A Critique of the Fuzzball Program

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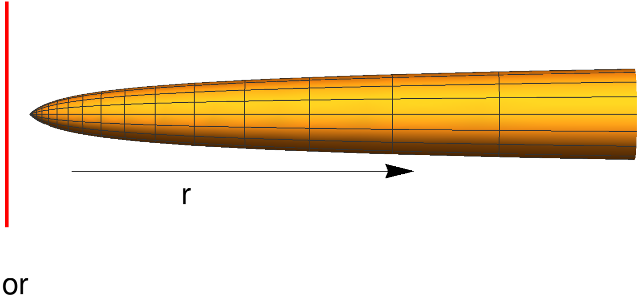
Outline

Fuzzballs

- Fuzzballs are classical solutions with the same charges as the black-hole. Look like black-holes at long-distances. Differ where the horizon would have been. (Fuzzballs have no horizon.)

- Avoid no-hair theorem, because an extra-dimension shrinks to zero before we reach the horizon.



The Fuzzball proposal

- Fuzzballs have structure; the extra-dimension can shrink to zero, in various ways.
- Claim is that fuzzballs are the true microstates of the black-hole.
- Fuzzball program also claims that black-holes have no interior. (This feature also suggested as resolution to information paradox.)

Summary

- We will examine the viability of the fuzzball proposal.
- Will argue, on general statistical-mechanics consideration that black-hole microstates cannot be represented by distinct classical geometries.
- Also argue that fuzzballs cannot serve as reliable indicators of the nature of the black-hole interior.
- These general arguments are backed by specific calculations in various sets of fuzzball solutions.

Outline

Typical States

- Let H_E be the subspace of states in the energy band $[E - \Delta E, E + \Delta E]$.
- **Theorem:** Typical states picked with the Haar measure, $d\mu_\psi$, on this space are exponentially close to the microcanonical ensemble.

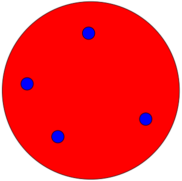
$$\int \langle \Psi | A | \Psi \rangle d\mu_\psi = \mathrm{Tr}(\rho A),$$

where $\rho = e^{-S} \mathbf{1}_{\mathbf{H_E}}$

- Deviations are exponentially suppressed

$$\int \left(\langle \Psi | A | \Psi \rangle - \mathrm{Tr}(\rho A) \right)^2 d\mu_\psi = \frac{\left(\mathrm{Tr}(\rho A^2) - (\mathrm{Tr} \rho A)^2 \right)}{e^S + 1}$$

Typicality of most states

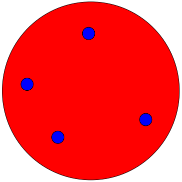


Most states are very close to typical. Volume of atypical states is exponentially suppressed.

Implications for the fuzzball program

- Almost all black-hole microstates correspond to a single special average geometry
- Average geometry must be the conventional black-hole when conventional black-hole has a regular horizon. [Otherwise, strong implications for AdS/CFT.]

Fuzzballs as a basis



- Perhaps fuzzballs form an atypical basis?
- But even a basis cannot be too atypical.

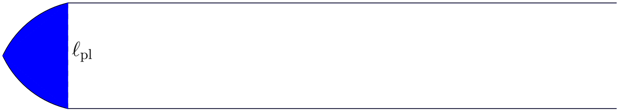
Limits on atypicality

- Let A be an operator where ratio of microcanonical standard deviation and expectation value is small: $\frac{\sigma}{\langle A \rangle} = \frac{1}{S^{\alpha}}$ for some positive number α. eg. Take A to be the metric operator at some point in space well away from the horizon.
- Let $|v_{\alpha_1}\rangle \ldots |v_{\alpha_M}\rangle$ be those elements of a basis where $\frac{\langle v_{\alpha_j}|A|v_{\alpha_j}\rangle - \langle A \rangle}{\langle A \rangle}$ remains finite in the thermodynamic limit.
- Then $\frac{M}{e^S}$ vanishes at least as fast as $O\left(\frac{1}{S^{\alpha}}\right)$.

Limits on atypicality

So, even if fuzzballs form an atypical basis, $1 - \mathrm{O}\left(\frac{1}{S^\alpha}\right)$ of fuzzball states must have metric expectation values within $\mathrm{O}\left(\frac{1}{S^\alpha}\right)$ of the black-hole away from the horizon.

So if fuzzballs are microstates, typical fuzzballs must resemble a black-hole almost exactly up to the horizon; and possibly deviate a Planck distance away.



Additional arguments for Planck-scale structure

- Previous argument relied on assuming that bulk metric was good observable with small fluctuations.

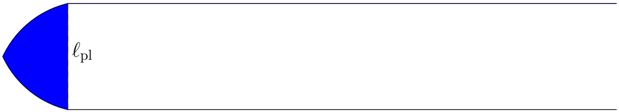
- But, even considering asymptotic observables, we expect this structure because black-hole microstates are expected to satisfy
  1. Vanishing gap between excitations in thermodynamic limit: $O\left(e^{S}\right)$ states in $O(S)$ energy means neighbouring states are separated by $O\left(e^{-S}\right)$. (Requires large red-shifts.)
  2. Eigenstate thermalization:

$$\langle v_j|A|v_i\rangle = A_i\delta_{ij} + Be^{\frac{-S}{2}}R_{ij}$$

     Requires most basis states to be close to the microcanonical average.

Planck-scale structure

Can classical solutions be used to argue for such Planck-scale structure?



Difference and Classicality Parameters

For any observable, define classicality parameter

$$\epsilon_A(r, x) = \left|\frac{\sigma(r, x)}{A^{\text{fuz}}(r, x)}\right|$$

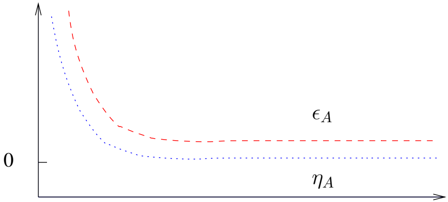
and difference parameter

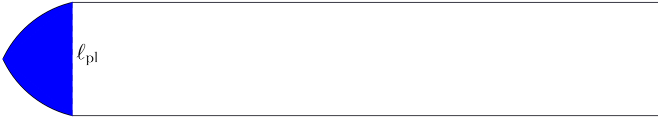
$$\eta_A(r, x) = \left|\frac{A^{\text{bh}}(r, x) - A^{\text{fuz}}(r, x)}{A^{\text{fuz}}(r, x)}\right|$$

These measure how reliable a classical solution is and how distinguishable it is from the black-hole.

Planck scale structure?

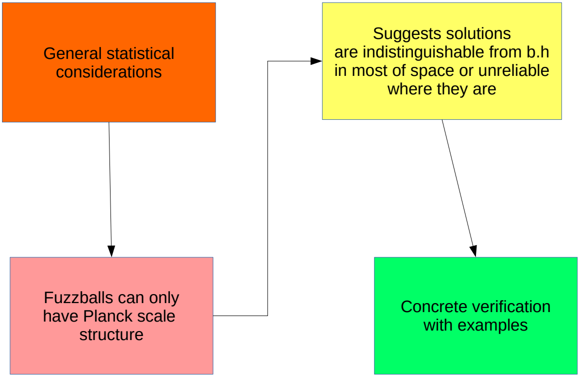
Expect that





So the solution is either indistinguishable from the conventional black-hole or unreliable.

Summary of logic



Summary of rest of talk

We now verify these expectations in

1. Original Lunin-Mathur two-charge solutions (claimed to correspond to 1/2-BPS states of the D1-D5 system)
2. Recently discovered three-charge solutions (Bena et al.) (claimed to correspond to 1/4-BPS black holes in the D1-D5 system.)

Outline

Lunin-Mathur Geometries

Claimed to be dual to 1/2-BPS sector of the D1-D5 CFT.

$$ds^2 = e^{-\frac{\phi}{2}} ds^2_{\rm str}, \quad e^{-\phi} = \frac{f_5}{f_1},$$

$$ds^2_{\rm str} = \frac{1}{\sqrt{f_1 f_5}} \left(-(dt + A)^2 + (dy + B)^2 \right) + \sqrt{f_1 f_5} d\vec{x}^2 + \sqrt{\frac{f_1}{f_5}} d\vec{z}^2,$$

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{ds}{|\vec{x} - \vec{F}(s)|^2}; \quad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|\vec{F}'(s)|^2}{|\vec{x} - \vec{F}(s)|^2}$$

$$A_i = \frac{Q_5}{L} dx^i \int_0^L \frac{F_i(s)}{|\vec{x} - \vec{F}(s)|^2} ds; \quad dB = *_4 dA$$

$$C = \frac{1}{f_1} (dt + A) \wedge (dy + B) + \mathcal{C}; \quad d\mathcal{C} = - *_4 df_5.$$

Conventional solution

- Conventional solution obtained by

$$f_1 \to 1 + \frac{Q_1}{\vec{x}^2}; \quad f_5 \to 1 + \frac{Q_5}{\vec{x}^2}$$

with

$$ds^2_{\text{str}} = \frac{1}{\sqrt{f_1 f_5}} \left(-dt^2 + dy^2\right) + \sqrt{f_1 f_5} d\vec{x}^2 + \sqrt{\frac{f_1}{f_5}} d\vec{z}^2,$$

- Conventional solution has vanishing horizon; different setting compared to macroscopic black holes.

Quantization of Lunin-Mathur Solutions

- Solutions were quantized by Rychkov. $F^k(s)$ becomes an operator

$$F^k(s) = \mu \sum_{n>0} \frac{1}{\sqrt{2n}} \left(a_n^k e^{\frac{-2\pi i n s}{L}} + (a_n^k)^\dagger e^{\frac{2\pi i n s}{L}} \right),$$

- States with right charges satisfy $\sum n a_n^\dagger a_n = N_1 N_5$.

- Also

$$\mu = \frac{g_s}{R\sqrt{V_4}}, \quad L = \frac{2\pi Q_5}{R} \quad Q_5 = g_s N_5; \quad Q_1 = g_s N_1 / V_4$$

$$S_{\text{fuzz}}(E) = 2\pi\sqrt{\frac{2N_1 N_5}{3}}$$

Not the full entropy, $S(E) = 2\pi\sqrt{2N_1 N_5}$, but at least scales correctly.

Physical Quantities Computed

- We will compute

$$\langle f_5 \rangle, \quad \langle f_1 \rangle, \quad \langle A_i \rangle$$

and

$$\langle f_5^2 \rangle, \quad \langle f_1^2 \rangle, \quad \langle A_i A_j \rangle$$

- We can compute "thermal" expectations

$$\langle O \rangle_\beta = \mathrm{Tr}\left(e^{-\beta H} O \right)$$

$$\text{where} \qquad \beta = \left(\frac{2\pi^2}{3 N_1 N_5} \right)^{\frac{1}{2}}$$

is the inverse-"temperature" at which $\langle H \rangle = N_1 N_5$.

- Precisely verify our expectations of η (difference parameter) and ϵ (classicality parameter)

Quantum Expectation Values

For one-point functions we find

$$\langle f_5 - 1 \rangle_\beta = Q_5 \frac{1 - e^{-\frac{r^2}{a}}}{r^2}$$

$$\langle f_1 - 1 \rangle_\beta = Q_1 \frac{\left(1 - e^{-\frac{r^2}{a}}\right)}{r^2}$$

$$\langle A_i \rangle_\beta = -\frac{Q_5}{r^4} \left(a x_i e^{-\frac{r^2}{a}} \left(1 - e^{\frac{r^2}{a}} + \frac{r^2}{a} \right) \right)$$

where

$$a = \frac{\pi^2 \mu^2}{3\beta}$$

Implications of one-point functions

- The "average" geometry differs from conventional geometry when $r^2 = a$.

- At $r^2 = a$, in units where $\alpha' = 1$, the compact-direction has radius

$$R^2_{\text{stretched}} = \frac{\pi}{2}\sqrt{\frac{2}{3}}\left(\frac{Q_1}{Q_5}\right)^{\frac{1}{4}} \frac{\ell_{\text{pl}}^4}{\sqrt{V_{\text{com}}}}$$

- Volume of the compact-manifold in string-frame should satisfy $V_{\text{com}} \geq 1$ and dilaton should be small $\frac{Q_1}{Q_5} \ll 1$.

$$\implies R_{\text{stretched}} \ll \ell_{\text{pl}}!$$

So the "quantum-corrected" fuzzball geometry corrects conventional geometry after compact direction has shrunk below Planck scale!

Expectations for η and ϵ

$$\langle f_5 \rangle_\beta = 1 + Q_5 \frac{1 - e^{-\frac{r^2}{a}}}{r^2}$$
$$\langle f_5 \rangle_{\text{bh}} = 1 + \frac{Q_5}{r^2}$$

- Away from $r^2 = a$, geometry is indistinguishable from the conventional geometry.
- Close to $r^2 = a$, quantum fluctuations expected to be large, so geometry is unreliable.

Quantum Fluctuations in f_5

We find

$$\langle (f_5 - 1)^2 \rangle = \int_0^L \frac{ds}{L}\frac{ds'}{L}\frac{e^{-\frac{r^2}{c}}}{c^2}\left[\text{Ei}\left(\frac{r^2}{c}\right) - 2\text{Ei}\left(\frac{(a-c)r^2}{ac}\right) + \text{Ei}\left(\frac{(a-c)r^2}{c(a+c)}\right)\right] + \frac{2ae^{-\frac{r^2}{a}}}{cr^2(a-c)} - \frac{(a+c)e^{-\frac{2r^2}{a+c}}}{cr^2(a-c)} - \frac{1}{cr^2}$$

where

$$c = \frac{\mu^2}{\beta}\left[Li_2\left(e^{-\frac{2i\pi(s-s')}{L}}\right) + Li_2\left(e^{\frac{2i\pi(s-s')}{L}}\right)\right]$$

and

$$Ei(x) = -\int_{-x}^{\infty} e^{-t}\mathcal{P}\left(\frac{1}{t}\right) dt$$

Integral over s, s' must be done numerically.

Classicality and Deviation Parameters for f_5: small r

$$\eta_5 = \left| \frac{\langle (f_5 - 1) \rangle_\beta - f_5^{\text{bh}} + 1}{\langle (f_5 - 1) \rangle_\beta} \right|$$

$$\epsilon_5 = \left| \frac{\left(\langle (f_5 - 1)^2 \rangle_\beta - \langle (f_5 - 1) \rangle_\beta^2 \right)^{\frac{1}{2}}}{\langle (f_5 - 1) \rangle_\beta} \right|$$

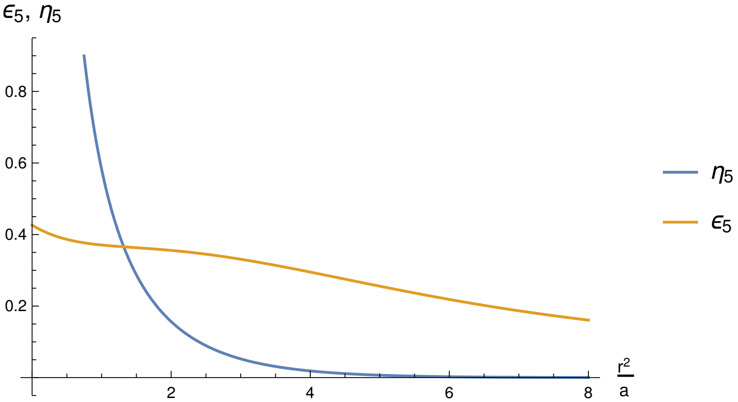
- Solution differs from the black-hole only around $r = O(a)$. For small r

$$\eta_5 = -\frac{a}{r^2} + \frac{1}{2}; \quad \epsilon_5 = 0.426 - 0.119\frac{r^2}{a};$$

- So, quantum fluctuations are $O(1)$ at $r = 0$.

Results for Difference and Classicality Parameters

Can compute fluctuations for larger *r* numerically.



Precisely as expected, solution is either indistinguishable from the conventional solution or unreliable.

Quantum fluctuations in A_i

$$\frac{1}{Q_5^2}\langle A_i A_j \rangle = \mathcal{A}\delta_{ij} + \mathcal{B} x_i x_j$$

where

$$\mathcal{A} = \frac{(c-a)e^{-\frac{r^2}{c}}\left(r^2(c-a)+3c(a+c)\right)\left(-2\text{Ei}\left(\frac{(a-c)r^2}{ac}\right)+\text{Ei}\left(\frac{(a-c)r^2}{c(a+c)}\right)+\text{Ei}\left(\frac{r^2}{c}\right)\right)}{12c^4} + \frac{ae^{-\frac{r^2}{a}}\left(-ac^2(a+3c)+r^4(a-c)^2+cr^2(c-a)(2a+3c)\right)}{6c^3r^4(a-c)} + \frac{(a+c)e^{-\frac{2r^2}{a+c}}\left(c^2r^2(a+c)^2+r^6\left(-(a-c)^2\right)+2cr^4(a-c)(a+c)\right)}{12c^3r^6(a-c)}$$

and

$$\mathcal{B} = \frac{\left(a^2+4ac+c^2\right)e^{-\frac{r^2}{c}}\left(-2\text{Ei}\left(\frac{(a-c)r^2}{ac}\right)+\text{Ei}\left(\frac{(a-c)r^2}{c(a+c)}\right)+\text{Ei}\left(\frac{r^2}{c}\right)\right)}{6c^4} + \frac{ae^{-\frac{r^2}{a}}\left(a^2\left(2c^2+cr^2+r^4\right)+ac\left(6c^2+5cr^2+4r^4\right)+c^2r^2\left(6c+r^2\right)\right)}{3c^3r^6(a-c)} - \frac{(a+c)e^{-\frac{2r^2}{a+c}}\left(a^2\left(2c^2+cr^2+r^4\right)+2ac\left(2c^2+3cr^2+2r^4\right)+c^2\left(2c^2+5cr^2+r^4\right)\right)}{6c^3r^6(a-c)}$$

Difference and Classicality Parameters for A_i

$$\eta_A = 1$$

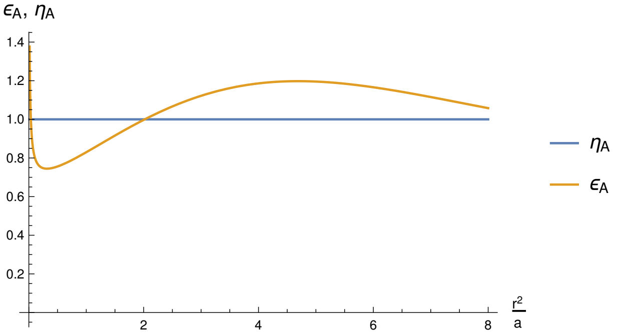
$$\epsilon_A = \frac{\left(\hat{x}^i \hat{x}^j \langle A_i A_j \rangle_\beta - \hat{x}^i \hat{x}^j \langle A_i \rangle_\beta \langle A_j \rangle_\beta\right)^{\frac{1}{2}}}{\hat{x}^i \hat{x}^j \langle A_i \rangle_\beta \langle A_j \rangle_\beta}$$

At small r, we have

$$\epsilon_A = 0.140 \frac{\sqrt{a}}{r} + 1.587 \frac{r}{\sqrt{a}}$$

Difference and Classicality Parameters for A_i

For arbitrary r, we can plot



Outline

Three-charge solutions

- Lunin-Mathur geometries correspond to solutions that have no horizon classically.
- Several solutions with same charges as macroscopic black-holes have been found.
- A recent larger class was found by Bena et al.

Three-charge solutions

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)(du+\omega+\frac{1}{2}\mathcal{F}(dv+\beta)) + \sqrt{\mathcal{P}}ds_4^2$$

$$u = (t-y)/\sqrt{2};\, v = (t+y)/\sqrt{2};\, y \sim y + 2\pi R_y;$$

$$ds_4^2 = \frac{\Sigma dr^2}{r^2+a^2} + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2;$$

$$\mathcal{P} = Z_1 Z_2 - Z_4^2; \quad \beta = \frac{a^2 R_y}{\sqrt{2}\Sigma}(\sin^2\theta d\phi - \cos^2\theta d\psi);$$

$$\Sigma = (r^2 + a^2\cos^2\theta)$$

Solutions are asymptotically AdS and labeled by integers n, m, k and parameters a, b, R_y. We only consider $k=1, m=0$, arbitrary n.

Three-charge solutions

- Charges are

$$J_L = \frac{\mathcal{N}}{2}\left(a^2 + \frac{m}{k}b^2\right); \quad J_R = \frac{\mathcal{N}}{2}a^2; \quad n_p = \frac{\mathcal{N}}{2}\frac{(m+n)}{k}b^2.$$

with $\mathcal{N} = \frac{n_1 n_5}{a^2+b^2/2}$. We will denote $\kappa = \frac{b}{a}$.

- The asymptotic AdS radius is

$$\frac{\lambda^4}{R_y^2} = a^2 + b^2/2.$$

- Useful to think of $b \sim \mathrm{O}(\lambda)$. Then "$a$" controls the size of the fuzzball.

Scalar Wightman Function

- We will compute

$$G(\omega, \gamma) = \int \langle \Psi | O(t, y) O(0, 0) | \Psi \rangle e^{i\omega t} e^{\frac{i\gamma y}{R_y}} dt dy$$

for a marginal scalar operator $O(t, y)$ on the boundary.

- Note this is a Wightman function.

Physical Quantity of Interest: Large γ Behaviour

- At large-γ one can prove for the thermal Wightman function that
$$\lim_{\gamma \to \infty} \frac{-\log |G_{\omega,k}|}{\gamma} \geq \frac{\beta}{2}$$
Here $\beta = \min(\beta_L, \beta_R)$.
- Black holes saturate this bound.
- Physically, the near-horizon region allows arbitrarily spacelike modes to propagate.

Do fuzzball solutions saturate this bound? If not, they violate the Eigenstate Thermalization Hypothesis.

Physical Quantity of Interest: Gap between successive excitations

- In a system with large-entropy, e^S states must fit in an $O(S)$ energy range.
- So gap between successive excitations is $O\left(e^{-S}\right)$
- True even in integrable systems; stronger expectation than eigenstate thermalization hypothesis.
- Only free-theories with degeneracy violate this bound.
- Gap can be measured by considering the support of $G(\omega, \gamma)$.

Propagation of a massless scalar

- Wave-equation $\Box\phi = 0$ is separable!
- We will consider propagation with no angular momentum on S^3 for simplicity.
- We set

$$\phi(r,t,y) = \frac{\psi_{\omega,\gamma}(r)}{\sqrt{r(r^2+a^2)}} e^{i\omega t} e^{i\gamma y}$$

Wave Equation

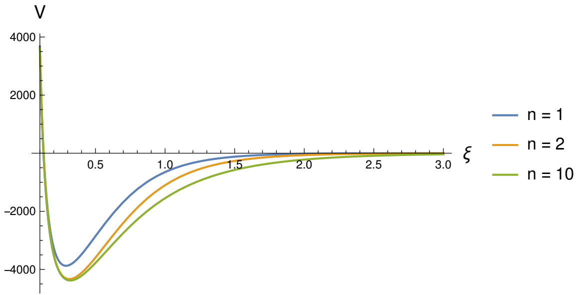
With $\xi = \frac{r}{a}$ and $b = a\kappa$, we have

$$\psi''_{\omega,k}(\xi) - V(\xi)\psi_{\omega,\gamma}(\xi) = 0$$

with

$$V(\xi) = \frac{1}{4\left(\xi^2+1\right)^2}\Big[6 + \frac{4\gamma^2-1}{\xi^2} + 4\gamma^2 + 3\xi^2 + \kappa^2\left(\kappa^2+2\right)(\omega-\gamma)^2\frac{\xi^{2n}}{\left(\xi^2+1\right)^n} - \left(\kappa^2(\omega-\gamma)+2\omega\right)^2\Big]$$

WKB Potential



- A graph of $V(\xi)$ vs ξ with $\gamma = 10, \omega = 0, \kappa = 4$ and different values of *n*.
- Black-hole potential would keep dropping to $-\infty$ near $\xi = 0$.

Energy gap

- At large γ, we can use the WKB approximation. We get the standard quantization condition

$$2\int_{\xi_1}^{\xi_2} |V(\zeta)|^{\frac{1}{2}} d\zeta = (2m+1)\pi$$

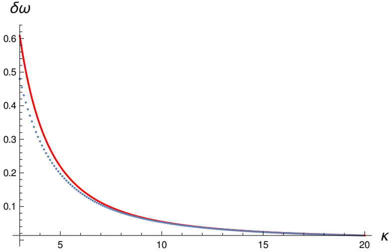
- At large κ we get

$$(\delta\omega)\kappa^2 g_n = \pi$$

where $g_n = \{0.5, 0.574, 0.610, 0.632, 0.648, \ldots\}$.

Numerical calculation of the energy gap

We can calculate the energy-gap by solving the scalar equation numerically. WKB approximation is excellent at large γ.



(Comparison between a numerical calculation (dots) of the gap between the first two allowed frequencies and analytic formula for $\gamma = 100$, $n = 2$.)

Energy-gap conclusions

The energy gap between successive excitations is $O(1)$ and too large for these states to be microstates of the black hole. $O(1)$ gap is suggestive of a phase of zero-entropy.

Large-γ falloff

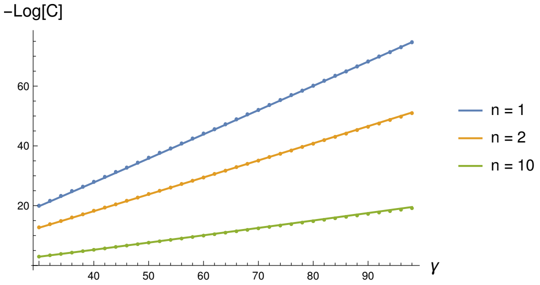
- At large γ, the Wightman function falls off faster than the black-hole. Does not saturate large-γ bound.

$$\lambda_{\text{fuzz}} = \lim_{\gamma \to \infty} \frac{-\log |G_{\omega,k}|}{\gamma} = \frac{\pi}{2\sqrt{n}} + \frac{(11n-1)\pi}{16 n^{\frac{3}{2}} \kappa^2}$$

- $$\lambda_{\text{fuzz}} - \frac{1}{2}\beta_L = \frac{\pi(3n+7)}{16\kappa^2 n^{3/2}} + \mathrm{O}\left(\frac{1}{\kappa^4}\right)$$

Large-γ falloff

Asymptotic falloff can also be verified numerically



Comparison between a numerical calculation (dots) of the asymptotic value C with the analytic formula for different values of n, γ with $\kappa = 5$.

Large-γ falloff conclusions

- For $\kappa = \frac{b}{a} = O(1)$, the Wightman function falls off too fast at large-γ, and suggests that if fuzzball states are black-hole microstates, they violate eigenstate thermalization.

- If these fuzzballs are microstates, some other fuzzballs must "oversaturate" the large-γ bound. We do not know of any geometry that oversaturates the bound.

Outline

Conclusions

- Fuzzballs that vary at $O(1)$ distance from the b.h. horizon cannot represent b.h. microstates.
- If fuzzballs are to represent even a basis of black-hole microstates, typical fuzzballs can vary from the conventional black-hole only Planck-length outside the horizon.
- But, in such geometries, quantum fluctuations become large near horizon. So the classical solution is unreliable where it is interesting.

Fuzzballs as stars

- All such problems arise, if we insist on fuzzballs as black-hole microstates.
- If we think of fuzzballs as stars in string-theory, they constitute an interesting class of solutions, which deserve investigation.

Additional Slides

$a \to 0$ limit

- What if $a = \ell_{\text{pl}}$? Then $\kappa \to \infty$, and the energy-gap and large-γ falloff tend to the black-hole answer.
- The $a \to 0$ solutions represent only a small class of microstates, since $J_L, J_R \propto a^2$.
- But can these solutions be microstates of the non-rotating D1-D5 system?

$a \to 0$ limit

First note that if we stay away from $r \sim \mathrm{O}\,(a)$, then

$$ds_6^2 \xrightarrow[a \to 0]{} \frac{\left(b^2 n - 2r^2\right)}{\sqrt{2} b R_y} dt^2 + \frac{\left(b^2 n + 2r^2\right)}{\sqrt{2} b R_y} dy^2 + \frac{b R_y}{\sqrt{2} r^2} dr^2 + \frac{\sqrt{2} b n}{R_y} dt dy + \frac{b R_y \cos^2(\theta)}{\sqrt{2}} d\psi^2 + \frac{b R_y \sin^2(\theta)}{\sqrt{2}} d\phi^2 + \frac{b R_y}{\sqrt{2}} d\theta^2$$

Change of variables to

$$\rho = \left(r^2 + \frac{b^2 n}{2}\right)^{\frac{1}{2}}$$

shows this is the metric of an extremal BTZ black hole.

$a \to 0$ limit

- But if we take $r = a\xi$ and expand around $\xi = 0$, we find a different metric! eg. for $n = 2$,

$$\sqrt{-g} = a^2\lambda^2\xi\sqrt{1-\xi^4}\cos(\theta)\sin(\theta)$$

- Now, if $a \sim \mathrm{O}\left(\ell_{\mathrm{pl}}\right)$ then $\delta a \sim a$. [ensemble fluctuations.]

- So we expect

$$\epsilon \sim \frac{\delta g}{g} \sim \frac{\delta g}{g\delta a}\delta a = \mathrm{O}(1)$$

if $\frac{\delta g}{\delta a} \sim \frac{g}{a}$ and $\frac{\delta a}{a} = \mathrm{O}(1)$.