# Sensitivity to the Neutrino Mass Hierarchy

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1 June 2016

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# Outline

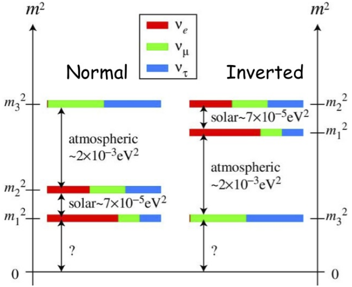
- Sensitivity to the Hierarchy
  - Non-nested hypotheses
  - Different Approaches
  - Possible Definitions of Sensitivity
- Reactor Neutrino Experiments
  - Mass Hierarchy Determination
  - Systematic Errors due to Non-Linearity

# The neutrino mass hierarchy

There are three light, mostly-active neutrino mass eigenstates called $\nu_1$, $\nu_2$ and $\nu_3$ with masses are $m_1$, $m_2$ and $m_3$. Define

$$\Delta m^2_{ij} = m^2_i - m^2_j$$

- Vacuum oscillations of ultrarelativistic $\nu \Rightarrow |\Delta m^2_{ij}|$
- Coherent interactions of solar neutrinos with the Sun $\Rightarrow \Delta m^2_{21} > 0$




The neutrino mass hierarchy (MH) is Sign($\Delta m^2_{31}$).

From vacuum oscillations, we know $|\Delta m^2_{ij}|$.

$$|\Delta m^2_{31}| = |\Delta m^2_{32}| \pm |\Delta m^2_{21}|$$

The hierarchies are two disjoint hypotheses

One hierarchy is true (realized in nature), the other is false

# Binary classification test

Define the classification function

$$\Delta\chi^2 = \chi^2_{\rm IH} - \chi^2_{\rm NH}$$

Where $\chi^2_{\rm NH/IH}$ are the $\chi^2$ values of the data to NH/IH using:

1) The best fit value of the nuisance parameters for *each* hierarchy
2) A penalty term in $\chi^2$ for each nuisance parameter

We will be interested in the binary classification test Sign($\Delta\chi^2$)

**Classification Rule**

If Sign($\Delta\chi^2$) is positive (negative) then the test result is that MH is normal (inverted)

Note that $\Delta\chi^2$ is *not* the quantity in the Wilks' theorem, because the last term is not necessarily the best fit:

It is the difference between two *disjoint* hypotheses, not two *nested* hypotheses

# Distribution of $\Delta\chi^2$

The binary classification function

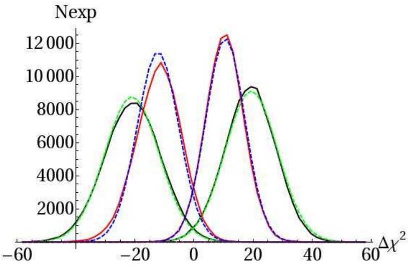
$$\Delta\chi^2 = \chi^2_{\mathrm{IH}} - \chi^2_{\mathrm{NH}}$$

does not follow a one-degree-of-freedom $\chi^2$ distribution
(for example: it is not always $> 0$)
Not the first case in physics: see, for ex. Cousins *et al.*, JHEP 2005
Discrimination between spin-0 and spin-2 resonances at LHC

**In the absence of degeneracies**, to a good approximation it follows a Gaussian distribution, with

$$\mu = \pm\overline{\Delta\chi^2} \qquad \sigma = 2\sqrt{\overline{\Delta\chi^2}}$$

Right: distribution of $\Delta\chi^2$ using different assumptions on the precision on $\Delta m^2_{32}$. Dashed curves: results of a JUNO MC simulation




EC, Evslin and Zhang JHEP 2014

# Disjoint Hypotheses: Approaches

The neutrino literature has so far used three statistical approaches to the MH determination:

1) **Cox 1961; 1962**:
Test each hypotheses (frequentist approach), then compare the likelihood. Possible to accept or reject each hypothesis separately
In the neutrino literature: (Qian et al. PRD 2012; Blennow et al. JHEP 2014)

2) **Hotelling 1940; Vuoung 1989**:
Consider the hypothesis that both $H_1$ and $H_2$ are equally effective. Test this new hypothesis (frequentist approach).
In the neutrino literature: (Capozzi, Lisi and Marrone PRD 2014)

3) **Jeffreys 1935; 1961; Kass and Raftery 1995:**
Bayesian Model Selection with the Bayes factor
In the neutrino literature: (Qian et al. PRD 2012; EC, Evslin and Zhang JHEP 2014; Blennow JHEP 2014)

**DISCLAIMER**

There are no ''right" an ''wrong" definitions of sensitivity of an experiments; we will discuss several possible definitions, stressing the relations with quantities of physical interest

# Comparison of Approaches

These approaches are each designed to answer different questions

1) **Cox 1961; 1962**:
   Tests the compatibility of the data with each hypothesis
   Often it will conclude that both or neither hypothesis is compatible at the desired significance

2) **Hotelling 1940; Vuoung 1989**:
   Tests how *robustly* the hierarchy has been determined
   Typically it yields a sensitivity of about half (measured in $\sigma$'s) of Cox's method (less than $2\sigma$ for JUNO )

3) **Jeffreys 1935; 1961; Kass and Raftery 1995:**
   Only method which incorporates the fact that precisely one MH is true (*model selection*, not *goodness of fit*)
   Requires a prior: for MH there is a very natural (symmetric) prior
   We will focus on this method for the rest of the talk

**Model selection**: Assume that precisely one of two disjoint models is correct and choose one of them.
**Goodness of Fit:** A frequentist analysis of the likelihoods of various datasets given a null hypothesis

# Mean Sensitivity

- *For a given experiment, what is the probability that the hierarchy which yields the lowest $\chi^2$, is the true hierarchy?*
  It only depends on $\overline{\Delta\chi^2}$ and the fact that *one* MH is true, but is independent of the priors on MH

$$p_s = \frac{1}{2}\left(1 + \mathrm{erf}\left(\sqrt{\frac{\overline{\Delta\chi^2}}{8}}\right)\right)$$

  Equivalent to a number of (1-sided) $\sigma$'s $s_s = \sqrt{\overline{\Delta\chi^2}}/2$ .

- **"Frequentist" model selection:** Let one MH be the null hypothesis (say NH), assume that precisely one MH is true. Then $\alpha$ (significance level)= $\beta$ (power). Same criterion was also used in Cousins *et al.*
- If our experiment finds the right MH with only $\Delta\chi^2 = 0.5$, it's not very convincing $\Rightarrow$ we want the posterior probability

# Bayes Factor

The contribution of an experiment to the determination of the MH can be summarized by the Bayes factor

$$K(\Delta\chi^2) = \frac{p(\Delta\chi^2|NH)}{p(\Delta\chi^2|IH)}$$

Using the Gaussian approximation above, one finds our main result

$$K(\Delta\chi^2) = \frac{\exp\left(-\frac{\left(\overline{\Delta\chi^2}-\Delta\chi^2\right)^2}{8\overline{\Delta\chi^2}}\right)}{\exp\left(-\frac{\left(\overline{\Delta\chi^2}+\Delta\chi^2\right)^2}{8\overline{\Delta\chi^2}}\right)} = \exp\left(\frac{\Delta\chi^2}{2}\right)$$

The posterior odds are then easily found via the standard result

$$\frac{p(\mathrm{NH}|\Delta\chi^2)}{p(\mathrm{IH}|\Delta\chi^2)} = \text{posterior odds} = K(\Delta\chi^2)(\text{prior odds})$$

$$= K(\Delta\chi^2)\frac{\pi(\mathrm{NH})}{\pi(\mathrm{IH})}$$

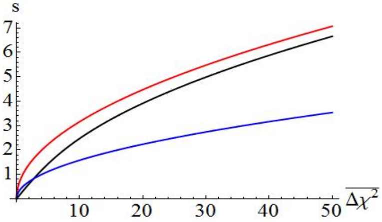
# Sensitivity of the Median Experiment

- *Given the median experiment ($\Delta\chi^2 = \overline{\Delta\chi^2}$), what is its sensitivity to the MH?*
  Using symmetric priors ($\pi(NH) = \pi(IH) = 0.5$) we have

$$p_v = p(\mathrm{NH}|\Delta\chi^2 = \overline{\Delta\chi^2}) = \frac{1}{K(\overline{\Delta\chi^2})^{-1} + 1} = \frac{1}{1 + e^{-\overline{\Delta\chi^2}/2}}$$

$$s_v = \sqrt{2}\mathrm{erf}^{-1}\left(\frac{1 - e^{-\overline{\Delta\chi^2}/2}}{1 + e^{-\overline{\Delta\chi^2}/2}}\right)$$

- Gives a better representation of the typical sensitivity of the experiment to the MH
- However, only 50% of the time the experiment will achieve such precision or better

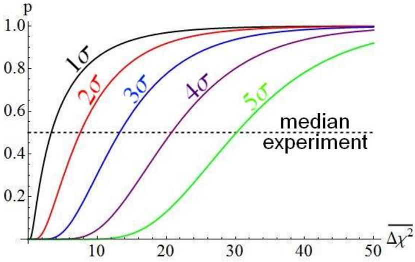



Black: $s_v$, Blue: $s_s$, Red: $\sqrt{\overline{\Delta\chi^2}}$

# Probability of (at least) $s\sigma$ Sensitivity

- *What is the probability $p(s)$ that the hierarchy will be determined with a sensitivity of at least $s\sigma$?*
  Using again symmetric priors we have

$$p(s) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\overline{\Delta\chi^2} - 4\,\mathrm{arctanh}\left(\mathrm{erf}\left(\frac{s}{\sqrt{2}}\right)\right)}{\sqrt{8\overline{\Delta\chi^2}}}\right)\right)$$



# Systematic Errors in Reactor Neutrino Experiments

# Reactor neutrinos

Antineutrino flavors can only be determined via charged current interactions

$$\overline{\nu}_l + X \to \bar{l} + Y$$

Reactor neutrinos are $\overline{\nu}_e$, with energies almost always $\leq 12MeV$.
As they travel, they oscillate into $\overline{\nu}_\mu$ and $\overline{\nu}_\tau$
Reactor $\overline{\nu}_e$ energies are $\geq$ the rest mass of $e$, but $\leq \mu$ and $\tau$
(and sufficient additional energy is not available in the target nuclei)

- Charged current interactions are only possible for $\overline{\nu}_e$
- $\overline{\nu}_e \to \overline{\nu}_e$ probability $P_{\overline{ee}} = P_{ee}$ is observable, not $P_{\overline{e\mu}}$ or $P_{\overline{e\tau}}$

So reactor neutrinos only allow *disappearance experiments*, which are insensitive to the CP-violating phase $\delta$
NB: accelerator driven reactors (under development around the world) create $\overline{\nu}_\mu$ (via $\mu^+$ decay at rest) and so they yield $P_{\overline{\mu e}}$

# Disappearance probability

At these low energies, coherent weak interactions with the background are negligible and so

$$P_{ee} = 1 - \sin^2(2\theta_{12})\cos^4(\theta_{13})\sin^2\left(\frac{1.27\Delta m^2_{21}L}{E}\right)$$
$$- \cos^2(\theta_{12})\sin^2(2\theta_{13})\sin^2\left(\frac{1.27\Delta m^2_{13}L}{E}\right)$$
$$- \sin^2(\theta_{12})\sin^2(2\theta_{13})\sin^2\left(\frac{1.27\Delta m^2_{23}L}{E}\right)$$

As $|\Delta m^2_{31}| \sim |\Delta m^2_{31}| \sim 30|\Delta m^2_{21}|$ one can easily isolate the oscillation in the first term

However the $\Delta m^2_{31}$ and $\Delta m^2_{32}$ oscillations cannot be disentangled, one observes their sum

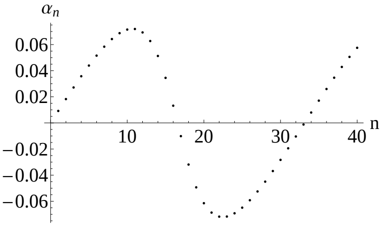
Intuitively the beating of these two similar frequencies determines sign($|\Delta m^2_{32} - \Delta m^2_{31}|$) and so the hierarchy (positive $\leftrightarrow$ IH)

# How JUNO and RENO 50 Determine MH

The MH can be determined studying the peaks of 1-3 and 2-3 oscillations at intermediate baselines. Indeed locations of the peaks is given by (EC, Evslin and Zhang, JHEP 2013)

$$\frac{L}{E} = \frac{\pi}{1.27\Delta m^2_{13}}(n \pm \alpha_n)$$

- $\alpha_n > 0$ $(< 0) \Rightarrow$ NH (IH)
- However, at small $n$ $\alpha_n$ is almost linear $\Rightarrow$ MH degenerate with shift in $\Delta m^2_{13}$
- It is necessary to measure the peaks at small and large n $\Rightarrow$ Intermediate baselines



# Why MH determination is difficult?

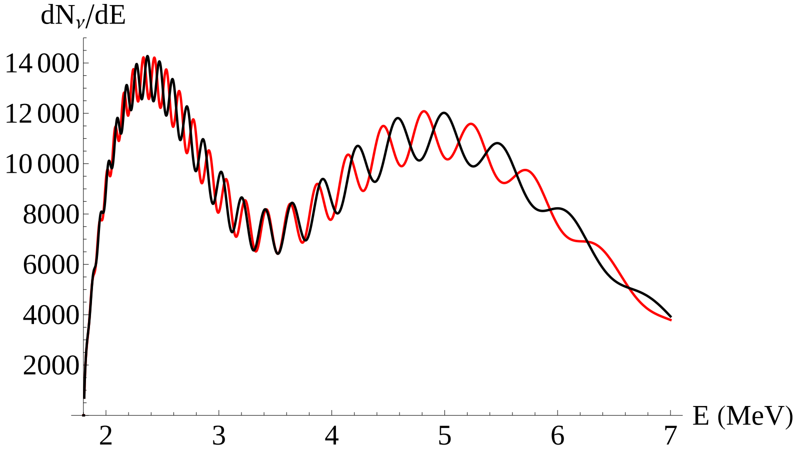



Figure: Expected spectra for normal and inverted hierarchy at 58km

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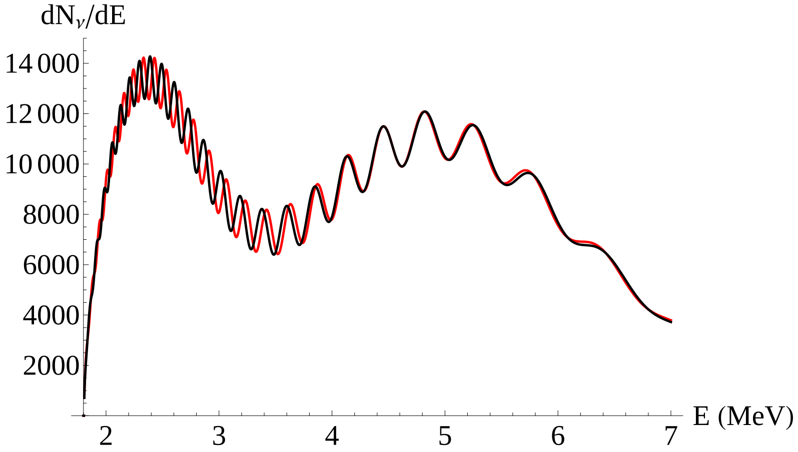


Figure: Expected spectra for normal and inverted hierarchy at 58km. Inverted hierarchy: $\Delta m^2_{23}$ shifted (by $\simeq 1.5\sigma$'s)

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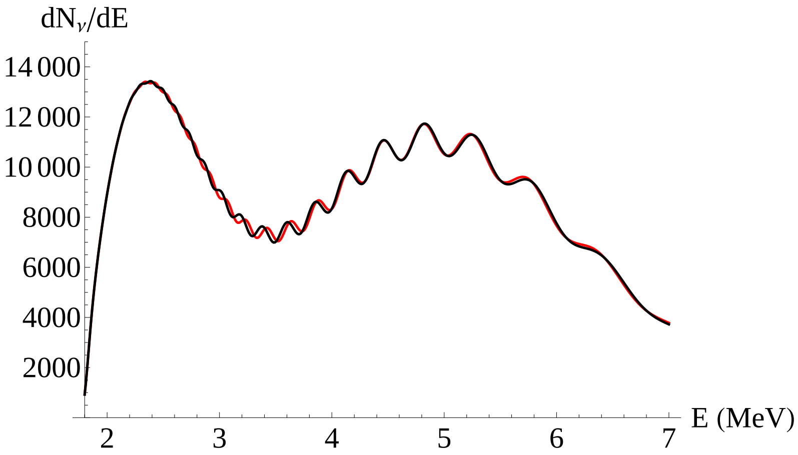



Figure: Expected spectra for normal and inverted hierarchy at 58km. Inverted hierarchy: $\Delta m^2_{23}$ shifted (by $\simeq 1.5\sigma$'s). Finite energy resolution

# Non-linearity

The MH determination in reactor neutrino experiments rely on the study of the interference between 1-3 and 2-3 oscillations
$\Rightarrow$ a systematic error in the energy reconstruction could have a huge impact on the final results of the experiments

$$E_{true} = E_{true}(E_{rec})$$

Several works on this topic in the literature (among the others: Li *et al.*, PRD 2013; EC *et al*, PRD 2014; Capozzi, Lisi and Marrone, PRD 2015)

The usual procedure is to introduce nuisance parameters in the $\chi^2$:

$$E_{true} = (1 + a_0 + a_1 E_{rec} + a_2 E_{rec}^2) E_{rec}$$

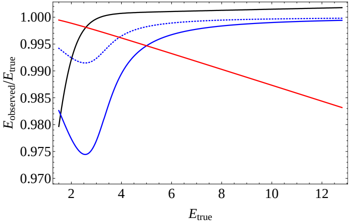
**PROBLEM**

The unknown linear response is an unknown function $\Rightarrow$ infinite-dimensional space.
Is it possible to describe it appropriately with a finite number of nuisance parameters?

# Non-linearity

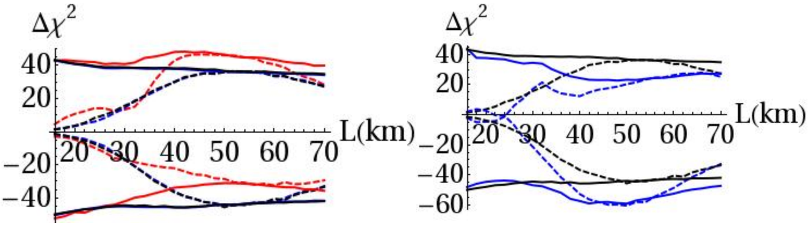
We considered different models of non-linearity:

- Quadratic model: $\Delta E \propto E^2$
- Exponential model: $\Delta E$ decrease exponentially




- Worst-Case Model: (Qian *et al.*, PRD 2013) the non-linearity mimics exactly the opposite hierarchy for a given value of baseline and mixing parameters (considered also multiplied by a factor $1/3$ to be of the order of 1%)

While the quadratic model does not change at all the $\Delta\chi^2$ (non-linearity perfectly fitted by the nuisance parameters), both the exponential and the worst-case models affect the final result.




Left Panel. Black: no NL, Blue: Quadratic model, Red: Exponential model. Right Panel: Worst-Case/3. Solid Curves: two detectors, Dashed Curves: one detector

How to estimate this kind of systematic error?

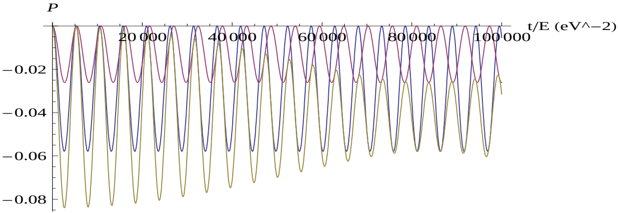
# Conclusions

- The two hierarchies are *disjoint* hypotheses $\Rightarrow \Delta\chi^2$ does not follow a one-degree-of-freedom $\chi^2$ distribution
- We presented different possible definitions of the sensitivity of an experiment
- There is no right or wrong definition, **BUT** it is important always to specify the convention chosen to avoid misunderstanding
- It is difficult to find a good way to parametrize the systematic errors due to non-linearity: infinite dimensional space!
- Even a small discrepancy between the model used and the non-linear response of the detector can significantly affect the final results $\Rightarrow$ How to estimate this kind of systematic error?

# Backup Slides

# MH from beats between 1-3 and 2-3 oscillations

The second term in $P_{ee}$ is




The purple and blue curves are the $\Delta m^2_{32}$ and $\Delta m^2_{31}$ terms and the yellow curve is their sum

Note that the sum is roughly a 2 flavor oscillation, with mass eigenstates 1 and 2 combined

However the frequency isn't quite constant:
It is roughly $|\cos^2(\theta_{12})\Delta m^2_{31} + \sin^2(\theta_{12})\Delta m^2_{32}|/4E$ for the first 10 oscillations but $|\Delta m^2_{31}|/4E$ near the 16th oscillation

The frequency increases with the oscillations **iff** MH is normal

# Uncertainties on the Reactor Neutrino Spectrum

Problem for the measurement of the mixing angles.

## Possible Solution to avoid this problem (ex: Daya Bay)

- Introduce one nuisance parameters for every bin of the energy spectrum
- They are completely uncorrelated, but fully correlated among the detectors (if more than one detector is present)
- A penalty term for each (uncorrelated) nuisance parameter could be introduced if additional information on the spectrum is present (ex: from other experiments)

Possible overestimation. However, in the case of Daya Bay: several detectors are present, at different baselines ⇒ the contribution to the total error is small (not important if it is overestimated).

Recent measurement from reactor neutrino experiments were in tension with the theoretical models (the ''bump'' Daya Bay: An *e et al.*, PRL 2016; Double Chooz: Suekane *et al.*, arXiv:1601.08041; RENO: Seo, arXiv:1410.7987)

# Uncertainties on the Reactor Neutrino Spectrum

Alternative Method (1 detector)

| | |
|---|---|
| Spectrum | $\phi(E)$ |
| Fractional Spectrum Deformation | $\delta\phi(E)/\phi(E)$ |
| Precision on $\delta\phi(E)/\phi(E)$ (binned spectrum, for bin i) | $\sigma_i$ |

For example, consider the determination of $\theta_{12}$

Assume the deformation to be

$$\frac{\delta\phi(E)}{\phi(E)} = \frac{P_{ee}(\theta_{12}, L, E) - P_{ee}(\theta'_{12}, L, E)}{P_{ee}(\theta_{12}, L, E)}$$

and choose $\theta'_{12}$ such as

$$\sum_i \int_{E_i}^{E_{i+1}} \frac{\delta\phi(E)^2}{\phi(E)^2\sigma_i^2} = 1$$

$\sigma_{12}^{syst}$ due to the uncertain on the spectrum is $|\theta_{12} - \theta'_{12}|$

# Nuisance parameters for $\Delta\chi^2$

The $\Delta\chi^2$ is related to the ratio of the likelihood of getting the experimental result **D** given the normal or inverted hierarchy

$$\Delta\chi^2 = -2\ln\frac{p(\mathbf{D}|\hat{\boldsymbol{\nu}}, NH)\pi(\hat{\boldsymbol{\nu}}, NH)}{p(\mathbf{D}|\hat{\boldsymbol{\nu}}', IH)\pi(\hat{\boldsymbol{\nu}}'|IH)}$$

Where $\hat{\boldsymbol{\nu}}$ and $\hat{\boldsymbol{\nu}}'$ are the **best fit values** of the nuisance parameters obtained for the NH and IH separately.

We are interested in the $\Delta\chi^2$ pdf for a given hierarchy $p(\Delta\chi^2|MH)$.

However, using a Bayesian approach, the nuisance parameters should be **integrated**, not minimized

$$p(\Delta\chi^2|MH) = \int p(\Delta\chi^2|MH, \boldsymbol{\nu})\pi(\boldsymbol{\nu}, MH)\mathrm{d}\boldsymbol{\nu}$$

**Laplace Method (Kass and Raftey, 1995)** $\Rightarrow$ if there are no degeneracies the integral can be approximated by the minimization over the nuisance parameters + penalty terms $\Rightarrow$ Verified with MC simulations.

Not valid if the minima of the nuisance parameters are degenerate (ex: accelerator experiments, depends on $\delta_{CP}$)