ODE/IM correspondence and Bethe ansatz for affine Toda field equations

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Introduction

- The ODE/IM correspondence is a relation between the spectral analysis aproach to ordinary differential equations (ODE), and the "functional relations" approach to 2d quantum integrable model (IM). [Dorey-Tateo 1998]
- This is an example of non-trivial correspondence between classical and quantum integrable models

massless and massive ODE correspondence

- Dorey-Tateo (1998) studied the spectral determinant of the second order differential equation

$$\left(-\frac{d^2}{dx^2}+x^{2M}-E\right)\psi(x,E)=0$$

and its relation to the A_{2M-1}-type Thermodynamic Bethe-ansatz (TBA) equations. (massless TBA)

- Lukyanov-Zamolodchikov (2010) studied the relation for the linear problem associated with the sinh-Gordon equation

$$\varphi_{tt}-\varphi_{xx}+\sinh\varphi=0$$

and the quantum XXZ model. In the conformal limit, this relation reduces to the above ODE/IM correspondence .

- This suggests existence of Lie algebraic structure behind this mysterious correspondence.

- (pseudo)ODE for classical Lie algebras $X = ABCD$ [Dorey-Dunning-Masoero-Suzuki-Tateo, 2006]
- ODE/IM correspondence based on the system of the first order differential equations for classical affine Lie algebra $(X^{(1)})^\vee$ [Sun, 2010]

Motivation

- Understand mathematical structure and physical meaning of the ODE/IM correspondence
  Making (complete) dictionaries
  use this relation to understand the relation between integrable models
- Application to the AdS/CFT correspondence
  Gluon scattering amplitudes, form factors, correlation functions at strong coupling [Alday-Maldacena-Sever-Vieira, Hatsuda-KI-Sakai-Satoh]
- Application to $\mathcal{N} = 2$ SUSY gauge theories [Nekrasov-Sahashvili]
  Gauge-Bethe correspondence
  quantum spectral curve

We will

- Introduce the modified affine Toda field equation for an affine Lie algebra $\hat{\mathfrak{g}}$ and its linear problem (including exceptional Lie algebras)
- Study the ODE/IM correspondence and its conformal limit and derive the corresponding Bethe ansatz equations

ODE/IM correspondence

[Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov]

- ODE
$$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^{2M}-E\right]y(x,E,\ell)=0$$
$x\in\mathbf{C}$, E complex, ℓ: real, $M>0$
- large, real positive x : we have two (divergent and convergent) solutions
$$y(x,E,\ell)\sim\frac{x^{-\frac{M}{2}}}{\sqrt{2i}}\exp\left(-\frac{x^{M+1}}{M+1}\right)\quad(x\to\infty)$$
subdominant (small) solution in the sector $|\arg x|<\frac{\pi}{2M+2}$
- small x asymptotics:
$$y(x,E,\ell)\sim x^{\ell+1},x^{-\ell}$$

- The ODE is invariant under the rotation $x \to ax$, $E \to a^{2M}E$:

$$\left[\frac{1}{a^2}\left(-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}\right)+a^{2M}(x^{2M}-E)\right]y(ax,a^{2M}E,\ell)=0$$

  for $a=\omega=\exp(\frac{2\pi i}{2M+2})$.
- Symanzik rotation

$$y_k(x,E,\ell)=\omega^{\frac{k}{2}}y(\omega^{-k}x,\omega^{2k}E,\ell)$$

  is also the solution of the ODE. ($k \in \mathbf{Z}$)
- y_k is subdominant in the sector $\mathcal{S}_k=\left\{x||\arg x-\frac{2k\pi}{2M+2}|<\frac{\pi}{2M+2}\right\}$

- $\{y_k, y_{k+1}\}$ forms a basis of the solutions.
- The Wronskian $W[f,g] := fg' - f'g$
  - If f, g are the solutions, then $W[f,g]$ is a constant, independent of x.
  - $W[f,g] = -W[g,f]$
  - $W_{k_1,k_2}(E,\ell) := W[y_{k_1}, y_{k_2}]$
- $W_{0,1} = 1$ (evaluated by asymptotic behaviour of y_0 and y_1)
- Periodicity (Symanzik rotation)

$$W_{k_1+1,k_2+1}(E,\ell) = W_{k_1,k_2}(\omega^2 E,\ell)$$

- $$W_{k,k+1}(E,\ell) = 1$$

- $\{y_0, y_1\}$ are chosen as a fixed basis

$$y_k = -\frac{W_{1,k}}{W_{0,1}} y_0 + \frac{W_{0,k}}{W_{0,1}} y_1 = -T_{k-2}^{[k+1]} y_0 + T_{k-1}^{[k]} y_1$$

- T-function

$$T_s(E,\ell) := \left(\frac{W_{0,s+1}(E,\ell)}{W_{0,1}}\right)^{[-(s+1)]}, \quad s \in \mathbf{Z}$$

$$f(E,\ell)^{[m]} := f(\omega^m E, \ell)$$

- The Plücker relation

$$W[y_{k_1}, y_{k_2}]W[y_{k_3}, y_{k_4}] = W[y_{k_1}, y_{k_4}]W[y_{k_3}, y_{k_2}] + W[y_{k_3}, y_{k_1}]W[y_{k_4}, y_{k_2}]$$

for $(k_1, k_2, k_3, k_4) = (1, s+2, 0, s+1)$ leads to the functional relation called the T-system

$$T_s^{[+1]} T_s^{[-1]} = T_{s-1} T_{s+1} + 1 \quad s \in \mathbf{Z}$$

- The Y-functions: $Y_s = T_{s-1} T_{s+1}$ define the Y-system:

$$Y_s^{[+1]} Y_s^{[-1]} = (1 + Y_{s+1})(1 + Y_{s-1})$$

The boundary conditions for the T-system

- generic M and ℓ ($T_s \propto W_{0,s+1}$)

$$T_{-1} = 0, \quad T_0 = 1$$

  T_s ($s \geq 0$) are non-zero.

- $\ell = 0$, $2M + 2 = n \geq 4$: integer
  no singularity (monodromy) at $x = 0$

$$y_n(x) \propto y_0(e^{-2\pi i} x) = y_0(x)$$

  $T_{n-1} = 0 \Longrightarrow A_{n-2}$-type T-system

- $\ell \neq 0$, $2M + 2 = n \geq 4$: integer
  There is a monodromy around $z = 0$. $\Longrightarrow D_{n-2}$-type T-system

Baxter's T-Q relation

The basis of the ODE around $x = 0$

$$\psi_+(x, E, \ell) := x^{\ell+1} + \cdots, \quad \psi_-(x, E, \ell) := x^{-\ell} + \cdots$$

- Monodromy around the origin: $\psi_+(e^{2\pi i}x) = e^{2\pi i(\ell+1)}\psi_+(x)$
- Q-function: $Q_\pm(E, \ell) = W[y_0, \psi_\pm](E, \ell)$

$$W[y_k, \psi_\pm](E, \ell) = \omega^{\pm(\ell+\frac{1}{2})} W[y_0, \psi_\pm](\omega^{2k}E, \ell)$$

Baxter's T-Q relation: $-T_{-3}y_0 = y_{-1} + y_1$

$$(-T_{-3})Q_\pm(E, \ell) = \omega^{\mp(\ell+\frac{1}{2})} Q_\pm(\omega^{-2}E, \ell) + \omega^{\pm(\ell+\frac{1}{2})} Q_\pm(\omega^2 E, \ell)$$

- $E = E_n$ such that $T(E_n, \ell) = 0$ (y_0, y_1 become linearly dependent)

$$\frac{Q_\pm(\omega^{-2}E_n, \ell)}{Q_\pm(\omega^2 E_n, \ell)} = -\omega^{\pm(2\ell+1)}$$

Bethe ansatz equation of the twisted six-vertex model

Y-system and TBA equation

- [Zamoldchikov] The Y-system can be transformed into the non-linear integral equations called the Thermodynamic Bethe-ansatz (TBA) equation. $\epsilon_a(\theta) = \log Y_a(\theta)$: pseudo-energy $(E = \exp(2M\theta/(M+1)))$

$$\epsilon_a(\theta) = m_a L e^{\theta} - \sum_b \int_{-\infty}^{+\infty} \phi_{ab}(\theta - \theta') \log(1 + e^{-\epsilon_b(\theta')}) d\theta'$$

- free energy

$$F(L) = -\frac{1}{4\pi} \sum_a \int_{-\infty}^{+\infty} m_a e^{\theta} \log(1 + e^{-\epsilon_a(\theta)}) d\theta = -\frac{\pi c_{eff}}{6L}$$

- The UV effective central charge becomes

$$c_{eff}^{UV} = 1 - \frac{6\left(\ell + \frac{1}{2}\right)^2}{M+1}$$

which agrees with the one of the twisted six-vertex model.

modified sinh-Gordon equation

modified Sinh-Gordon equation [Lukyanov-Zamolodchikov 1003.5333]

$$\partial_z \partial_{\bar{z}} \phi - e^{2\phi} + p(z)\bar{p}(\bar{z})e^{-2\phi} = 0, \quad p(z) = z^{2M} - s^{2M}$$

- zero curvature condition $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$A = \begin{pmatrix} \frac{1}{2}\partial\phi & -e^{\lambda}e^{\phi} \\ p(z)e^{\lambda}e^{\phi} & -\frac{1}{2}\partial\phi \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -\frac{1}{2}\bar{\partial}\phi & -e^{-\lambda}e^{\phi} \\ \bar{p}(\bar{z})e^{-\lambda}e^{\phi} & \frac{1}{2}\bar{\partial}\phi \end{pmatrix}$$

  λ: spectral parameter

- asymptotic behavior of $\phi(z, \bar{z})$ at $\rho \to 0, \infty$ ($z = \rho e^{i\theta}$)
  - $\phi(\rho, \theta) \to M \log \rho$ ($\rho \to \infty$)
  - $\phi(\rho, \theta) \to \ell \log \rho$ ($\rho \to 0$)
    We can introduce a new parameter ℓ for the boundary condition of ϕ at $\rho = 0$.

linear problem and its asymptotic solutions

- linear problem $(\partial + A)\Psi = (\bar{\partial} + \bar{A})\Psi = 0$
- linear problem is invariant under
  Symanzik rotation Ω: $\theta \to \theta + \frac{\pi}{M}$, $\lambda \to \lambda - \frac{i\pi}{M}$
- At $\rho \to \infty$, the subdominant solution is

$$\Psi \sim \begin{pmatrix} e^{\frac{iM\theta}{2}} \\ e^{-\frac{iM\theta}{2}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1}\cosh(\lambda + i(M+1)\theta)\right)$$

- $\rho \to 0$ basis $\Psi_+(\rho,\theta|\lambda) \to \begin{pmatrix} 0 \\ e^{(i\theta+\lambda)\ell} \end{pmatrix}$, $\Psi_-(\rho,\theta|\lambda) \to \begin{pmatrix} e^{(i\theta+\lambda)\ell} \\ 0 \end{pmatrix}$
- $$\Psi = Q_-(\lambda)\Psi_+ + Q_+(\lambda)\Psi_-$$

  $Q_\pm(\lambda)$ defines the Q-function satisfying the Bethe ansatz equation.
- T-functions and the Y-functions are also defined. They satisfy the D-type Y-system.

From MShG to ODE

- Take the light-cone limit $\bar{z} \to 0$. The linear system reduced to a holomorphic differential equation. $(\partial + A_z)\Psi = 0$.
- Under the gauge transformation by $U = diag(e^{\phi}, e^{-\phi})$, it becomes

$$(\partial_z + \tilde{A}_z)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \tilde{A}_z = \begin{pmatrix} \partial\phi & e^{\lambda} \\ p(z)e^{\lambda} & -\partial\phi \end{pmatrix}$$

- linear system $\Longrightarrow$ ODE (Miura transformation)

$$\left[(\partial_z - \partial_z\phi)(\partial_z + \partial_z\phi) - e^{2\lambda}p(z)\right]\psi_1(z) = 0$$

- conformal limit:
  $z \to 0$, $\lambda \to \infty$ with fixed $x = ze^{\frac{\lambda}{M+1}}, E = s^{2M}e^{\frac{2\lambda M}{1+M}}$, $\phi \sim \ell \log x$

$$\left[-(\partial_x - \frac{\ell}{x})(\partial_x + \frac{\ell}{x}) + x^{2M}\right]\psi = \left[-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2M}\right]\psi = E\psi$$

  This is the ODE of [Dorey-Tateo, BLZ]

affine Toda field equations and ODE/IM correspondence

two-dimensional affine Toda field Theory based on $\hat{\mathfrak{g}}$:
r-component scalar fields: $\phi(z,\bar{z})=(\phi^1,\ldots,\phi^r)$
complex coordinates: $z=\frac{1}{2}(x^0+ix^1)$, $\quad \bar{z}=\frac{1}{2}(x^0-ix^1)$ $(z=\rho e^{i\theta})$

$$\mathcal{L}=\frac{1}{2}\partial^\mu\phi\cdot\partial_\mu\phi-\left(\frac{m}{\beta}\right)^2\sum_{i=0}^{r}n_i\left[\exp(\beta\alpha_i\cdot\phi)-1\right],$$

affine Toda field equation:

$$\partial_z\partial_{\bar{z}}\phi+\left(\frac{m^2}{\beta}\right)\sum_{i=0}^{r}n_i\alpha_i\exp(\beta\alpha_i\phi)=0\,.$$

- without the potential term $e^{\beta\alpha_0\phi}$, the theory is conformally invariant (e.g. Liouville theory)
- with the potential term $e^{\beta\alpha_0\phi}$, it becomes massive theory. The equation of motion changes under the conformal transformation.

modified affine Toda field equation

conformal transformation ($\rho^\vee$: co-Weyl vector)

$$z \to \tilde{z} = f(z), \quad \phi \to \tilde{\phi} = \phi - \frac{1}{\beta}\rho^\vee \log(\partial f \bar{\partial}\bar{f}),$$

modified affine Toda equations:

$$\partial\bar{\partial}\phi + \left(\frac{m^2}{\beta}\right)\left[\sum_{i=1}^{r} n_i\alpha_i \exp(\beta\alpha_i\phi) + p(z)\bar{p}(\bar{z}) n_0\alpha_0 \exp(\beta\alpha_0\phi)\right] = 0,$$

$$p(z) = (\partial f)^h, \quad \bar{p}(\bar{z}) = (\bar{\partial}\bar{f})^h.$$

Lax formalism

- The modified affine Toda equation can be expressed as the zero-curvature condition: $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$A = \frac{\beta}{2}\partial\phi \cdot H + me^{\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + p(z)\sqrt{n_0^{\vee}}E_{\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\},$$

$$\bar{A} = -\frac{\beta}{2}\bar{\partial}\phi \cdot H - me^{-\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{-\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + \bar{p}(\bar{z})\sqrt{n_0^{\vee}}E_{-\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\}$$

λ: spectral parameter

- linear problem: $(\partial + A)\Psi = 0$ and $(\bar{\partial} + \bar{A})\Psi = 0$.
- gauge transformation: $\Psi \to U\Psi$, $A \to UAU^{-1} + U\partial U^{-1}$

asymptotic behavior of $\phi(z, \bar{z})$

$$\partial\bar{\partial}\phi + \left(\frac{m^2}{\beta}\right)\left[\sum_{i=1}^{r} n_i\alpha_i \exp(\beta\alpha_i\phi) + p(z)\bar{p}(\bar{z})n_0\alpha_0 \exp(\beta\alpha_0\phi)\right] = 0$$

- Motivated by the previous works[Dorey et al.], we fix

$$p(z) = z^{hM} - s^{hM}, \quad \bar{p}(\bar{z}) = \bar{z}^{hM} - \bar{s}^{hM}$$

  h is the Coxeter number, and M is some positive real parameter
- For large $|z|$, the asymptotic solution is

$$\phi(z, \bar{z}) = \frac{M}{\beta}\rho^{\vee}\log(z\bar{z}) + \cdots$$

- For small $|z|$, the field diverges logarithmically

$$\phi(z, \bar{z}) = g\log(z\bar{z}) + \cdots .$$

  g is an r-component vector that controlls the small z-behavior.

- periodicity

$$\phi(\rho, \theta + \frac{2\pi}{hM}) = \phi(\rho, \theta)$$

- The linear problem is invariant under the Symanzik rotation:

$$\hat{\Omega}_k : \begin{cases} z \to z e^{\frac{2\pi k i}{hM}} \\ s \to s e^{\frac{2\pi k i}{hM}} \\ \lambda \to \lambda - \frac{2\pi k i}{hM} \end{cases}$$

for $k \in \mathbf{Z}$.

$A_r^{(1)}$ modified affine Toda equations

[KI-Locke, Adamopuolou-Dunning]

- The linear problem $(\partial_z + A_z)\Psi = (\partial_{\bar{z}} + A_{\bar{z}})\Psi = 0$, $\Psi = {}^t(\psi_1, \cdots, \psi_{r+1})$
- the fundamental representation with highest weight ω_1
- flat connection

$$\tilde{A}_z = \begin{pmatrix} \beta h_1 \partial\phi & me^{\lambda} & 0 & \cdots & 0 \\ 0 & \beta h_2 \partial\phi & me^{\lambda} & & \vdots \\ & & \ddots & & \\ \vdots & & & \beta h_r \partial\phi & me^{\lambda} \\ me^{\lambda} p(z) & & & 0 & \beta h_{r+1}\partial\phi \end{pmatrix}.$$

weights are $h_1 = \omega_1$, $h_i = \omega_i - \omega_{i+1}$, $h_{r+1} = -\omega_r$

- the linear problem gives a single $(r+1)$-th order differential equation

$$D(h_{r+1}) \cdots D(h_1)\tilde{\psi}_1 = (-me^{\lambda})^h p(z)\tilde{\psi}_1.$$

$D(h) \equiv \partial + \beta h \cdot \partial\phi$

- scalar Lax operator (Gelfand-Dickii, Drinfeld-Sokolov reduction)

Massive ODE/IM correspondence

- For large $|z|$ the small solution is

$$\Xi(\rho,\theta|\lambda) \sim C \begin{pmatrix} e^{-\frac{irM\theta}{4}} \\ e^{-\frac{i(r-2)M\theta}{4}} \\ \vdots \\ e^{\frac{irM\theta}{4}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1}\, m\cosh(\lambda + i\theta(M+1))\right)$$

- For small $|z|$, the solution $\Psi^{(i)}$ with components

$$(\Psi^{(i)})_j \sim \delta_{ij}(\bar{z}/z)^{\frac{\beta}{2}h_i\cdot g} + \cdots .$$

- we can expand Ξ as

$$\Xi = \sum_{i=0}^{r} Q_i(\lambda)\Psi^{(i)}.$$

$Q_i(\lambda)$ is the Q-functions satisfying the $A_r^{(1)}$ type Bethe equations.

Conformal Limit and ODE/IM correspondence

- First we take the light-cone limit $\bar{z} \to 0$ and we define the conformal limit $z \to 0$, $\lambda \to \infty$ with fixed

$$x = (me^{\lambda})^{1/(M+1)} z, \quad E = s^{hM} (me^{\lambda})^{hM/(M+1)}$$

- The differential equation becomes

$$\left[D_x(h_{r+1}) \cdots D_x(h_1) - (-1)^h p(x, E) \right] \psi(x, E, g) = 0$$

where $D_x(a) = \partial_x + \beta \frac{a \cdot g}{x}$ and $p(x, E) \equiv x^{hM} - E$.

- This is the ODE for A_r-type Lie algebra Suzuki, Dorey-Dunning-Tateo

- subdominant solution on the real positive axis

$$\psi(x,E,g) \sim C x^{-\frac{rM}{2}} \exp\left(-\frac{x^{M+1}}{M+1}\right)$$

- small x

$$\chi^{(i)} \sim x^{\mu_i} + \mathcal{O}(x^{\mu_i+h})$$

$\mu_i = i - \beta g \cdot h_{i+1}$

- Symanzik rotation

$$\psi_k(x,E,g) = \psi(\omega^k x, \Omega^k E, g)$$

with $\Omega = \exp(i\frac{2\pi M}{M+1})$ and $\omega = \exp(i\frac{2\pi}{h})$

- $$\psi(x,E,g) = \sum_{i=0}^{r} Q^{(i)}(E)\chi^{(i)}(x,E,g)$$

Bethe ansatz equation

Dorey-Dunning-Masoero-Suzuki-Tateo

- auxiliary functions: $\psi^{(a)} = W^{(a)}[\psi_{\frac{1-a}{2}}, \cdots, \psi_{\frac{a-1}{2}}]$ $(a = 2, \cdots, r)$
- A_n ψ-system (Plücker relations)

$$\psi^{(a-1)}\psi^{(a+1)} = W[\psi^{(a)}_{-\frac{1}{2}}, \psi^{(a)}_{\frac{1}{2}}], \quad \psi^{(0)} = \psi^{(n)} = 1$$

- quantum Wronskian relation

$$Q^{(a+1)}Q^{(a-1)} = \omega^{\frac{1}{2}(\mu_a - \mu_{a-1})} Q^{(a)}_{-\frac{1}{2}} \bar{Q}^{(a)}_{\frac{1}{2}} - \omega^{\frac{1}{2}(\mu_{a-1} - \mu_a)} Q^{(a)}_{\frac{1}{2}} \bar{Q}^{(a)}_{-\frac{1}{2}}$$

- Bethe ansatz equation

$$\omega^{\mu_{i-1} - \mu_i} \frac{Q^{(i-1)}_{-1/2}(E^{(i)}_n) Q^{(i)}_1(E^{(i)}_n) Q^{(i+1)}_{-1/2}(E^{(i)}_n)}{Q^{(i-1)}_{1/2}(E^{(i)}_n) Q^{(i)}_{-1}(E^{(i)}_n) Q^{(i+1)}_{1/2}(E^{(i)}_n)} = -1\,.$$

where $E^{(i)}_n$ are zeros of $Q^{(i)}(E)$.

Other affine Lie algebras

[KI-Locke,1312.6759]

$A_r^{(1)}$	$D(\mathbf{h})\psi = (-me^{\lambda})^h p(z)\psi$
$D_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial^{-1}D(\mathbf{h})\psi = 2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$B_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$A_{2r-1}^{(2)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = -2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$C_r^{(1)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = (me^{\lambda})^h p(z)\psi$
$D_{r+1}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r+1}(me^{\lambda})^{2h}p(z)\partial^{-1}p(z)\psi$
$A_{2r}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = -2^{r}\sqrt{2}(me^{\lambda})^h p(z)\psi$
$G_2^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 8(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$D_4^{(3)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi + (\omega+1)2\sqrt{3}(me^{\lambda})^4 D(\mathbf{h}^\dagger)p(z)$ $-(\omega+1)2\sqrt{3}(me^{\lambda})^4 pD(\mathbf{h}) - 8\sqrt{3}\omega(me^{\lambda})^3 D(-h_1)\sqrt{p}\partial\sqrt{p}D(h_1)$ $+(\omega-1)^3 12(me^{\lambda})^8 p\partial^{-1}p\}\psi = 0$

$D(\mathbf{h}) = D(h_r)\cdots D(h_1)$, $D(\mathbf{h}^\dagger) = D(-h_1)\cdots D(-h_r)$ for $\mathbf{h} = (h_r,\cdots,h_1)$

Langlands duality

Langlands (GNO) dual: $\hat{\mathfrak{g}}$: simple roots $\alpha_i \Longleftrightarrow \hat{\mathfrak{g}}^\vee$: $\alpha_i^\vee$ simple coroots

- $\hat{\mathfrak{g}}^\vee = \hat{\mathfrak{g}}$ for $\hat{\mathfrak{g}} = X_r^{(1)}$ $(X = ADE)$
- $\hat{\mathfrak{g}}^\vee = X_r^{(s)}$ for non-simply laced $\hat{\mathfrak{g}}$ (twisted affine Lie algebra) $(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$, $(C_r^{(1)})^\vee = D_{r+1}^{(2)}$, $(F_4^{(1)})^\vee = E_6^{(2)}$, $(G_2^{(1)})^\vee = D_4^{(3)}$, $(A_{2r}^{(2)})^\vee = A_{2r}^{(2)}$
- In Dorey-Dunning-Masoero-Suzuki-Tateo (2007), they found a set of pseudo-differential equations associated to the Langlands dual of classical Lie algebras

affine Toda equation	ODE(Dorey et al.)
$A_r^{(1)}$	A_r
$(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$	B_r
$(C_r^{(1)})^\vee = D_{r+1}^{(2)}$	C_r
$D_r^{(1)}$	D_r

ψ-system for $\hat{\mathfrak{g}}^\vee$ and the Bethe ansatz equations

- The (pseudo) ODE is rather complicated for exceptional type.
- In the conformal case, one can derive the same ψ-system from the solutions of the linear problem. This was shown in the case of classical Lie algebra [Sun,1201.1614])
- In [Ki-Locke, 1502.00906], we have generalized this construction of ψ-system to linear problem associated to a modified affine Toda equations for $\hat{\mathfrak{g}}^\vee$. (see also [Masoero-Raimondo-Valeri])

We consider the affine Toda field equations for $\hat{\mathfrak{g}}$

- Applying the gauge transformation

$$U_A = z^{M\rho^\vee \cdot H} e^{-\beta\phi\cdot H/2}$$

gives a simple form of the linear problem in the large ρ limit,

$$\tilde{A} = me^{\lambda} z^M \Lambda_+, \quad \tilde{\bar{A}} = me^{-\lambda} \bar{z}^M \Lambda_-$$

$$\Lambda_\pm = \sqrt{n_0^\vee} E_{\pm\alpha_0} + \sum_{i=1}^{r} \sqrt{n_i^\vee} E_{\pm\alpha_i}$$

- We will consider the fundamental representations $V^{(a)}$ with the highest weight ω_a $(a = 1, \cdots, r)$ of $\hat{\mathfrak{g}}$.
  $\mathbf{e}_i^{(a)}$: a basis of $V^{(a)}$ associated with the weights $h_i^{(a)}$ and $\mathbf{e}_1^{(a)}$ is the vector for $h_1^{(a)} = \omega_a$

- asymptotic form for a subdominant solutions along the positive real axis

$$\Psi^{(a)}(z,\bar{z}|\lambda) = \exp\left(-2\mu^{(a)}\frac{\rho^{M+1}}{M+1}m\cosh\left(\lambda+i\theta(M+1)\right)\right)e^{-i\theta M\rho^{\vee}\cdot H}\boldsymbol{\mu}^{(a)}.$$

$\mu^{(a)}$ and $\boldsymbol{\mu}^{(a)}$ are the eigenvalues of $\Lambda_{+}^{(a)} = (\Lambda_{-}^{(a)})^{T}$ with the largest real part and its eigenvector in module $V^{(a)}$.

- For small z,

$$\mathcal{X}_i^{(a)} = e^{-(\lambda+i\theta)\beta g\cdot h_i^{(a)}}\mathbf{e}_i^{(a)} + \mathcal{O}(\rho) \text{ as } \rho\to 0$$

- Q-functions

$$\Psi^{(a)}(z,\bar{z}|\lambda,g) = \sum_{i=1}^{\dim(V^{(a)})} Q_i^{(a)}(\lambda,g)\ \mathcal{X}_i^{(a)}(z,\bar{z}|\lambda,g)$$

ψ-system

- embedding map:

$$\iota : \bigwedge^{2} V^{(a)} \to \bigotimes_{b=1}^{r} \left(V^{(b)}\right)^{B_{ab}} .$$

- the highest weight states in $\bigwedge^2 V^{(a)}$

$$\mathbf{e}_1^{(a)} \wedge E_{-\alpha_a} \mathbf{e}_1^{(a)}$$

$$2\omega_a - \alpha_a = \sum_{b=1}^{r} B_{ab}\omega_b$$

A_{ab}: Cartan matrix of $\mathfrak{g}$ and B_{ab} the incidence matrix

$$B_{ab} = 2\delta_{ab} - A_{ab} .$$

- take the anti-symmetric product of for $\Psi^{(a)}_{\pm 1/2}$ and decompose it by the embedding map such that large ρ asymptotics matches

ψ-system for $\hat{\mathfrak{g}}^\vee$

$\hat{\mathfrak{g}} = A, D, E$ [Sun, Masoero-Raimondo-Valeri, KI-Locke]

- ψ-system

$$\iota\left(\Psi^{(a)}_{-1/2} \wedge \Psi^{(a)}_{1/2}\right) = \bigotimes_{b=1}^{r} \left(\Psi^{(b)}\right)^{B_{ab}}.$$

- This follows from that the eigenvalues $\mu^{(a)}$ satisfy the equations

$$2\cos(\pi/h)\,\mu^{(a)} = \sum_{b=1}^{r} B_{ab}\,\mu^{(b)},$$

For $A^{(1)}_r$, $\mu^{(a)} = \sin\frac{\pi a}{r+1} / \sin\frac{\pi}{r+1}$

Bethe-ansatz equations for affine Toda field equations

- conformal limit We first take the light-cone limit $\bar{z} \to 0$. Then consider the limit $\lambda \to \infty$ and $z, s \to 0$ with fixed

$$x = (me^{\lambda})^{1/(M+1)} z, \quad E = s^{hM} (me^{\lambda})^{hM/(M+1)},$$

- the solution $\psi^{(a)}$ is expanded as

$$\psi^{(a)}(x, E) = Q^{(a)}(E)\,\chi_1^{(a)}(x, E) + \tilde{Q}^{(a)}(E)\,\chi_2^{(a)}(x, E) + \cdots,$$

with $\chi_i^{(a)} \sim x^{\lambda_i^{(a)}}$ $(\lambda_i^{(a)} = \rho^\vee \cdot (\omega_a - h_i^{(a)}) - \beta h_i^{(a)} \cdot g)$

- Substituting the ψ-system we obtain the quantum Wronskian relations. For $A_r^{(1)}$ case, for example, we find that

$$\omega^{-\frac{1}{2}(\lambda_1^{(a)} - \lambda_2^{(a)})} Q_{-1/2}^{(a)} \tilde{Q}_{1/2}^{(a)} - \omega^{\frac{1}{2}(\lambda_1^{(a)} - \lambda_2^{(a)})} Q_{1/2}^{(a)} \tilde{Q}_{-1/2}^{(a)} = Q^{(a-1)} Q^{(a+1)}.$$

where $Q_k^{(a)}(E) \equiv Q^{(a)}(\omega^{hMk} E)$ $(\omega = \exp(2\pi i/h(M+1)))$

- Let us denote the zeros of $Q^{(a)}(E)$ as $E_k^{(a)}$. Then the quantum Wronskian relations yields the Bethe ansatz equations.
- $A_r^{(1)}$, $D_r^{(1)}$, $E_r^{(1)}$:

$$\prod_{b=1}^{r} \frac{Q^{(b)}_{A_{ab}/2}}{Q^{(b)}_{-A_{ab}/2}}\Bigg|_{E_k^{(a)}} = -\omega^{1+\beta\alpha_a\cdot g}.$$

- For the modified affine Toda field equations for $\hat{\mathfrak{g}}^\vee$ with an untwisted affine Lie algebra $\hat{\mathfrak{g}}$, one obtains the Bethe ansatz equations for $\hat{\mathfrak{g}}$ found by Reshetikhin and Wiegmann (1987) and Kuniba-Suzuki (1995) based on $U_q(\hat{\mathfrak{g}})$.

ODE/IM and CFT

- Dorey-Dunning-Tateo [0712.2010] argued that the ODE

$$(-\partial^2 + x^{2M} - E)\psi(x, E, \ell) = 0$$

  corresponds to the non-unitary minimal model $M_{2,2M+2}$ with central charge $c = 1 - \frac{6M^2}{M+1}$ perturbed by the operator $\phi_{1,M}$ with conformal dimension $\Delta_{1,M} = \frac{1-4M^2}{16(M+1)}$. [Bazhanov-Lukyanov-Zamolodchikov].
- The $M_{2,M}$ is realized by the fractional level coset CFT:

$$\frac{su(2)_L \times su(2)_1}{su(2)_{L+1}}, \quad L = \frac{1}{M} - 2$$

ODE/IM and CFT(2)

- For the ODE analysis and the BAE, Dorey et al. claimed (conjectured) that the ODE for $\mathfrak{g} = ABCD$ with

$$P_K(x) = (x^{h^\vee (b-a)/(aK)} - E)^K$$

corresponds to the fractional coset model

$$\frac{\hat{\mathfrak{g}}_L \times \hat{\mathfrak{g}}_K}{\hat{\mathfrak{g}}_{L+K}}$$

where

$$L = \frac{Ka}{b-a} - h^\vee, \quad Ku = b - a$$

where $u = 1, 2, \cdots$

Summary

ODE/IM correspondence for $\hat{\mathfrak{g}}^\vee$ modified affine Toda field equations

ODE	$\Longleftrightarrow$ ODE/IM	BAE $U_q(\hat{\mathfrak{g}})$	$\Longleftrightarrow$	CFT
$\Uparrow$ Conformal limit		$\Uparrow$ UV limit		$\Uparrow$
Linear problem of affine Toda equation $\hat{\mathfrak{g}}^\vee$	$\Longleftrightarrow$ massive ODE/IM	BAE $U_q(\hat{\mathfrak{g}})$	$\Longleftrightarrow$	massive IM

Outlook

- general $p(z)$ Bazhanov-Lukyanov
- ODE/IM for $B_r^{(1)}$, $C_r^{(1)}$, $G_2^{(1)}$, $F_4^{(1)}$
  $B_2^{(1)}$ [Ito-Shu] null-polygonal Wilson loop and form factor in AdS_4
- Gaudin-type Bethe equations [Feigin-Frenkel]
  KZ and Drinfeld-Sokolov reduction
- ODE and the quantum spectral curve[KI and Shu, work in progress]
  2d/4d correspondence for the Argyres-Douglas theory of ADE type