operatorname{diag}\{\theta_{k,1}, \ldots, \theta_{k,N}\}$ $(k = 1, \ldots, n)$ satisfying a consistency relation $\sum_{k=1}^{n} \operatorname{Tr} \Theta_k = 0$
- $2n$ connection matrices $C_{k,\pm} \in \mathrm{GL}\left(N, \mathbb{C}\right)$ satisfying the constraints

$$\begin{aligned} M_{1\to k} &:= C_{k,-} e^{2\pi i \Theta_k} C_{k,+}^{-1} = C_{k+1,-} C_{k+1,+}^{-1}, \qquad k = 1, \ldots n-2, \\ M_{1\to n-1} &:= C_{n-1,-} e^{2\pi i \Theta_{n-1}} C_{n-1,+}^{-1} = C_{n,-} e^{-2\pi i \Theta_n} C_{n,+}^{-1}, \\ M_{1\to n} &:= \mathbf{1} = C_{n,-} C_{n,+}^{-1} = C_{1,-} C_{1,+}^{-1}, \end{aligned}$$

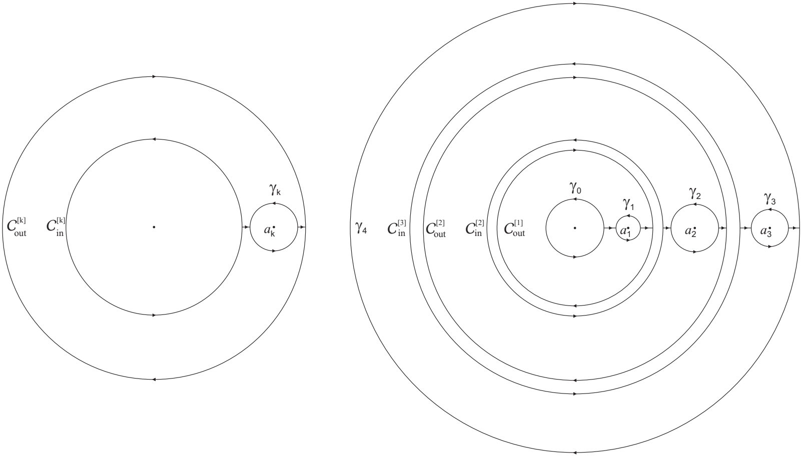
Jump matrix J

$$\begin{aligned} J\left(z\right)\Big|_{\ell_k} &= M_{1\to k}^{-1}, \qquad k = 1, \ldots, n-1, \\ J\left(z\right)\Big|_{\gamma_k} &= \left(a_k - z\right)^{-\Theta_k} C_{k,\pm}^{-1}, \qquad \Im z \gtrless 0, \quad k = 1, \ldots, n-1, \\ J\left(z\right)\Big|_{\gamma_n} &= \left(-z\right)^{\Theta_n} C_{n,\pm}^{-1}, \qquad \Im z \gtrless 0. \end{aligned}$$

Auxiliary 3-point RHPs



- we are going to associate to the n-point RHP $n-2$ 3-point RHPs assigned to different trinions



Contour $\Gamma^{[k]}$ (left) and $\hat{\Gamma}$ for $n = 5$ (right)

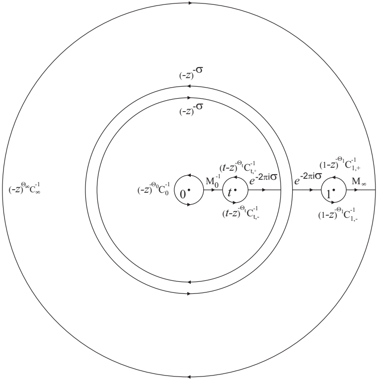
- $\hat{\Psi}(z) = \begin{cases} \Psi(z) & \text{outside the annuli,} \\ (-z)^{-\mathfrak{S}_k} S_k^{-1} \Psi(z) & \text{inside.} \end{cases}$
- jumps on the boundary circles $\mathcal{C}_{k-1}^{\text{out}}$, $\mathcal{C}_k^{\text{in}}$ mimic regular singularities characterized by counterclockwise monodromies $M_{1\to k}$

Example ($n = 4$)

$$\tau(t) = t^{\frac{1}{2}\operatorname{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right)} \det\left(\mathbf{1} + K\right),$$

with

$$K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \operatorname{End}(\mathcal{V})$$



where the operators $\mathsf{a} : \mathcal{V}_- \to \mathcal{V}_+$ and $\mathsf{d} : \mathcal{V}_+ \to \mathcal{V}_-$ are

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}(z, z')\, g(z')\, dz', \quad \mathsf{a}(z, z') = \frac{\Psi^{ext}(z)\, \Psi^{ext}(z')^{-1} - \mathbf{1}}{z - z'},$$

$$(\mathsf{d}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{d}(z, z')\, g(z')\, dz', \quad \mathsf{d}(z, z') = \frac{\mathbf{1} - \Psi^{int}(z)\, \Psi^{int}(z')^{-1}}{z - z'}.$$

For $N=2$:

$$\mathrm{a}\left(z,z'\right)=\frac{\left(1-z'\right)^{2\theta_1}\begin{pmatrix} K_{++}\left(z\right) & K_{+-}\left(z\right) \\ K_{-+}\left(z\right) & K_{--}\left(z\right)\end{pmatrix}\begin{pmatrix} K_{--}\left(z'\right) & -K_{+-}\left(z'\right) \\ -K_{-+}\left(z'\right) & K_{++}\left(z'\right)\end{pmatrix}-\mathbf{1}}{z-z'},$$

$$\mathrm{d}\left(z,z'\right)=\frac{\mathbf{1}-\left(1-\frac{t}{z'}\right)^{2\theta_t}\begin{pmatrix} \bar{K}_{++}\left(z\right) & \bar{K}_{+-}\left(z\right) \\ \bar{K}_{-+}\left(z\right) & \bar{K}_{--}\left(z\right)\end{pmatrix}\begin{pmatrix} \bar{K}_{--}\left(z'\right) & -\bar{K}_{+-}\left(z'\right) \\ -\bar{K}_{-+}\left(z'\right) & \bar{K}_{++}\left(z'\right)\end{pmatrix}}{z-z'},$$

with

$$K_{\pm\pm}\left(z\right)={}_2F_1\left[\begin{matrix}\theta_1+\theta_\infty\pm\sigma,\theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma\end{matrix}\;;z\right],$$

$$K_{\pm\mp}\left(z\right)=\pm\frac{\theta_\infty^2-\left(\theta_1\pm\sigma\right)^2}{2\sigma\left(1\pm 2\sigma\right)}\,z\,{}_2F_1\left[\begin{matrix}1+\theta_1+\theta_\infty\pm\sigma,1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma\end{matrix}\;;z\right],$$

$$\bar{K}_{\pm\pm}\left(z\right)={}_2F_1\left[\begin{matrix}\theta_t+\theta_0\mp\sigma,\theta_t-\theta_0\mp\sigma \\ \mp 2\sigma\end{matrix}\;;\frac{t}{z}\right],$$

$$\bar{K}_{\pm\mp}\left(z\right)=\mp\, t^{\mp 2\sigma}e^{\mp i\eta}\frac{\theta_0^2-\left(\theta_t\mp\sigma\right)^2}{2\sigma\left(1\mp 2\sigma\right)}\,\frac{t}{z}\,{}_2F_1\left[\begin{matrix}1+\theta_t+\theta_0\mp\sigma,1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma\end{matrix}\;;\frac{t}{z}\right].$$

Idea of the proof

- For a circle $\mathcal{C}\subset\mathcal{A}$ define

$$\tilde{\Psi}(z)=\begin{cases}\Psi^{ext}(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C},\\ \Psi^{int}(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}.\end{cases}$$

- contour $\tilde{\Gamma}=\mathcal{C}$ (single circle), jump $J:\mathcal{C}\to\mathrm{GL}(N,\mathbb{C})$ is

$$J(z)=\Psi^{int}(z)^{-1}\Psi^{ext}(z)=\tilde{\Psi}_{+}(z)\tilde{\Psi}_{-}(z)^{-1}$$

Given the symbol $J(z)=\sum_{k\in\mathbb{Z}}J_k z^k$, $z\in\mathcal{A}$, define

$$T_K[J]=(J_{k-k'}),\ H_K[J]=(J_{k+k'+1}),\ \bar{H}_K[J]=(J_{-k-k'-1}),\quad k,k'=0,\dots,K-1.$$

Theorem [Widom '76]. Let $C[J]=\dfrac{1}{2\pi i}\oint_C \ln\det J(z)\,d\ln z$, then the limit

$$D[J]=\lim_{K\to\infty}C[J]^{-K}\det T_K[J]$$

exists and is equal to

$$D[J]=\det T_\infty[J]T_\infty[J^{-1}]=\det\left(\mathbf{1}-H_\infty[J]\bar{H}_\infty[J^{-1}]\right).$$

Corollary. For symbols admitting 1st factorization, the Widom's constant $D[J]$ may be rewritten as

$$D[J] = \det\left(\mathbf{1} + K\right), \qquad K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \mathrm{End}\left(\mathcal{V}_+ \oplus \mathcal{V}_-\right),$$

where the operators $\mathsf{a} : \mathcal{V}_- \to \mathcal{V}_+$, $\mathsf{d} : \mathcal{V}_+ \to \mathcal{V}_-$ are defined by

$$\mathsf{a} = \Psi^{ext}\Pi_+ {\Psi^{ext}}^{-1}\Big|_{\mathcal{V}_-}, \qquad d = \Psi^{int}\Pi_- {\Psi^{int}}^{-1}\Big|_{\mathcal{V}_+}.$$

Theorem [Widom '74]. For symbols admitting left and right factorizations, the log-derivatives of the Dyson's constant wrt parameters are given by

$$\partial_t \ln D[J] = \frac{1}{2\pi i}\oint_C \mathrm{Tr}\left(J^{-1}\partial_t J\left[\partial_z\left(\tilde{\Psi}_-\right)\tilde{\Psi}_-^{-1} + {\Psi^{ext}}^{-1}\partial_z\left(\Psi^{ext}\right)\right]\right) dz.$$

- Dyson's constant = tau function !!!

Step 2: Write U in the Fourier basis and expand Fredholm determinant using von Koch formula:

$$\det(\mathbf{1}+K) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det K_{\mathfrak{Y}}, \qquad U \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$$

- multi-indices of principal minors

$$\det K_{\mathfrak{Y}} = \det \begin{pmatrix} 0 & \mathsf{a}_J^I \\ \mathsf{d}_I^J & 0 \end{pmatrix}$$

  incorporate color indices $\alpha = 1, \ldots N$ and (half-)integer Fourier indices

- combinatorial expansion

$$\det(\mathbf{1}+K) = \sum_{(I,J)\in \mathsf{Conf}_+} \det \mathsf{a}_J^I \det \mathsf{d}_I^J,$$

  with balance condition $|I| = |J|$

- elements of Conf_+ are in bijection with N-tuples of Young diagrams of zero total charge
- in the case $N = 2$

$$\det(\mathbf{1}+K) = \sum_{(I,J)\in \mathbb{Y}^{\mathbf{2}}\times\mathbb{Z}} \det \mathsf{a}_J^I \det \mathsf{d}_I^J$$

Step 3: Explicit computation of elementary determinants $\det \mathrm{a}^I_J$, $\det \mathrm{d}^J_I$ of Plemelj operators

- in the case $N = 2 \Longrightarrow$ Cauchy determinants $\det \dfrac{1}{x_i - y_j}$
- rewrite resulting factorized expressions using lengths of rows/columns instead of positions of particles/holes of different colors

Cauchy-Plemelj operators

- associate to every trinion $\mathcal{T}_k$ with $k = 2, \ldots, n-3$ the spaces of vector-valued functions

$$\mathcal{H}^{[k]} = \bigoplus_{\epsilon=\mathrm{in},\mathrm{out}} \left(\mathcal{H}^{[k]}_{\epsilon,+} \oplus \mathcal{H}^{[k]}_{\epsilon,-} \right), \qquad \mathcal{H}^{[k]}_{\epsilon,\pm} = \mathbb{C}^N \otimes \mathcal{V}_{\pm}\left(\mathcal{C}^{\epsilon}_k\right).$$

- elements $f^{[k]} \in \mathcal{H}^{[k]}$ will be written as

$$f^{[k]} = \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix}.$$

- define an operator $\mathcal{P}^{[k]} : \mathcal{H}^{[k]} \to \mathcal{H}^{[k]}$ by

$$\mathcal{P}^{[k]} f^{[k]}(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k \cup \mathcal{C}^{\mathrm{out}}_k} \frac{\Psi^{[k]}_{+}(z)\, \Psi^{[k]}_{+}(z')^{-1} f^{[k]}(z')\, dz'}{z - z'}$$

Lemma. We have $\left(\mathcal{P}^{[k]}\right)^2 = \mathcal{P}^{[k]}$ and $\ker \mathcal{P}^{[k]} = \mathcal{H}^{[k]}_{\mathrm{in},+} \oplus \mathcal{H}^{[k]}_{\mathrm{out},-}$. Moreover, $\mathcal{P}^{[k]}$ can be explicitly written as

$$\mathcal{P}^{[k]} : \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix} \mapsto \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a}^{[k]} & \mathsf{b}^{[k]} \\ \mathsf{c}^{[k]} & \mathsf{d}^{[k]} \end{pmatrix} \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix},$$

where the operators $\mathsf{a}^{[k]}$, $\mathsf{b}^{[k]}$, $\mathsf{c}^{[k]}$, $\mathsf{d}^{[k]}$ are defined by

$$\left(\mathsf{a}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_k^{\mathrm{in}}} \left[\Psi_+^{[k]}(z)\, \Psi_+^{[k]}(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}_k^{\mathrm{in}},$$

$$\left(\mathsf{b}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_k^{\mathrm{out}}} \Psi_+^{[k]}(z)\, \Psi_+^{[k]}(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}_k^{\mathrm{in}},$$

$$\left(\mathsf{c}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_k^{\mathrm{in}}} \Psi_+^{[k]}(z)\, \Psi_+^{[k]}(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}_k^{\mathrm{out}},$$

$$\left(\mathsf{d}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_k^{\mathrm{out}}} \left[\Psi_+^{[k]}(z)\, \Psi_+^{[k]}(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}_k^{\mathrm{out}}.$$

- introduce the total space

$$\mathcal{H} := \bigoplus_{k=1}^{n-2} \mathcal{H}^{[k]}.$$

- there is a splitting

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

$$\mathcal{H}_\pm := \mathcal{H}^{[1]}_{\mathrm{out},\pm} \oplus \left(\mathcal{H}^{[2]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[2]}_{\mathrm{out},\pm}\right) \oplus \ldots \oplus \left(\mathcal{H}^{[n-3]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[n-3]}_{\mathrm{out},\pm}\right) \oplus \mathcal{H}^{[n-2]}_{\mathrm{in},\mp}.$$

- combine the 3-point projections $\mathcal{P}^{[k]}$ into an operator $\mathcal{P}_\oplus : \mathcal{H} \to \mathcal{H}$ given by the direct sum

$$\mathcal{P}_\oplus = \mathcal{P}^{[1]} \oplus \ldots \oplus \mathcal{P}^{[n-2]}.$$

- similarly, define another projection $\mathcal{P}_\Sigma : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{P}_\Sigma f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\Sigma} \frac{\hat{\Psi}_+(z)\, \hat{\Psi}_+(z')^{-1} f(z')\, dz'}{z - z'}, \quad \mathcal{C}_\Sigma := \bigcup_{k=1}^{n-3} \mathcal{C}^{\mathrm{out}}_k \cup \mathcal{C}^{\mathrm{in}}_{k+1}.$$

- it is easy to show that $\mathcal{P}_{\Sigma}\mathcal{P}_{\oplus} = \mathcal{P}_{\oplus}$ and $\mathcal{P}_{\oplus}\mathcal{P}_{\Sigma}{=}\mathcal{P}_{\Sigma}$
- the space
$$\mathcal{H}_{\mathcal{T}} := \operatorname{im}\mathcal{P}_{\oplus} = \operatorname{im}\mathcal{P}_{\Sigma}.$$
can be thought of as the subspace of functions on the union of boundary circles $\mathcal{C}_k^{\mathrm{in}}$, $\mathcal{C}_k^{\mathrm{out}}$ that can be continued inside $\bigcup_{k=1}^{n-2}\mathcal{T}_k$ with monodromy and singular behavior of the n-point fundamental matrix solution $\Phi\left(z\right)$
- varying the positions of singular points, one obtains a trajectory of $\mathcal{H}_{\mathcal{T}}$ in the infinite-dimensional Grassmannian $\mathrm{Gr}\left(\mathcal{H}\right)$ defined with respect to the splitting $\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$
- each of the subspaces $\mathcal{H}_{\pm}$ may be identified with $N\left(n-3\right)$ copies of the space $L^2\left(S^1\right)$ of functions on a circle; the factor $n-3$ corresponds to the number of annuli and N is the rank of the appropriate RHP

- introduce operators $\mathcal{P}_{\oplus,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ and $\mathcal{P}_{\Sigma,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ given by restrictions of $\mathcal{P}_\oplus$ and $\mathcal{P}_\Sigma$ to $\mathcal{H}_+$
- define $L \in \mathrm{End}\,(\mathcal{H}_+)$ defined by

$$L := \mathcal{P}_{\oplus,+}{}^{-1} \mathcal{P}_{\Sigma,+}$$

- there exists a basis in which $L^{-1} = \mathbf{1} - K$, with

$$K = \begin{pmatrix} U_1 & V_1 & 0 & . & 0 \\ W_1 & U_2 & V_2 & . & 0 \\ 0 & W_2 & U_3 & . & . \\ . & . & . & . & V_{n-4} \\ 0 & 0 & . & W_{n-4} & U_{n-3} \end{pmatrix}, \ \vec{g} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{n-3} \end{pmatrix}, \ \tilde{g}_k = \begin{pmatrix} g^{[k]}_{\mathrm{out},+} \\ g^{[k+1]}_{\mathrm{in},-} \end{pmatrix},$$

$$U_k = \begin{pmatrix} 0 & \mathrm{a}^{[k+1]} \\ \mathrm{d}^{[k]} & 0 \end{pmatrix}, \ V_k = \begin{pmatrix} \mathrm{b}^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \ W_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{c}^{[k+1]} \end{pmatrix}$$

Definition

The tau function associated to the Riemann-Hilbert problem for Ψ is defined as

$$\tau(a) := \det\left(L^{-1}\right)$$

Theorem

We have

$$\tau(a) = \Upsilon(a)^{-1}\tau_{\mathrm{JMU}}(a),$$

where $\tau_{\mathrm{JMU}}(a)$ is defined up to a prefactor independent of a by

$$d_a \ln \tau_{\mathrm{JMU}} = \sum_{1\leq k<l\leq n-1} \operatorname{Tr} A_k A_l \, d\ln(a_k - a_l),$$

and $\Upsilon(a) = \prod_{k=2}^{n-2} a_k^{\bar{\Delta}_k - \bar{\Delta}_{k-1} - \Delta_k}$, with $\Delta_k = \frac{1}{2}\operatorname{Tr}\Theta_k^2$, $\bar{\Delta}_k = \frac{1}{2}\operatorname{Tr}\mathfrak{S}_k^2$

Fourier basis

Let us represent the elements of $\mathcal{H}_{\mathcal{C}}$ by their Laurent series inside $\mathcal{A}$,

$$f\left(z\right)=\sum_{p\in\mathbb{Z}'}f^{p}z^{-\frac{1}{2}+p},\qquad f^{p}\in\mathbb{C}^{N},$$

and write integral kernels of 3-point projection operators $\mathrm{a}^{[k]}$, $\mathrm{b}^{[k]}$, $\mathrm{c}^{[k]}$, $\mathrm{d}^{[k]}$ as

$$\mathrm{a}^{[k]}\left(z,z'\right):=\frac{\Psi_{+}^{[k]}\left(z\right)\Psi_{+}^{[k]}\left(z'\right)^{-1}-\mathbf{1}}{z-z'}=\sum_{p,q\in\mathbb{Z}'_{+}}\mathrm{a}^{[k]}{}_{-q}^{\;p}\,z^{-\frac{1}{2}+p}z'^{-\frac{1}{2}+q},\qquad z,z'\in\mathcal{C}_{k}^{\mathrm{in}},$$

$$\mathrm{b}^{[k]}\left(z,z'\right):=\quad-\frac{\Psi_{+}^{[k]}\left(z\right)\Psi_{+}^{[k]}\left(z'\right)^{-1}}{z-z'}=\sum_{p,q\in\mathbb{Z}'_{+}}\mathrm{b}^{[k]}{}_{q}^{p}\,z^{-\frac{1}{2}+p}z'^{-\frac{1}{2}-q},\quad z\in\mathcal{C}_{k}^{\mathrm{in}},z'\in\mathcal{C}_{k}^{\mathrm{out}}$$

$$\mathrm{c}^{[k]}\left(z,z'\right):=\quad\frac{\Psi_{+}^{[k]}\left(z\right)\Psi_{+}^{[k]}\left(z'\right)^{-1}}{z-z'}=\sum_{p,q\in\mathbb{Z}'_{+}}\mathrm{c}^{[k]}{}_{-q}^{-p}\,z^{-\frac{1}{2}-p}z'^{-\frac{1}{2}+q},\ z\in\mathcal{C}_{k}^{\mathrm{out}},z'\in\mathcal{C}_{k}^{\mathrm{in}}$$

$$\mathrm{d}^{[k]}\left(z,z'\right):=\frac{\mathbf{1}-\Psi_{+}^{[k]}\left(z\right)\Psi_{+}^{[k]}\left(z'\right)^{-1}}{z-z'}=\sum_{p,q\in\mathbb{Z}'_{+}}\mathrm{d}^{[k]}{}^{-p}_{\;\;q}z^{-\frac{1}{2}-p}z'^{-\frac{1}{2}-q},\quad z,z'\in\mathcal{C}_{k}^{\mathrm{out}}.$$

Von Koch's formula

Let $A \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$ be a matrix indexed by a discrete and possibly infinite set $\mathfrak{X}$. The basic tool for expanding $\tau(a)$ is the formula

$$\det(\mathbf{1}+A) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det A_{\mathfrak{Y}},$$

where $\det A_{\mathfrak{Y}}$ denotes the $|\mathfrak{Y}|\times|\mathfrak{Y}|$ principal minor obtained by restriction of A to a subset $\mathfrak{Y}\subseteq\mathfrak{X}$.

In our case : A is K in the Fourier basis. Elements of $\mathfrak{X}$ are multi-indices which encode the following data:

- positions of the blocks $\mathrm{a}^{[k]}$, $\mathrm{b}^{[k]}$, $\mathrm{c}^{[k]}$, $\mathrm{d}^{[k]}$ in K
- a half-integer Fourier index of the appropriate block;
- a color index in $\{1,\ldots,N\}$.

Combine Fourier and color indices into one multi-index

$$\imath = (p,\alpha) \in \mathfrak{N} := \mathbb{Z}'\times\{1,\ldots,N\}$$

Unordered sets $\{\imath_1,\ldots,\imath_m\}\in 2^{\mathfrak{N}}$ of such multi-indices are denoted by I or J. Given $M\in\mathbb{C}^{\mathfrak{N}\times\mathfrak{N}}$, we denote by M_I^J its restriction to rows I and columns J.

Principal minor

$$
\begin{pmatrix}
0 & \left(a^{[2]}\right)^{I_1}_{J_1} & \left(b^{[2]}\right)^{I_1}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
\left(d^{[1]}\right)^{J_1}_{I_1} & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & \left(a^{[3]}\right)^{I_2}_{J_2} & \left(b^{[3]}\right)^{I_2}_{I_3} & \cdot & \cdot & 0 & 0 \\
0 & \left(c^{[2]}\right)^{J_2}_{J_1} & \left(d^{[2]}\right)^{J_2}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \left(c^{[3]}\right)^{J_3}_{J_2} & \left(d^{[3]}\right)^{J_3}_{I_3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \left(b^{[n-3]}\right)^{I_{n-2}}_{I_{n-3}} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot & 0 & 0 & \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & \left(a^{[n-2]}\right)^{I_{n-3}}_{J_{n-3}} \\
0 & 0 & 0 & 0 & \cdot & \cdot & \left(c^{[n-3]}\right)^{J_{n-3}}_{J_{n-4}} & \left(d^{[n-3]}\right)^{J_{n-3}}_{I_{n-3}} & 0
\end{pmatrix}
$$

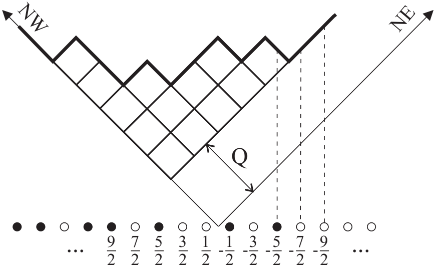
- vanishes unless balance condition $|I_k| = |J_k|$ is satisfied
- factorization into a product of elementary determinants

$$
Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right) := (-1)^{|I_k|} \det \begin{pmatrix} \left(a^{[k]}\right)^{I_{k-1}}_{J_{k-1}} & \left(b^{[k]}\right)^{I_{k-1}}_{I_k} \\ \left(c^{[k]}\right)^{J_k}_{J_{k-1}} & \left(d^{[k]}\right)^{J_k}_{I_k} \end{pmatrix}
$$

Corollary: Fredholm determinant $\tau(a)$ is given by

$$\tau(a) = \sum_{(\vec{I},\vec{J})\in \mathsf{Conf}_+} \prod_{k=1}^{n-2} Z_{I_k,J_k}^{I_{k-1},J_{k-1}}\left(\mathcal{T}^{[k]}\right)$$

- The set Conf_+ of proper balanced configurations $\left(\vec{I},\vec{J}\right)$ may be described in terms of Maya diagrams and charged partitions
- A Maya diagram is a map $\mathrm{m} : \mathbb{Z}' \to \{-1,1\}$ subject to the condition $\mathrm{m}(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- $\mathrm{charge(m)} = \sharp(\mathrm{particles}) - \sharp(\mathrm{holes})$
- balanced configurations (I_k, J_k) are in one-to-one correspondence with N-tuples of Maya diagrams of zero total charge



- here the charge $Q(\mathrm{m}) = 2$ and the positions of particles and holes are given by $\mathrm{p}(\mathrm{m}) = \left(\frac{13}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}\right)$ and $\mathrm{h}(\mathrm{m}) = \left(-\frac{5}{2}, -\frac{1}{2}\right)$
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q^{(\alpha)} = 0 \right\}.$$

Alternative combinatorial notation :

$$Z^{\vec{Y}_{k-\mathbf{1}}, \vec{Q}_{k-\mathbf{1}}}_{\vec{Y}_k, \vec{Q}_k}\left(\mathcal{T}^{[k]}\right) := Z^{I_{k-\mathbf{1}}, J_{k-\mathbf{1}}}_{I_k, J_k}\left(\mathcal{T}^{[k]}\right),$$

Theorem

Fredholm determinant $\tau(a)$ can be written as a combinatorial series

$$\tau(a) = \sum_{\vec{Q}_1,\ldots \vec{Q}_{n-3}\in\mathfrak{Q}_N}\ \sum_{\vec{Y}_1,\ldots \vec{Y}_{n-3}\in\mathbb{Y}^N}\ \prod_{k=1}^{n-2} Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}\left(\mathcal{T}^{[k]}\right)$$

- elementary determinants $Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}$ are constructed from matrix elements of 3-point Plemelj operators in Fourier basis
- in rank $N=2$, they are given by Cauchy matrices conjugated by diagonal factors $\Rightarrow$ explicitly computable !!!
- the result coincides with dual Nekrasov partition function for $U(2)$ linear quiver gauge theory with $\epsilon_1+\epsilon_2=0$
- series representation for general solution of PVI/Garnier system
- rank $N \Rightarrow$ a sum of $N-1$ Cauchy matrices
  (unless additional spectral conditions are imposed)

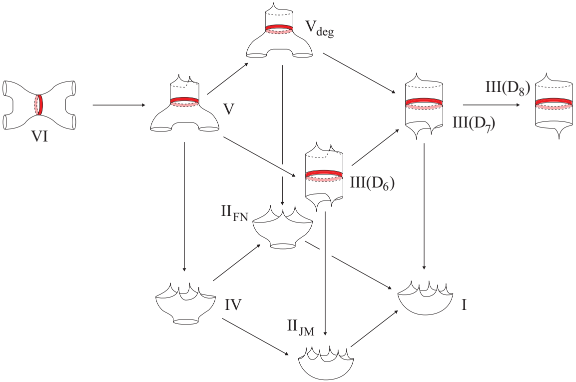
Conclusions

1. Isomonodromic tau functions of Fuchsian systems can be written as block Fredholm determinants whose kernels are built of fundamental solutions of 3-point Fuchsian systems
2. Expanding these determinants in Fourier basis leads to combinatorial series over tuples of partitions
3. The coefficients of the series can be computed explicitly when 3-point solutions have hypergeometric representations (in particular for $N = 2$)

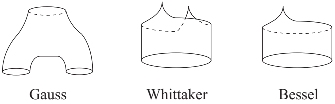
Generalizations

1. Irregular case [Nagoya, '15]
2. q-Painlevé equations [Bershtein, Shchechkin, '16]
3. ...

Other Painlevé equations



Chekhov-Mazzocco-Rubtsov confluence diagram



Some solvable RHPs in rank $N = 2$