Scattering Forms from Geometries at Infinity

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Focus Week on Quantum Gravity and Holography

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Motivations

Scattering Amplitudes: crucial for particle physics, and only known observable of quantum gravity in asymptotically flat spacetime

Natural holographic question: is there a ”theory at infinity” that computes S-matrix without local evolution? much harder than in AdS

∂ of AdS = ordinary space with time & locality” $\implies$ local QFT

No such luxuries for asymptotics of flat spacetime: no time/locality! Mystery: what principles a “theory of S-matrix” should be based on?

This is why S-matrix program failed! New strategy in its revival: look for fundamentally new laws ($\leftrightarrow$ new mathematical structures)
$\rightarrow$ S-matrix as the answer to entirely different kinds of questions
$\rightarrow$“discover” unitarity and causality, as derived consequences!

Geometric structures

Fascinating geometric structures underlying scattering amplitudes (particles, strings *etc.*) in some auxiliary space, encouraging this p.o.v.

- $\mathcal{M}_{g,n}$: **perturbative string theory**, amplitudes = correlators in worldsheet CFT $\rightarrow$ **twistor string theory& scattering equations**, similar worldsheet picture without stringy excitations
- Generalized $G_+(k,n)$: **the amplituhedron for** $\mathcal{N}=4$ **SYM**

Both geometries have "factorizing" boundary structures: locality and unitarity naturally emerge (without referring to the bulk)

What questions to ask, directly in the "kinematic space", to generate local, unitary dynamics? Avatar of these geometries?

Amplitudes as Forms

Scattering amps as differential forms on kinematic space $\to$ a new picture for amplituhedron [Arkani-Hamed, Hugh, Trnka] & much more!

Forms on momentum twistor space "bosonize" superamplitude in $\mathcal{N}=4$ SYM: replacing η_i by $dZ_i \implies \Omega_n^{(4k)}$ for N^{k}MHV tree

(tree) Ampliuehedron = "positive region" $\cap$ $4k$-dim subspaces
$\Omega_n^{(4k)}|_{\rm subspace}$ = **canonical form** of **positive geometry** [Arkani-Hamed, Bai, Lam]

Same for momentum-space forms combining helicity amps ($|h| \leq 1$)

This talk: identical structure for wide variety of theories in any dim:

- Bi-adjoint ϕ^3 from kinematic and worldsheet associahedra
- "Geometrize" color & its duality to kinematics, YM/NLSM *etc.*

Kinematic Space

The kinematic space, $\mathcal{K}_n$, for n massless momenta p_i ($D \geq n{-}1$) is spanned by Mandelstam variables $s_{i\,j}$'s subject to $\sum_{j \neq i} s_{i\,j} = 0$, thus $\dim \mathcal{K}_n = \binom{n}{2} - n = \frac{n(n-3)}{2}$; for any subset I, $s_I = \sum_{i<j \in I} s_{i\,j}$

Planar variables $s_{i,i+1,\cdots,j}$ for an ordering $(12 \cdots n)$ are dual to $n(n{-}3)/2$ diagonals of a n-gon with edges $p_1, p_2, \cdots, p_n$

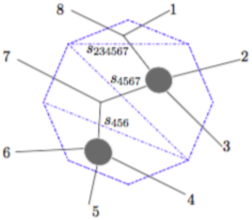
A planar cubic tree graph consists of $n-3$ *compatible* planar variables as poles, and it is dual to a full triangulation of the n-gon

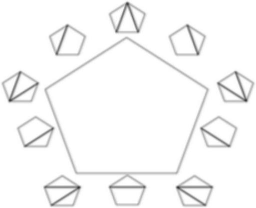
Claim: all the $\frac{n(n-3)}{2}$ planar variables form a basis of $\mathcal{K}_n$

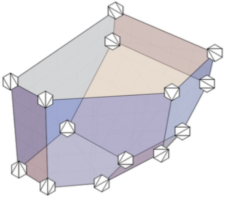
e.g. $\{s_{12} = s, s_{23} = t\}$ for $\mathcal{K}_4$, $\{s_{12},\ s_{23},\ s_{34},\ s_{45},\ s_{51}\}$ for $\mathcal{K}_5$, $\{s_{12},\ \cdots,\ s_{61},\ s_{123},\ s_{234},\ s_{345}\}$ for $\mathcal{K}_6$

The Associahedron

The **associahedron** polytope encodes combinatorial "factorization": each co-dim d face represent a triangulation with d diagonals or planar tree with d propagators (vertices $\leftrightarrow$ planar cubic trees)







Universal factorization structures of any massless tree amps (in particular ϕ^3), but how to realize it directly in kinematic space?

Kinematic Associahedron

Positive region Δ_n: all planar variables $s_{i,i+1,\cdots,j} \geq 0$ (top-dimension)

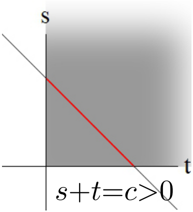
Subspace H_n: $-s_{i\,j} = c_{i,j}$ as *positive constants*, for all non-adjacent pairs $1 \leq i, j < n$; we have $\frac{(n-2)(n-3)}{2}$ conditions $\implies \dim H_n = n-3$.

Kinematic Associahedron is their intersection! $\mathcal{A}_n := \Delta_n \cap H_n$

$$\textbf{Proof}: \; s_{i\,j} = s_{i,\cdots,j} - s_{i,\cdots,j-1} - s_{i+1,\cdots,j} + s_{i+1,\cdots,j-1} = -c_{i,j} < 0$$

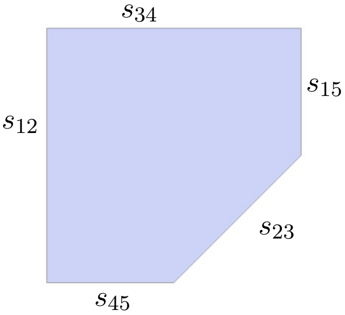
$\implies$ no boundaries for crossing diagonals are allowed, *e.g.* $s_{12} = s_{23} = 0$ forbidden ($-c_{1,3} = s_{13} \geq 0$ leads to contradiction)

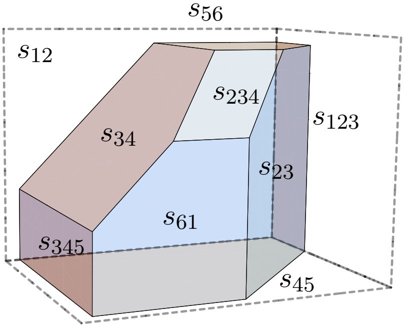
Equivalently, one can show $\mathcal{A}_n$ factorizes to $\mathcal{A}_L \otimes \mathcal{A}_R$ on every face!



$$e.g.\ \mathcal{A}_4 = \{s>0, t>0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_5 = \{s_{12}, \cdots, s_{51} > 0\} \cap \{s_{13}, s_{14}, s_{24} = \text{const} < 0\}$$





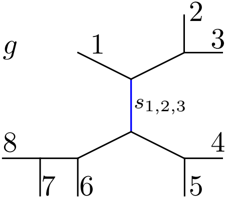
Planar Scattering Forms

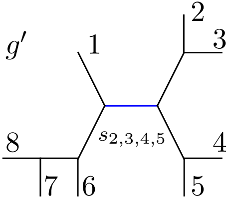
The *planar scattering form* for ordering $(12\cdots n)$ is a sum of rank-$(n-3)$ $d\log$ forms for $\mathbf{Cat}_{n-2}$ planar cubic graphs with $\text{sign}(g)=\pm 1$:

$$\Omega_n^{(n-3)} := \sum_{\text{planar } g} \text{sign}(g) \bigwedge_{a=1}^{n-3} d\log s_{i_a, i_a+1, \cdots, j_a}$$

Projectivity: invariant under *local* GL(1) transf. $s_{i,\cdots,j} \to \Lambda(s) s_{i,\cdots,j}$

$\Longrightarrow$ **Sign-flip rule**: $\text{sign}(g) = -\text{sign}(g')$ for any two planar graphs g, g' related by a *mutation*, *i.e.* exchange of channel in a 4pt subgraph





Projectivity is equivalent to requiring that the form only depends on *ratios* of variables, *e.g.* $\Omega_4^{(1)} = \frac{ds}{s} - \frac{dt}{t} = d\log\frac{s}{t}$ and

$$\begin{aligned}\Omega_5^{(2)} = & \quad \frac{ds_{12}}{s_{12}} \wedge \frac{ds_{34}}{s_{34}} + \frac{ds_{23}}{s_{23}} \wedge \frac{ds_{45}}{s_{45}} + \cdots + \frac{ds_{51}}{s_{51}} \wedge \frac{ds_{23}}{s_{23}} \\ = & \quad d\log\frac{s_{12}}{s_{23}} \wedge d\log\frac{s_{12}}{s_{45}} + d\log\frac{s_{12}}{s_{51}} \wedge d\log\frac{s_{34}}{s_{23}} \\ \Omega_6^{(2)} = & \quad \sum_{g=1}^{14} \pm \wedge (d\log s)^3 = \sum \pm\, d\log \mathrm{ratio}'s\end{aligned}$$

It follows immediately that $\Omega^{(n-3)}$ is cyclically invariant up to a sign $i \to i{+}1$: $\Omega_n^{(n-3)} \to (-1)^{n-3}\, \Omega_n^{(n-3)}$, and it factorizes correctly *e.g.*

$$s_{1,\cdots,m} \to 0: \qquad \Omega_n \to \Omega_{m+1} \wedge d\log s_{1,\cdots,m} \wedge \Omega_{n-m+1}$$

Projectivity is a remarkable property of $\Omega_n^{(n-3)}$, not true for each diagram or any proper subset of planar Feynman diagrams.

Canonical Form of $\mathcal{A}_n$

Unique form of any positive geometry= "volume" of the dual: $\Omega(A)$ has $d\log$ singularities on all boundaries ∂A with $\text{Res} = \Omega(\partial A)$

For simple polytopes: $\sum_v \pm \wedge\, d\log F$ for faces $F = 0$ adjacent to v

Canonical form of $\mathcal{A}_n$ = Pullback of Ω_n to $H_n \propto$ planar ϕ^3 amplitude!

$$e.g. \qquad \Omega(\mathcal{A}_4) = \Omega_4^{(1)}|_{H_4} = \left(\tfrac{ds}{s} - \tfrac{dt}{t}\right)\Big|_{-u=c>0} = \left(\tfrac{1}{s} + \tfrac{1}{t}\right) ds$$

$$\Omega(\mathcal{A}_5) = \Omega_5^{(2)}|_{H_5} = \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right) ds_{12} \wedge ds_{34}$$

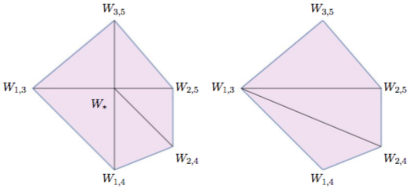
$$\Omega(\mathcal{A}_n) = \sum \text{sgn}(g)\ \wedge d\log s_{i,\cdots,j}(\mathbf{s}, c) = d^{n-3}\mathbf{s}\ m(12\cdots n|12\cdots n)$$

Similarly for $m(\alpha|\beta)$: "volume" of degenerate $\mathcal{A}_n$ (faces at infinity)

Triangulations & New Rep. of ϕ^3 Amps

Geometric picture: Feynman-diagram expansion = triangulation of the dual into $\mathbf{Cat}_{n-2}$ simplices by introducing the point at "infinity"

Triangulate the dual or itself in other ways → new rep. of ϕ^3 amps!



$$\Omega(\mathcal{A}_5) = d^2\mathbf{s}\ (\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}})$$

$$= d^2\mathbf{s}\ \left(\frac{s_{12}+s_{51}}{s_{12}s_{34}s_{51}} + \frac{s_{12}+s_{51}}{s_{12}s_{51}s_{23}} + \frac{s_{12}-s_{45}+s_{23}}{s_{12}s_{23}s_{45}}\right)$$

$= \text{sum of 3 triangles of } \mathcal{A}_5 \text{ itself}$

Similar to "local" or "BCFW" triangulations of the amplituhedron: manifest new symmetries of ϕ^3 obscured by Feynman diagrams!

Wordsheet Associahedron

A well-known associahedron: minimal blow-up of the open-string worldsheet $\mathcal{M}_{0,n}^{+} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford]

This is non-trivial in σ's but becomes manifest *e.g.* using cross ratios

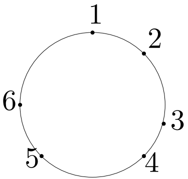
The *canonical form* of $\overline{\mathcal{M}}_{0,n}^{+}$ is the "Parke-Taylor" form [see also Mizera]

$$\omega_n^{\mathrm{WS}} := \frac{1}{\mathrm{vol}\,[\mathrm{SL}(2)]} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}} := \mathrm{PT}(1,2,\cdots,n)\; d\mu_n$$

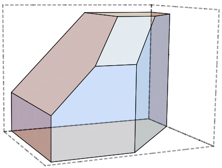
Beautifully "planar scattering form" of $\overline{\mathcal{M}}_{0,n}^{+}$ in cross-ratio space.

How to connect old and new: $\overline{\mathcal{M}}_{0,n}^{+} \leftrightarrow \mathcal{A}_n$ and their canonical forms?

Scattering Equations as a Diffeomorphism



scattering equations $\longrightarrow$ as a map from $\overline{\mathcal{M}}_{0,n}$ to $\mathcal{A}_n$



With pullback to H_n ($-s_{i,j} = c_{i,j}$), scattering equations ($\sum_{b\neq a} \frac{s_{a\,b}}{\sigma_a - \sigma_b} = 0$) provide a one-to-one map from $\overline{\mathcal{M}}^+_{0,n}$ to $\mathcal{A}_n$:

$$s_{a,a+1} = \sigma_{a,a+1} \sum_{1<i+1\leq a\leq j<n} \frac{c_{i,j}}{\sigma_{i,j}} \qquad \text{for } a = 1, \ldots, n{-}3 \ (\sigma_n \to \infty)$$

and similarly for $s_{a,a+1,\cdots,b}$: positive iff $\{\sigma\} \in \mathcal{M}^+_{0,n}$ (on H_n)!

One (out of $(n{-}3)!$) positive solution iff positive kinematics $\{s\} \in \Delta_n$.

Pushforward from Worldsheet

Theorem: diffeomorphism $A \to B \implies$ pushforward $\Omega(A) \to \Omega(B)$

$$y = f(x) \text{ as diffeom. from } A \text{ to } B: \qquad \Omega(B)_y = \sum_{x=f^{-1}(y)} \Omega(A)_x$$

$\implies$ canonical form of $\mathcal{A}_n$ is the pushforward of $\omega_n^{\rm WS}$ by summing over $(n-3)!$ sol. of scattering eqs. (equivalent to CHY)

$$\sum_{\rm sol.} d\mu_n \, {\rm PT}(\alpha)|_{H(\alpha)} = m(\alpha|\alpha)\, d^{n-3}\mathbf{s} \quad \text{or} \quad \sum_{\rm sol.} \omega_n^{\rm WS}(\alpha) = \Omega_{\phi^3}^{(n-3)}(\alpha)$$

$$\textbf{General}: \quad \sum_{\rm sol.} d\mu_n I_n := \Omega_n[I] \;\to\; \Omega_n[I]|_{H_\alpha} = d^{n-3}\mathbf{s} \int_{\rm CHY} {\rm PT}(\alpha) \times I_n$$

General Scattering Forms

General Scattering Forms: sum over all cubic graphs with numerators

$$\Omega[N] = \sum_g N(g) \bigwedge_{I=1}^{n-3} d\log s_I \,, \quad e.g.\ N_s\, d\log s + N_t\, d\log t + N_u\, d\log u$$

N_g are “kinematic numerators” that can depend on other data, for all $(2n{-}5)!!$ cubic tree graphs, *e.g.* 15 for $n = 5$ and 105 for $n = 6$.

Ω_{ϕ^3}: $N_g = 0$ for non-planar graphs and $N_g = \pm 1$ for planar ones.

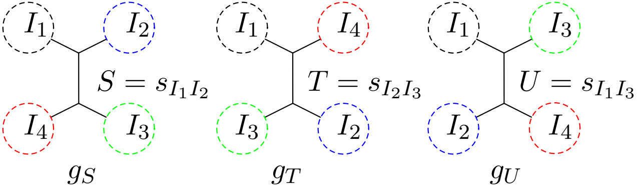
Natural Qs: what constraints can we put on these forms and what physics information do they contain? How can we relate them to scattering amplitudes of some theories? $\rightarrow$ Projectivity is the key!

Projectivity and BCJ duality

Projectivity: require $\Omega[N]$ to be well-defined in projectivized space (treating s_I's independently), *i.e.* covariant under $s_I \to \Lambda(s) s_I$

$\implies$ kinematic numerators can be chosen to satisfy Jacobi identities

$$N(g_S) + N(g_T) + N(g_U) = 0\,, \quad e.g.\ N_s + N_t + N_u = 0$$



A "geometric" origin of BCJ duality [BCJ 08], but how to get amps?

Color is Kinematics I

$f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \quad \leftrightarrow \quad \mathrm{d}s_{12} \wedge \mathrm{d}s_{45}$

Duality between *color factors* and *differential forms on $\mathcal{K}_n$* for cubic graphs: $C(g)$ and $W(g)$ satisfy the same algebra! Recall

$$C(g) := \prod_{v=1}^{n-2} f^{a_v b_v c_v} \implies C(g_S)+C(g_T)+C(g_U)=0\,, \forall \text{ triplet}$$

$$\textbf{Claim}: \; W(g) := \pm \bigwedge_{I=1}^{n-3} ds_I \implies W(g_S)+W(g_T)+W(g_U)=0$$

This is a basic fact of $\mathcal{K}_n$ directly follows from mom-conservation:

$$s_{I_1 I_2}+s_{I_2 I_3}+s_{I_1 I_3}=s_{I_1}+s_{I_2}+s_{I_3} \implies (ds_{I_1 I_2}+ds_{I_2 I_3}+ds_{I_1 I_3})\wedge\cdots=0$$

Fundamental link between color & kinematics (forms on $\mathcal{K}_n$) $\implies$ Duality between color-dressed amps and scattering forms:

$$\mathbf{M}_n[N] \qquad \leftrightarrow \qquad \Omega^{(n-3)}[N]$$

$$\mathbf{M}_n[N]=\sum_{\text{cubic } g} N(g)C(g)\prod_{I\in g}\frac{1}{s_I}$$

$$\Omega^{(n-3)}[N]=\sum_{\text{cubic } g} N(g)W(g)\prod_{I\in g}\frac{1}{s_I}$$

Scattering forms are color-dressed amps without color factors!

Color is Kinematics II

More is true for U(N)/SU(N): partial amps as pullback of forms

$$\textbf{trace decomp}.\ \mathbf{M}_n[N] = \sum_{\beta \in S_n/Z_n} \text{Tr}(\beta(1), \ldots, \beta(n)) M_n[N;\beta]$$

$$\Longrightarrow\ \textbf{partial amp}.\quad M_n[N;\beta] = \sum_{\beta-\text{planar } g} N(g|\beta) \prod_{I \in g} \frac{1}{s_I}$$

Completely parallel: Partial amplitude = pullback of scattering form to subspace $H(\beta) = \{s_{\beta(i)\,\beta(j)} = \text{const.}\}$ for non-adjacent $1 \leq i < j < n$

$$\Omega^{(n-3)}[N]|_{H[\beta]} = \left(\sum_{\beta\text{-pl. } g} N(g|\beta) \prod_{I \in g} \frac{1}{s_I} \right) dV[\beta] = M_n[N;\beta] dV[\beta]$$

Gravity Amplitudes and Double Copy

How about theories without color, such as gravity amplitude? A 0-form or equivalently top form, $\Omega^{\rm top} = M_n \times d^{n(n-3)/2} s$

Define dual forms for every scattering form: *e.g.* the dual for ϕ^3

$$\tilde{\Omega}_{\phi^3}(1,2,\cdots,n) := \bigwedge_{1 \leq i < j-1 < n-1} ds_{i,j} \,,$$

$\Longrightarrow$ pullback to partial amp, *e.g.* $M_n^{\rm YM}(\alpha) d^{n(n-3)/2} s = \Omega_{\rm YM} \wedge \tilde{\Omega}_{\phi^3}(\alpha)$

Natural language for BCJ double-copy : top form for *e.g.* gravity is literally the (wedge) product of a $\Omega_{\rm YM}$ and its dual $\tilde{\Omega}_{\rm YM}$:

$$\Omega^{\rm top}_{\rm GR} = \Omega^{(n-3)}_{\rm YM} \wedge \tilde{\Omega}^{(n-2)(n-3)/2}_{\rm YM} = d^{n(n-3)} s \sum_g \prod_{I \in g} \frac{N(g) \tilde{N}(g)}{s_I}$$

Scattering Forms for Gluons and Pions

Permutation invariant forms encoding full amps of gluon/pion scattering (*e.g.* can be directed constructed from "Feynman rules") $\rightarrow$ Remarkably rigid, fundamental objects for YM & NLSM (lowest dim):

$$\Omega^{(n-3)}_{\text{YM/NLSM}} = \sum_{g}^{(2n-5)!!} N_{\text{YM/NLSM}}(g) \bigwedge_{a=1}^{n-3} d\log S_{I_a}$$

$N_{\text{NLSM}}(\{s\})$ of degree-$(n-2)$ in $s_{i,j}$; $N_{\text{YM}}(\{\epsilon, p\})$ from contractions of momenta & polarizations with no more than $(n-2)$ $(\epsilon_i \cdot p_j)$ (rep. dependent and can be chosen to satisfy Jacobi) . For $n = 4$:

$$\Omega^{(1)}_{\text{NLSM}} = s dt - t ds = t du - u dt = u ds - s du$$
$$\Omega^{(1)}_{\text{YM}} = \frac{\mathbf{T}_8(\epsilon,p)}{stu}\,\Omega^{(1)}_{\text{NLSM}} = N_{\text{YM}}(g_{1234}) d\log\tfrac{s}{t} + N_{\text{YM}}(g_{1324}) d\log\tfrac{u}{t}$$

Uniqueness of YM and NLSM Forms

Gauge invariance: $\Omega_{\rm YM}$ invariant under every shift $\epsilon_i^\mu \to \epsilon_i^\mu + \alpha p_i^\mu$

Adler zero: $\Omega_{\rm NLSM}$ vanishes under every soft limit $p_i^\mu \to 0$

Key: forms are projective $\implies$ unique $\Omega_{\rm YM}$ and $\Omega_{\rm NLSM}$!

Stronger than the amp "uniqueness theorem" [Arkani-Hamed, Rodina, Trnka]: $(n-1)!$ parameters for amp vs. unique form up to an overall const.

Proof: (1). $\Omega^{(n-3)}_{\rm ansatz} = \sum_{\pi \in S_{n-2}} W(g_\pi) A_n(\pi)$ with "partial amps" $A_n(\pi)$ sum over π-planar graphs
(2). $A_n(\pi) = \alpha_\pi M_n(\pi)$ by amp "uniqueness", and $\alpha_\pi = \alpha$ since $A_n(\pi)$'s must satisfy BCJ relations by projectivity!

Direct proof/deeper reason for uniqueness of forms? Extended positive geometry for gluons ("geometrize" polarizations) & pions ?

YM and NLSM Forms from the Worldsheet

Despite lack of geometric meaning, non-$d\log$ projective forms can also be obtained as pushforward of non-$d\log$ worldsheet forms

$$\Omega[N] = \sum_\pi N(g_\pi)\Omega_{\phi^3}(\pi) \implies \Omega[N] = \sum_{\text{sol.}} \omega[N]\,,$$
$$\omega[N] \simeq d\mu_n\ \sum_\pi N(g_\pi)\ \mathrm{PT_n}(\pi)\,, \quad \simeq \text{ means = up to scattering eqs}$$

$\Omega_{\mathrm{YM/NLSM}}$ as pushforward of **canonical, rigid worldsheet objects**:

$$\Omega^{(n-3)}_{\mathrm{YM}} = \sum_{\text{sol.}} d\mu_n\ \mathrm{Pf}'\Psi_n(\{\epsilon, p, \sigma\}) \quad \Omega^{(n-3)}_{\mathrm{NLSM}} = \sum_{\text{sol.}} d\mu_n\ \det{}' A_n(\{s, \sigma\})$$

$\implies$ at this order, $\mathrm{Pf}'\Psi_n$ (or $\det' A_n$) is the unique gauge inv. (or Adler zero) worldsheet function, on support of scattering eqs!

Question: why $\mathrm{Pf}'\Psi_n$ also determines complete superstring amps?

Outlook

- "Factorization Polytopes" : relations to cluster associahedra, gen. permutohedra [Postnikov] & " MHV leading singularities" [Cachazo]
- Loops : "geometrize" color for loops; ambitwistor strings
- Four Dimensions : "amplituhedron" in momentum space; forms combining helicity amps & pushforward from twistor string
- Beyond amplitudes: Witten diagrams, cosmological polytopes *etc.*
- Towards a unified geometric picture for amplitudes & more

Thank you for your attention!